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PCF Theory

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To my parents

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Contents

Chapter 1

Introduction

The pcf theory (possible cofinalities theory), developed in late 1970's by Saharon Shelah, is a powerful theory with many remarkable applications in set theory. It became especially famous for introducing a completely new way of studying the arithmetic of cardinal numbers, which led to amazing results. Before we mention these results, let us first present their historical context.

In 1940 Gödel showed that CH (the continuum hypothesis: $2^{\aleph_0} = \aleph_1$) is consistent with ZFC axioms (Zermelo-Fraenkel axioms + axiom of choice). By introducing the method of forcing, Cohen showed in 1963 that the negation of CH is also consistent with ZFC. Thus, Cantor's conjecture that $2^{\aleph_0}$ is the first uncountable cardinal was proven to be independent of ZFC, that is, neither provable nor refutable from ZFC. This result completely changed the view on cardinal arithmetic and set off the wave of independence results. In 1970's Easton showed that if f is any function on regular cardinals, such that

1. $f(\lambda_1) < f(\lambda_2)$ for $\lambda_1 < \lambda_2$, and
2. the cofinality of $f(\lambda)$ is bigger than λ,

then it is consistent (with ZFC) to assume that $2^\lambda = f(\lambda)$, for all regular cardinals λ. For a long time it was believed that the same holds for singular cardinals, and hence, that no deep theorems about cardinal arithmetic can be proved within ZFC. So it came as a big surprise in 1974 when Silver produced a new theorem of cardinal arithmetic:

$$\text{if } 2^{\aleph_\alpha} = \aleph_{\alpha+1} \text{ for every } \alpha < \omega_1, \text{ then } 2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}.$$

It became clear that singular cardinals represented a new challenge, which was called *the singular cardinals problem.* In 1975 Galvin and Hajnal came up with the following theorem:

$$\text{if } \aleph_\delta \text{ is a strong limit singular cardinal with } cf(\aleph_\delta) > \aleph_0,$$
$$\text{then } 2^{\aleph_\delta} < \aleph_{(|\delta|^{cf(\delta)})^+}.$$

For example, if $\aleph_{\omega_1}$ is a strong limit, i.e. $2^{\aleph_\alpha} < \aleph_{\omega_1}$ for every $\alpha < \omega_1$, then $2^{\aleph_{\omega_1}} < \aleph_{(2^{\aleph_1})^+}$. Although these ZFC-theorems represented a new trend, set theory was still marked by the tendency to produce independence results, and move on in some sense, rathen than investigate ZFC. Commenting this, Shelah said:

> "...when I became interested in the subject, I saw a great deal of activity and suspected I had come into the game too late; shortly thereafter I seemed to be the only one still interested in getting theorems in ZFC."

However, he was wrong about being late. Making the following three deep observations, he established a new theory: *pcf theory*.

1. Instead of studying cardinal exponentiation, one could, more generally, study (reduced) products of infinite cardinals.

2. It would be useful to shift the focus from cardinalities to cofinalities (of products of cardinals).

3. The notion of cofinality can be generalized to the notion of possible cofinality (cofinality modulo some ultrafilter).

We define the reduced products and study the possible cofinalities in chapters 3 and 4. [Chapter 2 is rather a brief introduction to basic set theory and serves as an overview of the preliminaries needed for understanding later chapters. We refer to [3] for a detailed introduction.]

Applying the pcf theory to cardinal arithmetic, in 1978 Shelah proved a new theorem in ZFC:

$$\text{if } \aleph_\delta \text{ is a singular cardinal such that } \delta < \aleph_\delta, \text{ then } \aleph_\delta^{|\delta|} < \aleph_{(2^{|\delta|})^+}.$$

For example, $\aleph_\omega^{\aleph_0} < \aleph_{(2^{\aleph_0})^+}$. On the one hand, this theorem was special for involving singular cardinals $\aleph_\delta$ of countable cofinality (unlike Galvin - Hajnal theorem), but on the other hand, the upper bound $\aleph_{(2^{|\delta|})^+}$ could be arbitrarily large, by Easton's theorem. Nevertheless, after improving the pcf theory, in 1989 Shelah came up with a much stronger theorem:

$$\text{if } \delta \text{ is a limit ordinal such that } |\delta|^{cf(\delta)} < \aleph_\delta, \text{ then } \aleph_\delta^{cf(\delta)} < \aleph_{|\delta|^{+4}}.$$

For example, if $2^{\aleph_0} < \aleph_\omega$, then $\aleph_\omega^{\aleph_0} < \aleph_{\omega+4}$. Hence, $\aleph_\omega^{\aleph_0} \leq 2^{\aleph_0} + \aleph_{\omega_4}$. Moreover, if $\aleph_\omega$ is a strong limit cardinal, then $2^{\aleph_\omega} = \aleph_\omega^{\aleph_0}$, and thus, $2^{\aleph_\omega} < \aleph_{\omega+4}$. The above theorems will be proved in Chapter 5.

There have been written many papers about pcf theory. Especially great effort to explain and elaborate Shelah's original ideas and proofs was made by Abraham, Burke, Jech, Kojman and Magidor. However, the recent paper [1], by Abraham and Magidor, seems to be one of the best presentations of pcf theory. It explains the substantial parts of the theory separately and very clearly. For instance, exact upper bounds are explained very well.

Our aim is to make a detailed introduction to pcf theory and give a clear insight into the proof of the theorem $\aleph_\omega^{\aleph_0} \leq 2^{\aleph_0} + \aleph_{\omega_4}$. What follows can also be considered as a complement to the splendid work by Magidor and Abraham; we complete some of the proofs and give examples (and use a similar notation). We try to make the theory and its application to cardinal arithmetic as short and fluent as possible, doing a favour to those readers who want to learn pcf theory, but neither have too much time for it, nor want to see only sketches (for example, those who are only interested in the proof of the theorem above).

Unless stated otherwise, all theorems and results in the last three chapters are due to Shelah.

Chapter 2

Some basic set theory

2.1 Sets and numbers

The founder of set theory, Georg Cantor, defined sets to be collections of any objects (that can be thought of). However, the words *any* and *every* turned out to be relative. Russell's paradox[1] was a clear sign that a formal approach to set theory demands more precise definitions. One way to avoid troubles was to start with axioms and only consider 'worlds of objects'(also called models) in which these axioms are true. The most famous system of axioms for set theory is called ZFC[2].

We fix a model of ZFC which becomes our universe. By a *set* we understand any collection of objects in that universe. (If a collection of objects is (possibly) not in the universe, we use the word *class* for it.) The set of all subsets of a set A is called the *powerset* of A and is denoted by $P(A)$. A set A is *countable* if there exists an injective function $f : A \to \mathbb{N}$, otherwise it is *uncountable*.

2.1.1 Relations on sets

Let A be a set. Any subset R of $A \times A = \{(a, b) : a \in A,\ b \in A\}$ is called a *(binary) relation on A*. We usually write $a\ R\ b$ instead of $(a, b) \in R$. A

[1]Consider the collection of all objects which do not contain themselves. Is it contained in itself?

[2]Zermelo-Fraenkel axioms with the axiom of choice: there exists an empty set $\emptyset$ (can be thought of as a unit); there exists an infinite set; pairs, unions, powersets and certain subsets of sets exist (are sets); images of sets (under any function) are sets; two sets are same if and only if they have the same elements; every nonempty set has a $\in$-minimal element; every family of nonempty sets has a choice function. We refer to [3, Chapter 1] for a complete and formal description of ZFC axioms.

relation R is said to be

reflexive if $a\ R\ a$ for every $a \in A$;

irreflexive if $a \not R\ a$ for every $a \in A$;

symmetric if $a\ R\ b$ implies $b\ R\ a$ for every $a, b \in A$;

transitive if $a\ R\ b \wedge b\ R\ c$ implies $a\ R\ c$; and

total if either $a\ R\ b$ or $b\ R\ a$ or $a = b$ for every $a, b \in A$.

Definition 2.1.

1. A binary relation $\leq_P$ on a set P is called a *quasi ordering* of P if it is reflexive and transitive.

2. A binary relation $<_P$ on a set P is called a *strict partial ordering* of P if it is irreflexive and transitive.

3. A total strict partial ordering on a set P is called a *linear ordering* of P.

4. A binary relation on a set P is called an *equivalence relation* on P if it is reflexive, symmetric and transitive.

There can be at the same time both a quasi ordering $\leq_P$ and a strict partial ordering $<_P$ on a set P; we identify P with $(P, \leq_P, <_P)$.

Definition 2.2. Fix a set P and let $\leq_P$ and $<_P$ be a quasi ordering and a strict partial ordering of P, respectively. For nonempty sets $X, Y \subseteq P$, and $p \in P$, we say that

p is a *maximal* element of X if $p \in X$ and $p \not<_P x$ for every $x \in X$;

p is a *minimal* element of X if $p \in X$ and there is no $q \in X$ such that $q \leq_P p$ and $p \not\leq_P q$;

p is a *least* element of X (in the relation $\leq_P$) if $p \in X$ and $p \leq_P x$ for every $x \in X$;

p is an *upper bound* of X (or p bounds X) if $x \leq_P p$ for every $x \in X$;

p is a $<_P$-*upper bound* of X (or p $<_P$-bounds X) if $x <_P p$ for every $x \in X$;

X is *cofinal in* Y in the relation $<_P$ (resp. $\leq_P$) if for every $b \in Y$ there is some $a \in X$ such that $b <_P a$ (resp. $b \leq_P a$) [we also say 'cofinal in $(Y, <_P)$' instead of cofinal in the relation $<_P$];

X is *bounded* in Y if there is an upper bound for X in Y;

So p is a *minimal upper bound* of X if p is an upper bound of X and there is no upper bound q of X such that $q \leq_P p$ and $p \nleq_P q$; and p is a *least upper bound* of X if p is an upper bound of X and $p \leq_P q$ for every upper bound q of X (p is then also called a supremum of X (sup$X$)).

We say that p is an *exact upper bound* of X if p is a least upper bound of X and X is cofinal in $\{q \in P : q <_P p\}$ in the relation $\leq_P$.

Suppose that R is an equivalence relation on a set P. For each $p \in P$, we define the *equivalence class* $[p] := \{q \in P : p \; R \; q\}$ of p. Every element of P is then in some equivalence class ($p \in [p]$), and no element is in two different classes. The *quotient* P/R *of* P *modulo* R is the collection of all equivalence classes.

2.1.2 Ordinal numbers

A linearly ordered set $(P, <_P)$ is *well-ordered* if every nonempty subset of it has a least element (in the linear ordering). By a *proper initial segment* of a well-ordered set P we mean a subset of the form $\{x \in P : x <_P r\}$ for some $r \in P$. It holds[3] that any two well-ordered sets are comparible in the following sense; either they are isomorphic (with respect to the relation $<_P$) to each other, or one of them is isomorphic to an initial segment of the other one. If we define equivalence classes on the collection of all well-ordered sets by putting isomorphic well-ordered sets into the same class, we can think of ordinal numbers as the collection of the nicest representatives of these equivalence classes.

Definition 2.3. A set A is an *ordinal number* (an *ordinal*) if it is well-ordered by the relation $\in$ (is an element of), and if $a \subseteq A$ for every $a \in A$ (transitiveness).

Ordinals are usually denoted by lowercase greek letters α, β, etc., and the class (collection) of all ordinal numbers is denoted by *Ord*. A function f is called an *ordinal function* if range$(f) \subseteq Ord$. For ordinals α and β we also write $\alpha < \beta$ instead of $\alpha \in \beta$. We list some of the basic facts about ordinals without proving them. The proofs can be found in [3].

[3]See [3] for a proof.

Proposition 2.4. *The following hold for any ordinal number α:*

1. *The empty set $\emptyset$ is an ordinal;*
2. *if $\beta \in \alpha$, then β is also an ordinal;*
3. $\alpha = \{\beta : \beta \in \alpha\}$*;*
4. $\alpha + 1 := \alpha \cup \{\alpha\}$ *is also an ordinal;*
5. *If X is a nonempty set of ordinals, then $\bigcup X$ is also an ordinal;*
6. $<$ *is a linear ordering of the class Ord;*
7. *each well-ordering P is isomorphic to exactly one ordinal; this ordinal is then called the order-type of P.*

Ordinals of the form $\alpha \cup \{\alpha\}$ are called *successor ordinals*. All other ordinals are called *limit ordinals*. Finite ordinals are also known as *natural numbers* and are written as follows:

$$
\begin{aligned}
&0 = \emptyset, \\
&1 = 0 + 1 = \emptyset \cup \{\emptyset\} = \{\emptyset\}, \\
&2 = 1 + 1 = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}, \\
&\text{etc.}
\end{aligned}
$$

2.1.3 Cardinal numbers

Definition 2.5. An ordinal number α is a *cardinal number* (a *cardinal*) if there is no bijection between α and any $\beta < \alpha$.

We usually use κ, λ, μ... to denote cardinals. By the *cardinality* $|X|$ of a set X we mean the unique cardinal number κ for which there is a bijection $f : \kappa \to X$. (The existence of such a bijection is not trivial; it relies on the axiom of choice.) Note that each natural number is a cardinal number; the cardinality of a finite set is simply the number of its elements.

The infinite cardinals are called *alephs*. Since cardinals are linearly ordered by $<$, we can enumerate them by ordinal numbers; $\aleph_\alpha$ denotes the α-th infinite cardinal. The 0-th infinite cardinal $\aleph_0$ is the set of natural numbers. If α is a successor (limit) ordinal, then we say that $\aleph_\alpha$ is a *successor* (*limit*) cardinal. We also write $\aleph_\alpha^+$ for $\aleph_{\alpha+1}$.

The arithmetic operations on cardinals are defined as follows:

$$\kappa + \lambda := |A \cup B|, \quad \kappa \cdot \lambda := |A \times B|,$$

$$\kappa^\lambda := |A^B| = |\{f : f \text{ is a function from } B \text{ into } A\}|,$$

where A and B are any disjoint sets with cardinalities $|A| = \kappa$ and $|B| = \lambda$.

Proposition 2.6. *The following hold for any cardinals κ, λ:*

1. *If κ and λ are infinite cardinals, then $\kappa + \lambda = \kappa \cdot \lambda = max\{\kappa, \lambda\}$;*
2. *$+$ and $\cdot$ are associative, commutative and distributive;*
3. $(\kappa \cdot \lambda)^\mu = \kappa^\mu \cdot \lambda^\mu, \quad \kappa^{\lambda+\mu} = \kappa^\lambda \cdot \kappa^\mu, \quad (\kappa^\lambda)^\mu = \kappa^{\lambda \cdot \mu}$;
4. *$\kappa \leq \lambda$ implies $\kappa^\mu \leq \lambda^\mu$, and $0 < \lambda \leq \mu$ implies $\kappa^\lambda \leq \kappa^\mu$;*
5. *Cantor : $\kappa < 2^\kappa$. (If a set A has the cardinality $\kappa = |A|$, then $2^\kappa = |\{f : f$ is a function from A into $2\}| = |P(A)|$ is the cardinality of the powerset of A.)*

For a proof see [3].

We say that a set of ordinals A is *cofinal* in a set of ordinals B if for every $\beta \in B$ there is an $\alpha \in A$ such that $\beta < \alpha$. For any ordinal α define the *cofinality of* α, denoted as $cf(\alpha)$, to be the least cardinality of a subset of α which is cofinal in α. If α is a cardinal number and $cf(\alpha) = \alpha$, then α is called a *regular* cardinal. Otherwise, (that is, if $cf(\alpha) < \alpha$), α is called a *singular* cardinal. (We denote the class of regular cardinals by *Reg*.) One can show that for every α, $cf(cf(\alpha)) = cf(\alpha)$. Thus, $cf(\alpha)$ is always a regular cardinal.

The exponentiation of cardinal numbers, unlike addition and multiplication, which are trivial, is one of the main topics in set theory. In the following proposition we state some of the basic properties of the cardinal arithmetic.[4]

Proposition 2.7. *The following hold for any cardinals κ, λ:*

1. *If λ is infinite and $2 \leq \kappa \leq \lambda$, then $\kappa^\lambda = 2^\lambda$;*
2. *if $\lambda \geq cf(\kappa)$, then $\kappa < \kappa^\lambda$;*
3. *if I is any index set and $\kappa_i < \lambda_i$ for every $i \in I$, then $\sum_{i \in I} \kappa_i < \prod_{i \in I} \lambda_i$;*
4. *$(\kappa^+)^\lambda = \kappa^\lambda \cdot \kappa^+$ (Hausdorff formula).*

For a proof see [3].

[4] For a proof of the proposition we refer the reader to [3, page 51]. In chapter 5 there are deeper results regarding cardinal arithmetic.

2.1.4 Closed unbounded sets

Let $\kappa > \omega$ be a limit ordinal, and let $C \subseteq \kappa$. Any limit ordinal $\alpha < \kappa$ with $\sup C \cap \alpha = \alpha$ is called a *limit point* of C. We say that C is *closed unbounded* (in κ) if it contains all its limit points and is cofinal in κ. For example, the set of all limit ordinals in κ is a closed unbounded set. If $cf(\kappa) > \omega$, then the intersection of two closed unbounded sets is also closed unbounded.

Suppose that κ is a regular uncountable cardinal. A set $S \subseteq \kappa$ is said to be *stationary* (in κ) if $S \cap C \neq \emptyset$, for every closed unbounded set C in κ. An ordinal function f on S is *regressive* if $f(\alpha) < \alpha$, for every $\alpha \in S$. We are going to use the following fact.

Lemma 2.8 (Fodor)**.** *If f is a regressive function on a stationary set $S \subseteq \kappa$, then f is constant on some stationary set $T \subseteq S$.*

For a proof see [3, Theorem 8.7].

2.2 Ideals and filters

Ideals and filters are the central tools in the pcf theory. Usually ideals (resp. filters) are collections of small (resp. large) subsets of a given set A. Therefore, elements of an ideal are called null-sets, and all other subsets of A are called positive sets. We also define the notion of a maximal ideal (filter), and state the important properties.

Definition 2.9. A family $I \subseteq P(A)$ of subsets of a set A is called an *ideal on A* if it satisfies the following conditions:

1. $\emptyset \in I$;
2. if $X \in I$ and $Y \in I$, then $X \cup Y \in I$;
3. if $X, Y \subseteq A$, $X \in I$, and $Y \subseteq X$, then $Y \in I$.

A family $F \subseteq P(A)$ of subsets of a set A is called a *filter on A* if it satisfies the following conditions:

1. $\emptyset \notin F$ and $A \in F$;
2. if $X \in F$ and $Y \in F$, then $X \cap Y \in F$;
3. if $X, Y \subseteq A$, $X \in F$, and $X \subseteq Y$, then $Y \in F$.

By a *proper*[5] ideal on a set A we mean an ideal satisfying $A \notin I$. We say that a proper ideal I (resp. a filter F) on a set A is *maximal* if there is no ideal I' with $I \subsetneq I' \subsetneq P(A)$ (resp. no filter F' with $F \subsetneq F' \subsetneq P(A)$).

A proper ideal I (resp. a filter F) is a *prime* ideal (resp. *ultrafilter*) if for every $X \subseteq A$, either $X \in I$ (resp. $X \in F$), or $A \backslash X \in I$ (resp. $A \backslash X \in F$), but not both, where $A \backslash X = \{a \in A : a \notin X\}$ denotes the complement of X in A.

If I is a proper ideal on a set A, then the collection $F = \{X \subseteq A : A \backslash X \in I\}$ is a filter on A. It is called the *dual* filter of I. I is then called the dual ideal of F.

We say that a set $G \subseteq P(A)$ generates an ideal I, if I is the closure of G under subsets and finite unions. Similarly, we say that that a set $H \subseteq P(A)$ generates a filter F, if F is the closure of H under supersets and finite intersections.

Proposition 2.10.

1. *An ideal* (*resp. filter*) *is a prime ideal* (*resp. ultrafilter*) *if and only if it is maximal.*

2. (*Tarski*) *Every ideal* (*resp. filter*) *can be extended to a prime ideal* (*resp. ultrafilter*)*. Moreover,*

3. (*Stone*) *If I is an ideal* (*resp. F is a filter*) *on a set A and $X \in P(A) \backslash I$* (*resp. $X \in P(A) \backslash F$*)*, then there is a prime ideal J* (*resp. ultrafilter D*) *with $I \subseteq J$ and $X \notin J$* (*resp. $F \subseteq D$ and $X \notin D$*).

For a proof of (1) and (2) see [3, page 74].

Remark 2.11. We will use the following consequences of Proposition 2.10. (i) If I is an ideal on a set A and $Y \in P(A) \backslash I$, then there is an ultrafilter D on A such that $I \cap D = \emptyset$ and $Y \in D$. [Let D simply be the dual filter of a prime ideal $J \supseteq I$ with $Y \notin J$.] (ii) Moreover, if F is a filter[6] on A such that $I \cap F = \emptyset$, then I can be extended to a maximal ideal J on A such that $J \cap F = \emptyset$.

[5] Attention: By *proper* we do not mean $I \neq \{\emptyset\}$.

[6] It suffices that $a \cap b \neq \emptyset$, for every $a, b \in F$.

Chapter 3

Reduced products

In this chapter we define the reduced products of sets of ordinals and develop the 'theory' of exact upper bounds, which is the basis of the pcf theory, as we will see in chapters 4 and 5. Our main reference for this chapter is [1]. Though, all the theorems in it are due to Shelah, unless otherwise stated.

3.1 Definition

Let A denote a set of regular cardinals[1] in this chapter. For any sequence $S = \langle S_a : a \in A\rangle$ of nonempty sets of ordinals we define the *product of* S to be the set of all ordinal functions $f : A \to Ord$ with $f(a) \in S_a$, for each $a \in A$, i.e.

$$\prod_{a\in A} S_a := \{f : \ f \in Ord^A, \ \forall a \in A \ (f(a) \in S_a)\}.$$

If h is an ordinal function on A (with $h(a) > 0$, for each $a \in A$), then we just write $\prod h$ instead of $\prod_{a\in A} h(a)$. Similarly, if $S_a = a$ for every $a \in A$, then we just write $\prod A$ instead of $\prod_{a\in A} a$.

We define the following relations on the product $\prod_{a\in A} S_a$:

1. If I is an ideal on A, then for any functions $f, g \in \prod_{a\in A} S_a$:

$$\begin{aligned} f =_I g \quad &:\Longleftrightarrow \quad \{a \in A : f(a) \neq g(a)\} \in I; \\ f <_I g \quad &:\Longleftrightarrow \quad \{a \in A : f(a) \geq g(a)\} \in I; \\ f \leq_I g \quad &:\Longleftrightarrow \quad \{a \in A : f(a) > g(a)\} \in I. \end{aligned}$$

2. If F is a filter on A, then for any functions $f, g \in \prod_{a\in A} S_a$:

[1] The theory of reduced products can be developed for any index set A.

$$f =_F g \quad :\Longleftrightarrow \quad \{a \in A : f(a) = g(a)\} \in F;$$

$$f <_F g \quad :\Longleftrightarrow \quad \{a \in A : f(a) < g(a)\} \in F;$$

$$f \leq_F g \quad :\Longleftrightarrow \quad \{a \in A : f(a) \leq g(a)\} \in F.$$

The relations $\leq_I$ and $\leq_F$ are quasi orderings of $\prod_{a\in A} S_a$, and the relations $<_I$ (if I is a proper ideal) and $<_F$ are strict partial orderings of $\prod_{a\in A} S_a$.

The relation $=_I$ (resp. $=_F$) is an equivalence relation on $\prod_{a\in A} S_a$. The quotient $\prod_{a\in A} S_a/I$ (resp. $\prod_{a\in A} S_a/F$) is called the *reduced product of S modulo I* (resp. modulo F). Although $\prod_{a\in A} S_a/I$ (resp. $\prod_{a\in A} S_a/F$) consists of equivalence classes of functions in $\prod_{a\in A} S_a$, for our purposes we want to work with single functions - identifying equivalent ones. [Therefore, for simplicity, when we have $f <_I g$, by changing f on a null-set we can assume that $f(a) < g(a)$, for each $a \in A$, without any loss of generality. Also, if h is an ordinal function on A such that $\{a \in A : h(a) \notin S_a\} \in I$, then we consider h as an element of $\prod_{a\in A} S_a/I$ (because it is equivalent to some $h' \in \prod_{a\in A} S_a/I$). The same for a filter F.] This means that on $\prod_{a\in A} S_a/I$ (resp. $\prod_{a\in A} S_a/F$) we consider the relations $<_I$ and $\leq_I$ (resp. $<_F$ and $\leq_F$), which are actually defined on $\prod_{a\in A} S_a$.

Note that for functions f and g, $f \leq_I g$ does not imply that either $f <_I g$ or $f =_I g$. The converse is clearly true.

If I and F are dual to each other (see page 11), then for any functions f, $g \in \prod A$ we have: $f =_I g$ iff $f =_F g$, $f <_I g$ iff $f <_F g$, and $f \leq_I g$ iff $f \leq_F g$. For this reason, whenever I is dual to F, we identify $(\prod_{a\in A} S_a/I, <_I, \leq_I)$ with $(\prod_{a\in A} S_a/F, <_F, \leq_F)$ (or say that they have the same structure), and make no difference between the I-relations and the F-relations. Further, every filter has a dual ideal, hence, it suffices to develop the theory of reduced products for ideals. If $I = \{\emptyset\}$ we identify $\prod_{a\in A} S_a/\{\emptyset\}$ with $\prod_{a\in A} S_a$, and write $f < g$ instead of $f <_{\{\emptyset\}} g$.

If $J \supseteq I$ is another ideal on A, then $f <_I g$ implies $f <_J g$. We say that $<_J$ *extends* $<_I$. In particular, if D is an ultrafilter on A extending the dual filter of I, that is, $D \cap I = \emptyset$, then $<_D$ extends $<_I$.

Recall that a set $B \subseteq \prod_{a\in A} S_a/I$ is cofinal in $(\prod_{a\in A} S_a/I, <_I)$ iff for every function $f \in \prod_{a\in A} S_a/I$ there is a function $g \in B$ such that $f <_I g$. A sequence $f = \langle f_\xi : \xi < \lambda\rangle$ of functions in $\prod_{a\in A} S_a/I$ is said to be *$<_I$-increasing* iff for every $\xi_1, \xi_2 < \lambda$ we have $f_{\xi_1} <_I f_{\xi_2}$.

We say that a sequence $f = \langle f_\xi : \xi < \lambda\rangle$ of functions in $\prod_{a\in A} S_a/I$ is a *scale* for $\prod_{a\in A} S_a/I$ iff it is $<_I$-increasing and cofinal in $(\prod_{a\in A} S_a/I, <_I)$.

We need to further enrich our terminology before we can say why we are interested in working with reduced products. The following definitions can be easily generalized for any partial or quasi orderings; particularly for $\leq_I$.

Definition 3.1. Let I be an ideal on A and let $S = \langle S_a : a \in A \rangle$ be any sequence of nonempty sets of ordinals.

1. The *cofinality* $cf(\prod_{a\in A} S_a/I, <_I)$ of $(\prod_{a\in A} S_a/I, <_I)$ is defined as the least cardinality of a cofinal set in $(\prod_{a\in A} S_a/I, <_I)$.

2. The *true cofinality* $tcf(\prod_{a\in A} S_a/I, <_I)$ of $(\prod_{a\in A} S_a/I, <_I)$ is defined as the least cardinality of a linearly ordered set which is cofinal in $(\prod_{a\in A} S_a/I, <_I)$, if it exists. [In other words, the true cofinality is the minimal length of a scale. Note that the length of a scale is a regular cardinal iff it is minimal.]

Remark 3.2. (1) Suppose that each S_a has no maximal element. Then a set $B \subseteq \prod_{a\in A} S_a/I$ is cofinal in $(\prod_{a\in A} S_a/I, \leq_I)$ iff it is cofinal in $(\prod_{a\in A} S_a/I, <_I)$. [The 'if' direction is trivial: if $f \in \prod_{a\in A} S_a/I$ and for some $g \in B$ we have $f <_I g$, then also $f \leq_I g$. Conversely, suppose that $f \in \prod_{a\in A} S_a/I$, then also $f + 1 \in \prod_{a\in A} S_a/I$, where $f + 1(a) = f(a) + 1$. By assumption, there is some $g \in B$ with $f + 1 \leq_I g$. It follows that $f <_I g$.] This justifies our new terminology - cofinal in $\prod_{a\in A} S_a/I$.

(2) Every reduced product has a cofinality: Firstly, there is always a cofinal subset, namely the set itself; and secondly, there is always the least cardinal among a class of cardinals. Cofinality can be either a regular or a singular cardinal.

(3) The true cofinality does not always exist (see the examples below). If it exists, then it is a regular cardinal. [Otherwise there is a shorter scale.]

(4) If I is a maximal ideal on A, then $\prod A/I$ has a true cofinality. Namely, $<_I$ is a linear ordering of $\prod A/I$, and thus, every cofinal subset is a scale for $\prod A/I$. Similarly, if D is an ultrafilter on A, then $tcf(\prod A/D)$ always exists. We write $cf(\prod A/D)$ instead of $tcf(\prod A/D)$.

(5) If $J \supseteq I$ is another ideal on A, then, since $<_J$ extends $<_I$, we have that any $<_I$-increasing sequence of functions is also $<_J$-increasing. Further, any cofinal sequence in $(\prod_{a\in A} S_a/I, <_I)$ is also cofinal in $(\prod_{a\in A} S_a/J, <_J)$. In particular, if F is a filter on A extending the dual filter of I, then

$$cf(\textstyle\prod_{a\in A} S_a/F) \leq cf(\prod_{a\in A} S_a/I),$$

and

$$tcf(\textstyle\prod_{a\in A} S_a/F) = tcf(\prod_{a\in A} S_a/I),$$

if $tcf(\prod_{a\in A} S_a/I)$ exists.

(6) If h is an ordinal function on A (with $h(a) > 0$ is a limit ordinal, for each $a \in A$), then the reduced products $\prod h/I = \prod_{a\in A} h(a)/I$ and

$\prod_{a\in A} cf(h(a))/I$ have the same cofinality (and true cofinality, if it exists). [Choose for every $a \in A$ a cofinal set S_a in $h(a)$ of order type $cf(h(a))$. Then, on the one hand, $\prod_{a\in A} h(a)/I$ and $\prod_{a\in A} S_a/I$ are cofinally equivalent. That is, for every $f \in \prod_{a\in A} h(a)/I$ there is $g \in \prod_{a\in A} S_a/I$ with $f \leq_I g$, and vice versa, which means they have the same cofinality. On the other hand, $\prod_{a\in A} S_a/I$ can be identified with $\prod_{a\in A} cf(h(a))/I$ because for each $a \in A$, S_a has order-type $cf(h(a))$.]

Let h be an ordinal function on A (with $h(a) > 0$ is a limit ordinal, for each $a \in A$). We are interested in the existence and value of the true cofinality of $\prod h/I$; and conversely, we want to represent regular cardinals as true cofinalities of some reduced products.

Remark 3.2(6) tells us that we can concentrate on ordinal functions h, which take values in the class of infinite regular cardinals.

Suppose that λ is a regular cardinal and $f = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence of functions in $\prod A/I$, which has an exact upper bound h in $\prod A/I$. Then, (by Definition 2.2) f is cofinal in the set $\{g \in Ord^A : g <_I h\}$. But this is the same as to say f is cofinal in $\prod h/I$. It follows that f is a scale for $\prod h/I$. Finally, since λ is a regular cardinal, f must be a scale of minimal length, and hence we have that λ is the true cofinality of $\prod h/I = \prod cf(h)/I$.

This motivates the study of exact upper bounds in the next section. We first want to cite some examples and state a useful lemma.

Example 3.3. (1) If A ($|A|> 1$) is a set of regular cardinals and $I = \{\emptyset\}$, then $f =_I g$ means $f(a) = g(a)$, for every $a \in A$. The product $\prod A = \prod A/I$ does not have a true cofinality. [We argue indirectly. Suppose first that there is a scale $f = \langle f_\xi : \xi < \lambda\rangle$ for $\prod A$ of length $\lambda < \sup A$. Then there is $\kappa \in A$ with $\lambda < \kappa$. Since κ is a regular cardinal and $\lambda < \kappa$, the set $\{f_\xi(\kappa) : \xi < \lambda\}$ is bounded in κ. This means that f is not cofinal in $\prod A$, contradicting f being a scale. Suppose now that there is a scale $f = \langle f_\xi : \xi < \lambda\rangle$ for $\prod A$ of length $\lambda \geq \sup A$. Then for any $\kappa \in A$ with $\kappa < \sup A$ we have that the sequence $\langle f_\xi(\kappa) : \xi < \lambda\rangle$ is an increasing sequence of ordinals of length λ in κ. But this is impossible, since $\kappa < \lambda$.]

Example 3.4. [2] Let κ be a strong limit cardinal, i.e. $2^\alpha < \kappa$ for every $\alpha < \kappa$. Consider an increasing sequence $\langle\lambda_n\rangle_{n\in\mathbb{N}}$ of infinite regular cardinals with limit κ. One shows by a diagonalisation argument that $\kappa^+ \leq cf(\prod_{n\in\mathbb{N}} \lambda_n)$. [No set $F \subseteq \prod_{n\in\mathbb{N}} \lambda_n$ of cardinality $\leq \kappa$ is cofinal in $\prod_{n\in\mathbb{N}} \lambda_n$. For if $|F| \leq \kappa$, then $F = \bigcup_{n\in\mathbb{N}} F_n$, for some $F_n \subseteq \prod_{n\in\mathbb{N}} \lambda_n$ with $|F_n|< \lambda_n$. Choose a

[2]For readers familiar with forcing and large cardinals.

function g such that for each n, $g(\lambda_n) > f(\lambda_n)$, for every $f \in F_n$. Then for every $f \in F$, $g \not\leq f$. Hence F is not cofinal.] On the other hand, since $|\prod_{n\in\mathbb{N}} \lambda_n| = 2^\kappa$, we have $cf(\prod_{n\in\mathbb{N}} \lambda_n) \leq 2^\kappa$. Therefore, the problem of cofinality of such a product is related to the continuum function problem. We want to mention two types of models in which cofinalities of products have been studied: Prikry's model and Magidor's model. The following results are taken from [6].

- *Let κ be a measurable cardinal with normal measure U and let $\langle\kappa_n\rangle_{n\in\mathbb{N}}$ be a Prikry sequence for U. Then for every regular cardinal λ with $\kappa^+ \leq \lambda \leq 2^\kappa$ there exists an increasing sequence $\langle\lambda_n\rangle_{n\in\mathbb{N}} \in V[G]$ of regular cardinals with limit κ such that $cf(\prod_{n\in\mathbb{N}} \lambda_n) = \lambda$.*

- *Let κ be a supercompact cardinal and let N be Magidor's extension of V, which introduces a Prikry sequence $\langle\kappa_n\rangle_{n\in\mathbb{N}}$ together with collapsing to obtain $\kappa_n = \aleph_{(k+1)n}$, for $n \in \mathbb{N}$, and $2^{\aleph_\omega} = \aleph_{\omega+k}$ $(2 \leq k < \omega)$. Then for each $m = 1, ..., k$ we have (in N) $cf(\prod_{n\in\mathbb{N}} \kappa_n^{+m}) = \aleph_{\omega+m}$.*

For proofs we refer to [6].

Lemma 3.5. *Suppose that c is a function from A into the class of regular cardinals and $B = \{c(a) : a \in A\}$ is its range. Then the following hold.*

1. *If I is an ideal on A, then its Rudin-Keisler projection on B, defined by*

$$X \in J \quad \textit{iff} \quad X \subseteq B \textit{ and } c^{-1}X \in I,$$

   where $c^{-1}X = \{a \in A : c(a) \in X\}$, is an ideal on B.

2. *The function $h : \prod B/J \to \prod_{a\in A} c(a)/I$, defined by $h([f]_J) = [f \circ c]_I$, is injective and order-preserving.*

3. *If $|A| < minB$, then the image of h is cofinal in $\prod_{a\in A} c(a)/I$. Thus,*

$$tcf(\prod B/J) = tcf(\prod_{a\in A} c(a)/I),$$

   if one of the products has true cofinality.

For a proof see [1, Lemma 2.3].

3.2 Exact upper bounds

In this section we determine the conditions for the existence of exact upper bounds for $<_I$-increasing sequences $f = \langle f_\xi : \xi < \lambda \rangle$ of functions in Ord^A.

The remarkable study of these conditions is actually a small theory on exact upper bounds, which plays a very important role in the pcf theory. However, we only state the most important theorems of that theory; sketching most of the proofs, so that the reader can see the ideas behind the definitions. For details and complete proofs we refer to [1].

Let A again be a set of regular cardinals and let I be an ideal on A.

Definition 3.6. Suppose that λ is a regular cardinal and $f = \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of functions in Ord^A. Then f is said to be *strongly increasing* if there are null-sets $Z_\xi \in I$, for $\xi < \lambda$, such that whenever $\xi_1 < \xi_2 < \lambda$, then

$$a \in A \backslash (Z_{\xi_1} \cup Z_{\xi_2}) \Longrightarrow f_{\xi_1}(a) < f_{\xi_2}(a).$$

Definition 3.7. Suppose that λ is a regular cardinal and $f = \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of functions in Ord^A. For any regular cardinal $\kappa \leq \lambda$ we denote the following property of f by $(*)_\kappa$:

Whenever $X \subseteq \lambda$ is unbounded in λ, then for some
$X_0 \subseteq X$ of order-type κ, $\langle f_\xi : \xi \in X_0 \rangle$ is strongly increasing.

Note that if f has the $(*)_\kappa$ property for some $\kappa < \lambda$, then f also has the $(*)_{\kappa'}$ property for every regular $\kappa^{'} < \kappa$.

Let $S = \langle S_a : a \in A \rangle$ be a sequence of sets of ordinals. We denote the function $a \mapsto \sup S_a$ by *sup-of-S*.

Suppose that a function $f \in Ord^A$ is bounded by *sup-of-S*. Then we define the *projection of f onto S*, $proj(f, S)$, as the function $f^+(a) := \min(S_a \backslash f(a))$.

Definition 3.8. Suppose that λ is a regular cardinal and $f = \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of functions in Ord^A. For any regular cardinal $\kappa \leq \lambda$ we call the following property of f the *bounding projection property for κ*:

Whenever $S = \langle S_a : a \in A \rangle$ with $S_a \subseteq Ord$ and $|S_a| < \kappa$ is such
that the sequence f is $<_I$-bounded by the function $a \mapsto \sup S_a$,
then there exists $\xi < \lambda$ such that the projection $f^+_\xi = proj(f_\xi, S) \in \prod_{a \in A} S_a$
is an upper bound of f in the $<_I$ relation.
[Shortly: $f <_I$ *sup-of-S* $\Rightarrow \exists \xi < \lambda \ (f <_I f^+_\xi)$.]

Lemma 3.9. *Suppose that A is a set of regular cardinals, I is an ideal on A, $\lambda > |A|^+$ is a regular cardinal, and $f = \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of functions in Ord^A. Then for every regular cardinal κ such that $|A|^+ \leq \kappa \leq \lambda$ the following are equivalent*[3]*:*

1. *$(*)_\kappa$ holds for f;*
2. *f has the bounding projection property for κ;*
3. *f has an exact upper bound h such that: $\{a \in A : cf(h(a)) < \kappa\} \in I$.*

Proof. $1 \Rightarrow 2$. See [1, Theorem 2.12].

$2 \Rightarrow 3$. Sketch of the proof: One first shows that f has a minimal upper bound h (for a proof see Theorem 2.13 in [1]). We show that h then must be an exact upper bound of f, i.e. f is cofinal below h:

Suppose that $g <_I h$. We shall find $\xi < \lambda$ such that $g \leq_I f_\xi$. Assume w.l.o.g. that $g(a) < h(a)$, for every $a \in A$ (see page 14). Define a sequence $S = \langle S_a : a \in A \rangle$ by

$$S_a := \{g(a), h(a)\}$$

for every $a \in A$. Since $|S_a| < \kappa$ clearly holds and *sup-of-S* $= h$ is a $<_I$-upper bound of f (if it were not a $<_I$-upper bound of f, then it would not be an upper bound at all, because f is $<_I$-increasing), the bounding projection property for κ implies that there is $\xi < \lambda$ such that f_ξ^+ is a $<_I$-upper bound of f. Since h is minimal and $f_\xi^+ \leq_I h$, we have that $h \leq_I f_\xi^+$. Thus, $f_\xi^+ =_I h$. It follows that $g <_I f_\xi$, (because if $f_\xi \leq g$ holds on a positive set, then $f_\xi^+ = g < h$ on this positive set, and that contradicts $f_\xi^+ =_I h$). Thus, we have shown that h is an exact upper bound of f.

This exact upper bound is determined up to $=_I$. Since f is $<_I$-increasing, $h(a)$ can be 0 or a successor ordinal only on a null-set. Thus, we can assume that it is never 0 or a successor ordinal (see page 14).

It remains to show that $\{a \in A : cf(h(a)) < \kappa\} \in I$ holds. In order to get a contradiction, suppose that $P := \{a \in A : cf(h(a)) < \kappa\} \notin I$. For every $a \in P$, choose a set $S_a \subseteq h(a)$ cofinal in $h(a)$, such that order-type$(S_a) < \kappa$. For $a \in A \setminus P$ define $S_a := \{h(a)\}$. Then *sup-of-S*$= h$ is a $<_I$-upper bound of f and $|S_a| < \kappa$. Like above, the bounding projection property for κ implies that there is $\xi < \lambda$ such that $f_\xi^+ \in \prod_{a \in A} S_a$ is a $<_I$-upper bound of f. By the definition of S, we have $f_\xi^+ \leq_I h$ and $f_\xi^+ \upharpoonright P < h \upharpoonright P$. But since P is a positive set, this means that $f_\xi^+ \leq_I h$ and $h \not\leq_I f_\xi^+$, contradicting our assumption that h is a minimal upper bound of f (see Definition 2.2).

$3 \Rightarrow 1$. See [1, Theorem 2.15]. $\square$

[3] For the later chapters we only need the $1 \Rightarrow 2 \Rightarrow 3$ direction.

The condition $(*)_\kappa$ does not seem to be easily verifiable for a sequence of ordinal functions. However, we are only interested in the existence of sequences which have this property[4], and the following theorem gives us a strategy for constructing them.

Lemma 3.10. *Suppose that A is a set of regular cardinals and*

1. *I is a proper ideal on A;*

2. *κ and λ are regular cardinals such that $\kappa^{++} < \lambda$; and*

3. *$f = \langle f_\xi : \xi < \lambda \rangle$ is a $<_I$-increasing sequence of functions in Ord^A which satisfies the following requirement:*

   for every $\delta < \lambda$ with $cf(\delta) = \kappa^{++}$ there is a closed unbounded set $E_\delta \subseteq \delta$ such that for some δ' with $\delta \leq \delta' < \lambda$

$$sup\{f_\alpha : \alpha \in E_\delta\} <_I f_{\delta'}.$$

Then $(*)_\kappa$ holds for f.

For a proof see [1, Lemma 2.19].

From the following application we see that the condition on f in Lemma 3.10 is nothing but a strategy for constructing sequences of ordinal functions for which $(*)_\kappa$ holds (and which, thus, have exact upper bounds that are of big interest to us).

Definition 3.11. Suppose that I is a proper ideal on A and $S = \langle S_a : a \in A \rangle$ is a sequence of sets of ordinals. We say that the product $\prod_{a\in A} S_a/I$ is λ-*directed* (for a cardinal λ) iff every set $B \subseteq \prod_{a\in A} S_a/I$ with cardinality $|B| < \lambda$ has an upper bound in $\prod_{a\in A} S_a/I$.

Lemma 3.12. *Suppose that A is a set of regular cardinals and I is a proper ideal on A. Let λ be a regular cardinal such that $\prod A/I$ is λ-directed. Then there exists a $<_I$-increasing sequence $f = \langle f_\xi : \xi < \lambda \rangle$ of functions in $\prod A/I$ such that for every regular cardinal $\kappa < \lambda$, $(*)_\kappa$ holds for f iff*

$$\kappa^{++} < \lambda \text{ and } \{a \in A : a \leq \kappa^{++}\} \in I.$$

Proof. Define f as follows. At successor stages $\xi + 1 < \lambda$, let $f_{\xi+1} := f_\xi + 1$. Since A is a set of limit ordinals, we have $f_\xi + 1 \in \prod A/I$.

At limit stages $\delta < \lambda$ we consider two cases. If $cf(\delta) = \kappa^{++} < \lambda$, where κ is a regular cardinal such that $\{a \in A : a \leq \kappa^{++}\} \in I$, then fix some closed unbounded set $E_\delta \subseteq \delta$, and define

[4] For the purposes of pcf theory; see page 16 and Theorem 4.10.

$$f_\delta = \sup\{f_i : i \in E_\delta\}.$$

(For every $a > \kappa^{++}$ we have that $f_\delta(a) < a$, because each $a \in A$ is a regular cardinal. Thus, $f_\delta \in \prod A/I$, since $\{a \in A : a \leq \kappa^{++}\} \in I$.) If $cf(\delta)$ is not of that form, then let $f_\delta \in \prod A/I$ be any upper bound of $\langle f_\xi : \xi < \delta\rangle$ guaranteed by the λ-directedness assumption.
Lemma 3.10 implies that $(*)_\kappa$ holds for f, for every κ of the required form. □

The following theorem is an immediate consequence of Lemma 3.9 and Lemma 3.12.

Theorem 3.13. *Suppose that*

1. *A is a set of regular cardinals and I is a proper ideal on A;*
2. *λ is a regular cardinal such that $\prod A/I$ is λ-directed;*
3. *there is a regular cardinal κ with $|A|^+ \leq \kappa \leq \lambda$, such that $\kappa^{++} < \lambda$ and $\{a \in A : a \leq \kappa^{++}\} \in I$.*

Then there exists a $<_I$-increasing sequence $f = \langle f_\xi : \xi < \lambda\rangle$ of functions in $\prod A/I$ with an exact upper bound h such that $\{a \in A : cf(h(a)) < \kappa\} \in I$ for every regular cardinal κ with $|A|^+ \leq \kappa \leq \lambda$, such that $\kappa^{++} < \lambda$ and $\{a \in A : a \leq \kappa^{++}\} \in I$.

Proof. Follows immediately from Lemma 3.9 and Lemma 3.12. Note that the existence of a regular cardinal κ with $|A|^+ \leq \kappa \leq \lambda$, such that $\kappa^{++} < \lambda$ and $\{a \in A : a \leq \kappa^{++}\} \in I$, is needed (by Lemma 3.9) for the existence of an exact upper bound of f. □

Remark 3.14. From the proof of Lemma 3.10 we can see that f can be chosen to dominate any given sequence $g = \langle g_\xi : \xi < \lambda\rangle$ of functions in $\prod A/I$, i.e. such that for each $\xi < \lambda$, $g_\xi \leq_I f_\xi$.

In the next section and in Chapter 4 (Theorem 4.10) we will see applications of Theorem 3.13.

3.3 Representation theorems

We have seen on page 16 that if λ is a regular cardinal and $f = \langle f_\xi : \xi < \lambda\rangle$ is a $<_I$-increasing sequence of functions in $\prod A/I$, which has an exact upper bound h in $\prod A/I$, then

$$\lambda = tcf(\prod cf(h)/I). \tag{3.1}$$

Using this fact we can get stronger results for successors μ^+ of singular cardinals. We first consider μ^+ with $cf(\mu) = \omega$.

Theorem 3.15. *Suppose that μ is a singular cardinal of countable cofinality. Then there exists an unbounded set $B \subset \mu$ (of order-type ω) of regular cardinals such that*

$$\mu^+ = tcf(\prod B/J^{bd}),$$

where J^{bd} is the ideal of bounded (or finite) subsets of B.

Proof. Let $B_0 \subset \mu$ be any unbounded set (of order-type ω) of regular cardinals, and let I^{bd} be the proper ideal of bounded (or finite) subsets of B_0. We show that the product $\prod B_0/I^{bd}$ is μ-directed.

Suppose that $F \subseteq \prod B_0/I^{bd}$ has cardinality $\gamma < \mu$. We need to show that F has an upper bound in $\prod B_0/I^{bd}$. Define a function $g \in \prod B_0/I^{bd}$ by $g(a) := \sup\{f(a) : f \in F\}$ if $a > \gamma$, and $g(a) := 0$ if $a \leq \gamma$. [It holds that $g \in \prod B_0/I^{bd}$, since $|F| < \gamma$ and each $a \in B_0$ is regular.] Then for every $f \in F$ we have $\{a \in B_0 : f(a) > g(a)\} = \{a \in B_0 : a \leq \gamma\} \in I^{bd}$, that is, $f \leq_{I^{bd}} g$. Thus, we have shown that F has an upper bound in $\prod B_0/I^{bd}$.

But since μ is a singular cardinal, $\prod B_0/I^{bd}$ is also μ^+-directed: Suppose that $F \subseteq \prod B_0/I^{bd}$ has cardinality μ. There are subsets $F_n \subseteq \prod B_0/I^{bd}$, for $n \in \omega$, of cardinality $< \mu$ such that $F = \bigcup_{n\in\omega} F_n$. Since $\prod B_0/I^{bd}$ is μ-directed and $|F_n| < \mu$ ($n \in \omega$), there exist upper bounds f_n of F_n in $\prod B_0/I^{bd}$. Finally, there exists also an upper bound of $\langle f_n : n \in \omega\rangle$, which bounds every F_n, and thus, the whole F.

Let κ be a regular cardinal such that $\aleph_1 \leq \kappa < \mu^+$, then $\kappa^{++} < \mu < \mu^+$ holds (since μ is singular) and the set $\{a \in B_0 : a \leq \kappa^{++} < \mu\}$ is obviously bounded in μ.

Apply Theorem 3.13 to B_0, I^{bd} and μ^+. There exists a $<_{I^{bd}}$-increasing sequence $f = \langle f_\xi : \xi < \mu^+\rangle$ of functions in $\prod B_0/I^{bd}$ with an exact upper bound h such that

$$\{a \in B_0 : cf(h(a)) < \kappa\} \in I^{bd}, \tag{3.2}$$

for every regular cardinal κ with $\aleph_1 \leq \kappa \leq \mu^+$, such that $\kappa^{++} < \mu^+$ and $\{a \in B_0 : a \leq \kappa^{++}\} \in I^{bd}$. [Since the identity function id on B_0 is a $<$-upper bound of f, we have $h \leq_I id$ (otherwise h is not minimal), and thus, we can assume that $h(a) \leq id(a) = a$, for every $a \in B_0$.]

Note that the set of such regular cardinals κ is unbounded in μ. Therefore, it follows from (3.2) that also

$$B := \{cf(h(a)) : a \in B_0\} \subset \mu$$

is unbounded in μ. We can assume that B is of order-type ω and $\aleph_0 < \min B$ (modify B_0 if necessary).

We show that $\mu^+ = tcf(\prod B/I^{bd})$. By (3.1) we have

$$\mu^+ = tcf(\prod_{a \in B_0} cf(h(a))/I^{bd}). \tag{3.3}$$

Define a function $c : B_0 \to B$ by $c(a) := cf(h(a))$. Let J be the ideal on B defined by $X \in J$ iff $\{c^{-1}(x) : x \in X\} \in I^{bd}$. Lemma 3.5 implies that

$$tcf(\prod B/J) = tcf(\prod_{a \in B_0} cf(h(a))/I^{bd}) = \mu^+.$$

It remains to show that J is the ideal J^{bd} of bounded subsets of B. The inclusion $J \subseteq J^{bd}$ follows easily. If $X \in J$, then $\{c^{-1}(x) : x \in X\} \in I^{bd}$ is a finite set, and thus, X is a finite (and bounded) set.

In order to prove $J^{bd} \subseteq J$, suppose that $X \in J^{bd}$ is a bounded subset of $B = \{cf(h(a)) : a \in B_0\}$. Say, $\gamma < \mu$ is an upper bound of X. We need to show that the preimage of X, under the map c, is bounded in μ. In order to get a contradiction, suppose that it is not bounded. That is, there is an unbounded set $Y \subseteq B_0$ such that for every $a \in Y$, $c(a) = cf(h(a)) \in X$. Since X is finite, it follows that there is an unbounded set $Y_1 \subseteq Y$ and some $\delta \leq \gamma$ such that for every $a \in Y_1$, $c(a) = cf(h(a)) = \delta$.

We now easily get a contradiction. Let D be an ultrafilter on B_0 extending the dual filter of I^{bd}, such that $Y_1 \in D$ (which exists by Remark 2.11). Then, on the one hand, since $\leq_D$ extends $\leq_{I^{bd}}$, (3.3) implies that $\mu^+ = tcf(\prod_{a \in B_0} cf(h(a))/D)$ (by Remark 3.2(6)). But on the other hand, since $Y_1 \in D$, any sequence $\langle g_\xi : \xi < \delta \rangle$ with $g_\xi(a) = \xi$, for $a \in Y_1$, is a scale for $\prod_{a \in B_0} cf(h(a))/D$. This completes the proof. $\square$

We have a similar result for successors μ^+ of singular cardinals with $cf(\mu) > \omega$.

If X is a set of cardinals, then let $X^{(+)} := \{\alpha^+ : \alpha \in X\}$ denote the set of successors of cardinals in X.

Theorem 3.16. *Suppose that μ is a singular cardinal of uncountable cofinality. Then there exists a closed unbounded set (of limit cardinals) $C \subseteq \mu$ such that $|C| < minC$ and*

$$\mu^+ = tcf(\prod C^{(+)}/J^{bd}),$$

where J^{bd} is the ideal of bounded subsets of $C^{(+)}$.

Proof. Let $C_0 \subseteq \mu$ be any closed unbounded set of limit cardinals bigger than $cf(\mu)$, such that $|C_0| = cf(\mu)$. It follows that $|C_0| < \min C_0$.

All the limit points γ of C_0 are singular cardinals, since $|C_0| = cf(\mu) < \gamma$. So we can assume that C_0 consists only of singular cardinals. [The set of limit points of C_0 is also a closed unbounded set.]

By the same argument like in the proof of Theorem 3.15, the product $\prod C_0^{(+)}/J^{bd}$ is μ^+-directed, where J^{bd} is the ideal of bounded subsets of $C_0^{(+)}$.

Apply Theorem 3.13 to $C_0^{(+)}$, J^{bd} and μ^+. There exists a $<_{J^{bd}}$-increasing sequence $f = \langle f_\xi : \xi < \mu^+ \rangle$ of functions in $\prod C_0^{(+)}/J^{bd}$ with an exact upper bound h such that

$$\{a \in C_0^{(+)} : cf(h(a)) < \kappa\} \in J^{bd}, \tag{3.4}$$

for every regular cardinal κ with $cf(\mu)^+ \leq \kappa \leq \mu^+$, such that $\kappa^{++} < \mu^+$ and $\{a \in C_0^{(+)} : a \leq \kappa^{++}\} \in J^{bd}$; this means, for every regular cardinal $\kappa < \mu$. [We can assume, like in the previous proof, that $h(a) \leq a$, for every $a \in C_0^{(+)}$.]

Claim. There is a set

$$C \subseteq \{\alpha \in C_0 : h(\alpha^+) = \alpha^+\},$$

which is closed unbounded.

Proof of the claim. In order to get a contradiction, suppose that the set $\{\alpha \in C_0 : h(\alpha^+) = \alpha^+\}$ does not contain a closed unbounded set. Then there is some stationary set $S \subseteq C_0$ such that $S \cap \{\alpha \in C_0 : h(\alpha^+) = \alpha^+\} = \emptyset$. It follows that $h(\alpha^+) < \alpha^+$ for every $\alpha \in S$.

Since all cardinals in C_0 are singular, we have that $cf(h(\alpha^+)) < \alpha$, for every $\alpha \in S$. Hence, by Lemma 2.8, $cf \circ h$ is constant, and hence bounded by some $\kappa < \mu$, on a stationary set of α's in S. But this is in contradiction with (3.4).

Thus, we have proved that there exists a closed unbounded set $C \subseteq C_0$ such that $h(\alpha^+) = \alpha^+$ for every $\alpha \in C$. It follows easily that $\mu^+ = tcf(\prod C^{(+)}/J^{bd})$. Namely, the sequence $\langle f_\xi \restriction C^{(+)} : \xi < \mu^+ \rangle$ is a scale for $\prod C^{(+)}/J^{bd} = \prod_{\alpha \in C} h(\alpha^+)/J^{bd}$; it is cofinal, because $h \restriction C^{(+)}$ is an exact upper bound of $f \restriction C^{(+)}$, and it is $<_{J^{bd}}$-increasing. □

We will prove a stronger version of the last theorem in the next chapter.

Chapter 4

The pcf function

4.1 Definition

In Chapter 3 we have seen that some cardinals can be represented as true cofinalities of certain (reduced) products of sets. Now we want to change our point of view and investigate which cardinals can be realised as true cofinalities of some fixed product of sets - of course, modulo different ideals. This motivates the following definition.

Definition 4.1. [The pcf function] For any set A of regular (uncountable[1]) cardinals define

$$pcf(A) := \{\lambda : \lambda = tcf(\prod A/I) \text{ for some proper ideal } I \text{ on } A\}.$$

Let I be a proper ideal on A such that $\lambda = tcf(\prod A/I)$, and let D be any ultrafilter on A, extending the dual filter of I. Then, as we mentioned in Remark 3.2(6), $\lambda = cf(\prod A/D) = tcf(\prod A/I)$. Hence, the following formulation is equivalent to the one in Definition 4.1:

$$pcf(A) = \{\lambda : \lambda = cf(\prod A/D) \text{ for some ultrafilter } D \text{ on } A\}.$$

4.2 Basic properties

Let A and B be any sets of regular uncountable cardinals. We state the basic properties of the pcf function.

1. $pcf(A) \cap \min A = \emptyset$. Proof: Suppose that $\lambda \in \min A$. Then for every ultrafilter D on A, any sequence $f = \langle f_\xi : \xi < \lambda\rangle$ of functions in

[1] We will only consider uncountable cardinals, since the finite case is trivial.

$\prod A/D$ is $\leq_D$-bounded in $\prod A/D$ by the pointwise supremum of f_ξ's. Hence for every ultrafilter D on A, we have $\lambda < cf(\prod A/D)$. Thus, $\lambda \notin pcf(A)$.

2. $A \subseteq pcf(A)$.[2] Proof: Suppose that $\lambda \in A$. We need to find an ultrafilter D on A such that $\lambda = cf(\prod A/D)$. Let D be the principal ultrafilter on A, which concentrates on λ; that is, for every $X \subseteq A$, $X \in D$ iff $\lambda \in X$. Then any sequence $f = \langle f_\xi : \xi < \lambda\rangle$ of functions in $\prod A/D$, with $f_\xi(\lambda) = \xi$ for every $\xi < \lambda$, is a scale for $\prod A/D$, (because $\{\lambda\} \in D$). Thus, $\lambda = cf(\prod A/D)$.

3. If $A \subseteq B$, then $pcf(A) \subseteq pcf(B)$. Proof: Suppose that $\lambda \in pcf(A)$. Then there is an ultrafilter D on A such that $\lambda = cf(\prod A/D)$. Let $f = \langle f_\xi : \xi < \lambda\rangle$ be a scale for $\prod A/D$. Extend D (canonically) to an ultrafilter D' on B, that is, let $D' := \{b \subseteq B : b \cap A \in D\}$; and extend each f_ξ (arbitrarily) to a function f'_ξ with domain B. We claim that $f' = \langle f'_\xi : \xi < \lambda\rangle$ is a scale for $\prod B/D'$, and thus, that $\lambda \in pcf(B)$.

   We first show that it is cofinal. Let $g \in \prod B/D'$, then, for some $\xi < \lambda$, $g \upharpoonright A \leq_D f_\xi$, since f is cofinal in $\prod A/D$. This means that $\{\alpha \in A : g(\alpha) \leq f_\xi(\alpha)\} = \{\beta \in B : g(\beta) \leq f'_\xi(\beta)\} \cap A$ is an element of D. It follows by definition of D' that $\{\beta \in B : g(\beta) \leq f'_\xi(\beta)\} \in D'$, which means that $g \leq_{D'} f'_\xi$.

   To show that f' is $<_{D'}$-increasing, suppose that $\xi_1 < \xi_2 < \lambda$. Then $\{\alpha \in A : f_{\xi_1}(\alpha) \leq f_{\xi_2}(\alpha)\} = \{\beta \in B : f'_{\xi_1}(\beta) \leq f'_{\xi_2}(\beta)\} \cap A$ is an element of D. It follows by definition of D' that $\{\beta \in B : f'_{\xi_1}(\beta) \leq f'_{\xi_2}(\beta)\} \in D'$, which means that $f'_{\xi_1} \leq_{D'} f'_{\xi_2}$.

4. $pcf(A \cup B) = pcf(A) \cup pcf(B)$. Proof: The inclusion '$\supseteq$' follows by (3). We show that $pcf(A \cup B) \subseteq pcf(A) \cup pcf(B)$. Suppose that $\lambda \in pcf(A \cup B)$. Let D be an ultrafilter on $A \cup B$ such that $\lambda = cf(\prod A \cup B/D)$, and let $f = \langle f_\xi : \xi < \lambda\rangle$ be a scale for $\prod A \cup B/D$.

   It holds that either $A \in D$ or $B \in D$, for if $A \notin D$, then $A' \in D$, and hence $A' \subseteq B \in D$. Without any loss of generality, assume that $A \in D$.

   We show that $\lambda \in pcf(A)$. Let $D' := \{a \subseteq A : a \in D\}$ be the canonical restriction of D to A. D' is an ultrafilter on A. By elementary arguments, like in (3), one shows that $f \upharpoonright A = \langle f_\xi \upharpoonright A : \xi < \lambda\rangle$ is a scale for $\prod A/D'$, and thus, that $\lambda = cf(\prod A/D')$.

[2] Moreover, one of the main theorems of this chapter says that if A is an interval of regular cardinals, then $pcf(A)$ is also an interval of regular cardinals, and A is an initial segment of it (Theorem 4.10).

Next, we define the central tool for investigating further properties of the pcf function.

4.3 Ideal $J_{<\lambda}$

Let A be a set of regular cardinals, and let λ be a cardinal. We say that a subset $a \subseteq A$ *forces* $\prod A$ *to have cofinality less than* λ, and write *a forces cof*$<\lambda$, if for every ultrafilter D on A with $a \in D$, $cf(\prod A/D) < \lambda$.

Definition 4.2. For any cardinal λ define

$$J_{<\lambda}[A] := \{a \subseteq A : a \text{ forces } cof{<}\lambda\}.$$

$J_{<\lambda}[A]$ is an ideal on A: (i) $\emptyset \in J_{<\lambda}[A]$. (ii) Suppose that $a' \subseteq a \subseteq A$ and $a \in J_{<\lambda}[A]$. If D is any ultrafilter on A with $a' \in D$, then $a \in D$, and hence $cf(\prod A/D) < \lambda$. Thus, $a' \in J_{<\lambda}[A]$. (iii) Suppose that $a, a' \in J_{<\lambda}[A]$. If D is any ultrafilter on A with $a \cup a' \in D$, then either $a \in D$ or $a' \in D$, and hence $cf(\prod A/D) < \lambda$. Thus, $a \cup a' \in J_{<\lambda}[A]$.

Proposition 4.3.

1. *If $\lambda \leq min\ pcf(A)$, then $J_{<\lambda}[A] = \{\emptyset\}$.*
2. *If $\lambda > max\ pcf(A)$, then $J_{<\lambda}[A] = P(A)$.*
3. *If $\lambda \in pcf(A)$, then $J_{<\lambda}[A]$ is a proper ideal.*

Proof. (1) We argue indirectly. Suppose that there is $a \in J_{<\lambda}[A]$, $a \neq \emptyset$. Then there exists an ultrafilter D on A with $a \in D$. Since $a \in J_{<\lambda}[A]$, $cf(\prod A/D) < \lambda$. It follows that min $pcf(A) < \lambda$.
(2) By definition, $J_{<\lambda}[A] \subseteq P(A)$. To show that $P(A) \subseteq J_{<\lambda}[A]$, let $a \subseteq A$, $a \neq \emptyset$. Suppose that D is an ultrafilter on A with $a \in D$. Then $cf(\prod A/D) \leq$ max $pcf(A) < \lambda$. Thus, $a \in J_{<\lambda}[A]$.
(3) We need to show that $A \notin J_{<\lambda}[A]$. We argue indirectly. If $A \in J_{<\lambda}[A]$, then, by definition, for each ultrafilter D on A with $A \in D$, $cf(\prod A/D) < \lambda$. But $A \in D$ for every ultrafilter D on A. Therefore, $cf(\prod A/D) < \lambda$ for every ultrafilter D on A. Hence $\lambda \notin pcf(A)$. ☐

Note that $\lambda_1 < \lambda_2$ implies $J_{<\lambda_1}[A] \subseteq J_{<\lambda_2}[A]$; and if λ is a singular cardinal, then $J_{<\lambda}[A] = J_{<\lambda^+}[A]$.

If λ is a limit cardinal, then $J_{<\lambda}[A] = \bigcup_{\mu<\lambda} J_{<\mu}[A]$. [By the previous line, $\bigcup_{\mu<\lambda} J_{<\mu}[A] \subseteq J_{<\lambda}[A]$. For the converse inclusion consider some $a \in J_{<\lambda}[A] \setminus \bigcup_{\mu<\lambda} J_{<\mu}[A]$. It follows that there is an ultrafilter D on A such that $a \in D$ and $D \cap \bigcup_{\mu<\lambda} J_{<\mu}[A] = \emptyset$, which means that $cf(\prod A/D) < \lambda$ and $cf(\prod A/D) \geq \mu$, for every $\mu < \lambda$. Contradiction.]

Proposition 4.4. *If $A_0 \subseteq A$, then $J_{<\lambda}[A_0] = J_{<\lambda}[A] \cap P(A_0)$.*[3]

Proof. We first show that '$\subseteq$' holds. Let $a \in J_{<\lambda}[A_0]$. Clearly, $a \in P(A_0)$. Assume that D is an ultrafilter on A such that $a \in D$. Then also $A_0 \in D$, because $a \subseteq A_0$. Therefore, the restriction $D' = \{a \subseteq A_0 : a \in D\}$ of D to A_0 is an ultrafilter on A_0, and we have that $cf(\prod A/D) = cf(\prod A_0/D')$ (the restriction of any scale for $\prod A/D$ is a scale for $\prod A_0/D'$). It follows that $cf(\prod A/D) = cf(\prod A_0/D') < \lambda$, since $a \in D'$ and $a \in J_{<\lambda}[A_0]$. Hence, $a \in J_{<\lambda}[A]$.

To prove that '$\supseteq$' holds, suppose that $a \in J_{<\lambda}[A] \cap P(A_0)$, and let D be an ultrafilter on A_0 such that $a \in D$. Extend D to the ultrafilter $D' := \{b \subseteq A : b \cap A_0 \in D\}$ on A. Then $cf(\prod A_0/D) = cf(\prod A/D')$ (see page 26). Since $a \in D'$ and $a \in J_{<\lambda}[A]$, we have $cf(\prod A_0/D) = cf(\prod A/D') < \lambda$. Hence, $a \in J_{<\lambda}[A_0]$. $\square$

We say that A is *progressive* if $|A| < \min A$. Recall that a reduced product $\prod A/I$ is λ-directed (for a cardinal λ) iff every $F \subseteq \prod A/I$ with $|F| < \lambda$ has an upper bound in $\prod A/I$.

The following theorem, which has a number of consequences, states a crucial property of the ideals $J_{<\lambda}$.

Theorem 4.5. [λ-Directedness] *Assume that A is a progressive set of regular cardinals. Then $\prod A/J_{<\lambda}$ is λ-directed for every cardinal λ.*

For a proof see [1, Theorem 3.4].

Corollary 4.6. Suppose that A is a progressive set of regular cardinals. Then for every ultrafilter D on A

$$cf(\prod A/D) < \lambda \quad \text{iff} \quad J_{<\lambda} \cap D \neq \emptyset,$$

that is, iff some element of D forces *cof* $<\lambda$.

Proof. We prove the 'only if' direction indirectly. Suppose that $J_{<\lambda} \cap D = \emptyset$. It means that D extends the dual filter of I. Therefore, since $\prod A/J_{<\lambda}$ is λ-directed, $\prod A/D$ is λ-directed as well. It follows that $cf(\prod A/D) \geq \lambda$ (because any sequence of length $<\lambda$ of functions in $\prod A$ is bounded in $\prod A$). Conversely, if $J_{<\lambda} \cap D \neq \emptyset$, then, by definition of $J_{<\lambda}$, $cf(\prod A/D) < \lambda$. $\square$

Corollary 4.7. Suppose that A is a progressive set of regular cardinals. Then $\lambda \in pcf(A)$ iff $J_{<\lambda} \subsetneq J_{<\lambda^+}$.

[3] This proposition allows us to write $J_{<\lambda}$ instead of $J_{<\lambda}[A_0]$ and $J_{<\lambda}[A]$, whenever we are dealing with some fixed sets of cardinals A and A_0 with $A_0 \subseteq A$.

Proof. If $\lambda \in pcf(A)$, then there is an ultrafilter D such that $cf(\prod A/D) = \lambda < \lambda^+$. By Corollary 4.6, $J_{<\lambda^+} \cap D \neq \emptyset$. Let $a \in J_{<\lambda^+} \cap D$. Then $a \notin J_{<\lambda}$, because $a \in D$ and $cf(\prod A/D) \not< \lambda$. Hence $a \in J_{<\lambda^+} \backslash J_{<\lambda}$.

For the converse, suppose that $a \in J_{<\lambda^+} \backslash J_{<\lambda}$. Since $a \notin J_{<\lambda}$, there is an ultrafilter D with $a \in D$ such that $cf(\prod A/D) \geq \lambda$. It follows that $cf(\prod A/D) = \lambda$, because $a \in J_{<\lambda^+}$. Thus, $\lambda \in pcf(A)$. □

Corollary 4.8. Suppose that A is a progressive set of regular cardinals. Then

$$|pcf(A)| \leq |P(A)|.$$

Proof. By Corollary 4.7, whenever $\lambda \in pcf(A)$, then $J_{<\lambda} \subsetneq J_{<\lambda^+}$. It follows that $\langle J_{<\lambda}\rangle_{\lambda \in pcf(A)}$ is a strictly decreasing sequence of length $|pcf(A)|$ of subsets of A. Since such a sequence can have length at most $|P(A)|$, it holds that $|pcf(A)| \leq |P(A)|$. □

Corollary 4.9. Suppose that A is a progressive set of regular cardinals. Then the set $pcf(A)$ has a maximal element.

Proof. Since $\lambda_1 < \lambda_2$ implies $J_{<\lambda_1} \subseteq J_{<\lambda_2}$, we have that $\langle J_{<\lambda}\rangle_{\lambda \in pcf(A)}$ is an $\subseteq$-increasing sequence of ideals on A. It follows easily that the union

$$I := \bigcup_{\lambda \in pcf(A)} J_{<\lambda}$$

is an ideal on A as well. By Proposition 4.3(3), each $J_{<\lambda}$ in the sequence is proper, that is, $A \notin J_{<\lambda}$. Therefore, I is proper as well. Hence, by Proposition 2.10(3), I can be extended to a maximal proper ideal J. Let D be the dual (ultra)filter of J, and let $\mu = cf(\prod A/D)$. Then, since D is disjoint from $J_{<\lambda}$ for each $\lambda \in pcf(A)$, Corollary 4.6 implies that $cf(\prod A/D) \geq \lambda$, for each $\lambda \in pcf(A)$. Thus, $\mu = cf(\prod A/D) \in pcf(A)$ is the maximal element of $pcf(A)$. □

We say that a set X is an *interval of regular cardinals* if for some cardinals $\alpha < \beta$, $X = \{a \in Ord : a$ is a regular cardinal and $\alpha \leq a < \beta\}$.

Note that $pcf(A)$ is not necessarily an interval of regular cardinals. For instance, if $A = \{\aleph_{2n} : 1 < \ n < \omega\}$, then $\aleph_{2n+1}$, $n \in \omega$, can not be realized as true cofinality of A modulo some ultrafilter D. [Proof: We argue indirectly. Suppose that for some $n \in \omega$ there is an ultrafilter D such that $\aleph_{2n+1} = cf(\prod A/D)$. Let $f = \langle f_\xi : \xi < \lambda\rangle$ be a scale for $\prod A/D$. It holds that either $\{\aleph_{2k} : 1 < k \leq n\} \in D$ or $\{\aleph_{2k} : n < k < \omega\} \in D$. If the finite set $\{\aleph_{2k} : 1 < k \leq n\}$ is in D, then D is a principal ultrafilter concentrating on some cardinal below $\aleph_{2n+1}$, and thus, $cf(\prod A/D) < \aleph_{2n+1}$. This contradicts our assumption. If $\{\aleph_{2k} : n < k < \omega\} \in D$, then f is $<_D$-bounded by the pointwise supremum of f_ξ's. This contradicts f being a scale.]

The following theorem is an important result of the pcf theory which plays a crucial role in the applications of the pcf theory to cardinal arithmetic. See Chapter 5.

Theorem 4.10. *Suppose that A is a progressive interval of regular cardinals. Then $pcf(A)$ is also an interval of regular cardinals.*

Proof. Suppose that A is a progressive interval of regular cardinals. Recall that $pcf(A) \cap \min A = \emptyset$ and $A \subseteq pcf(A)$. Hence we need to show that every regular cardinal λ with $\sup A \leq \lambda < \max pcf(A)$ is in $pcf(A)$.

If $\sup A \notin A$, then $\sup A$ must be a limit point of A, and thus, a singular cardinal, since $|A| < \min A \leq \sup A$. Otherwise, $\sup A \in A \subseteq pcf(A)$.

Consider now a regular cardinal λ with $\sup A < \lambda < \max pcf(A)$. We show that $\lambda \in pcf(A)$.

Let A' be the first initial segment of A that is not in the ideal $J_{<\lambda}$ (it exists). Then all proper initial segments of A' are in $J_{<\lambda}$.

Claim. A' has no maximal element.

Proof of the claim. In order to get a contradiction, suppose that $\sup A' \in A'$. Then $A' \backslash \sup A' \in J_{<\lambda}$. Since $A' \notin J_{<\lambda}$, there is an ultrafilter D on A' such that $cf(\prod A'/D) \geq \lambda$. It follows that $A' \backslash \sup A' \notin D$, because $A' \backslash \sup A' \in J_{<\lambda}$. Hence $\{\sup A'\} \in D$; that is, D is a principal ultrafilter. Thus, we have $\lambda \leq cf(\prod A'/D) = \sup A'$. But this is a contradiction to $\lambda > \sup A$.

It follows that $\kappa^{++} < \sup A' < \lambda$ and $\{a \in A' : a \leq \kappa^{++}\} \in J_{<\lambda}$ (and $|A'|^+ \leq \kappa \leq \lambda$), for every cardinal $\kappa \in A'$. Further, $J_{<\lambda}$ is a proper ideal, and the product $\prod A'/J_{<\lambda}$ is λ-directed. Thus, we can apply Theorem 3.13 to A', $J_{<\lambda}$ and λ: there exists a $<_{J_{<\lambda}}$-increasing sequence $f = \langle f_\xi : \xi < \lambda\rangle$ of functions in $\prod A'/J_{<\lambda}$ with an exact upper bound h such that

$$\{a \in A' : cf(h(a)) < \kappa\} \in J_{<\lambda},$$

for every cardinal $\kappa \in A'$. In particular,

$$I_1 := \{a \in A' : cf(h(a)) < \min A'\} \in J_{<\lambda}. \tag{4.1}$$

The identity function id on A' is a $<$-upper bound for f. So, since h is minimal, we have $h \leq_{J_{<\lambda}} id$, that is,

$$I_2 := \{a \in A' : h(a) > id(a) = a\} \in J_{<\lambda}. \tag{4.2}$$

It follows from (4.1) and (4.2) that

$$\min A' \leq cf(h(a)) \leq a \tag{4.3}$$

holds for every $a \in A' \backslash (I_1 \cup I_2)$. Thus, by changing h on the null-set $I_1 \cup I_2$, we can assume that (4.3) holds for every $a \in A'$. But since A' is an interval of regular cardinals, we have

$$cf(h(a)) \in A' \text{ for every } a \in A'.$$

Recall that by (3.1) we have

$$\lambda = tcf(\textstyle\prod_{a \in A'} cf(h(a))/J_{<\lambda}).$$

Let $B := \{cf(h(a)) : a \in A'\}$, and let $c : A' \to B$ be the function defined by $c(a) := cf(h(a))$. Then $|A'| < \min B$. Hence, we can apply Lemma 3.5: there is an ideal J on B such that

$$tcf(\textstyle\prod B/J) = tcf(\prod_{a \in A'} cf(h(a))/J_{<\lambda}) = \lambda.$$

Thus, we have proved that $\lambda \in pcf(B) \subseteq pcf(A') \subseteq pcf(A)$. This completes the proof. $\square$

We have the following generalization of the last theorem.

Definition 4.11. Suppose that A is a set of regular cardinals. For every cardinal $\kappa < \min A$ define

$$pcf_\kappa(A) := \bigcup\{pcf(X) : \; X \subseteq A \text{ and } |X| = \kappa\}.$$

Theorem 4.12. *Suppose that A is an interval of regular cardinals. Then for every cardinal $\kappa < minA$, $pcf_\kappa(A)$ is also an interval of regular cardinals.*

For a proof see [1, Theorem 3.11].

The pcf function has (under weak assumptions) the following closure property.

Theorem 4.13. *Suppose that A is a progressive set of regular cardinals, and $B \subseteq pcf(A)$ is also progressive. Then*

$$pcf(B) \subseteq pcf(A).$$

In particular, if $pcf(A)$ is progressive, then $pcf(pcf(A)) = pcf(A)$.

For a proof see [1, Theorem 3.12].

4.4 Generators for $J_{<\lambda}$

We shall prove that for every $\lambda \in pcf(A)$ there is a set $B_\lambda[A] \subseteq A$, called generating set, such that

$$J_{<\lambda^+}[A] = J_{<\lambda}[A] + B_\lambda[A],$$

that is, $J_{<\lambda^+}[A]$ is generated by $J_{<\lambda} \cup \{B_\lambda[A]\}$. This property of ideals $J_{<\lambda}$ is called *normality*. Moreover, if A is progressive, then for every $X \subseteq A$,

$$X \subseteq B_{\lambda_1}[A] \cup \cdots \cup B_{\lambda_n}[A],$$

for some finite set $\{\lambda_1, ..., \lambda_n\} \subseteq pcf(X)$.[4]

Definition 4.14. Suppose that $\lambda \in pcf(A)$. A sequence $f = \langle f_\xi : \xi < \lambda \rangle$ of functions in $\prod A$, increasing in $<_{J_{<\lambda}}$, is a *universal sequence for* λ if it is a scale for $\prod A/D$ whenever D is an ultrafilter on A such that $cf(\prod A/D) = \lambda$.[5]

Theorem 4.15. *Suppose that A is a progressive set of regular cardinals. Then every $\lambda \in pcf(A)$ has a universal sequence.*

For a proof see [1, Theorem 4.2].

The universal sequences will be frequently used from now on.[6] Before we use them to prove the existence of generating sets, we state two other important consequences of Theorem 4.15.

Lemma 4.16. *Suppose that A is a progressive set of regular cardinals. The following are equivalent for every cardinal λ:*

1. $\lambda = max\ pcf(A)$
2. $\lambda = tcf(\prod A/J_{<\lambda})$
3. $\lambda = cf(\prod A/J_{<\lambda})$

Proof. 1 $\Rightarrow$ 2. We show that any universal sequence for λ is cofinal in $\prod A/J_{<\lambda}$. Argue indirectly. Suppose that $f = \langle f_\xi : \xi < \lambda \rangle$ is not cofinal in $\prod A/J_{<\lambda}$, i.e. there exists $h \in \prod A/J_{<\lambda}$ such that $\{a \in A : f_\xi(a) < h(a)\} \notin J_{<\lambda}$, for every $\xi < \lambda$. Note that, since f is $<_{J_{<\lambda}}$-increasing, we have for $\xi_1 < \xi_2 < \lambda$,

[4] These nice properties of the ideals $J_{<\lambda}$ play a key role in the applications of pcf theory to cardinal arithmetic.

[5] Notice that it suffices to say cofinal, instead of scale, since such an f is $<_D$-increasing.

[6] We refer the reader to [2, section 4] for another (motivating) approach to generators and universal sequences. The idea of a universal sequence arises from the attempt to dominate a sequence of scales.

$$\{a \in A : f_{\xi_1}(a) < h(a)\} \supseteq_{J_{<\lambda}} \{a \in A : f_{\xi_2}(a) < h(a)\}.$$

Thus, we can extend the ideal $J_{<\lambda}$ to a maximal ideal J such that $\{a \in A : f_\xi(a) < h(a)\} \notin J$, for every $\xi < \lambda$. Let D be the dual (ultra)filter of J. It follows that $cf(\prod A/D) = \lambda$, because $D \cap J_{<\lambda} = \emptyset$ and $\lambda = \max\ pcf(A)$. But f is not a scale for $\prod A/D$, since $f_\xi <_D h$ for every $\xi < \lambda$. Hence f is not universal for λ.

$2 \Rightarrow 3$. Trivial.

$3 \Rightarrow 1$. Suppose that $\lambda = cf(\prod A/J_{<\lambda})$. We first show that $\lambda \le \max\ pcf(A)$. The ideal $J_{<\lambda}$ is clearly a proper ideal (otherwise, if $A \in J_{<\lambda}$, then $cf(\prod A/J_{<\lambda}) = 1$). Hence, there is an ultrafilter D on A such that $D \cap J_{<\lambda} = \emptyset$. It follows that $cf(\prod A/D) \le cf(\prod A/J_{<\lambda}) = \lambda$ (see Remark 3.2(5)). But $cf(\prod A/D) < \lambda$ is impossible, because $D \cap J_{<\lambda} = \emptyset$. Thus, $cf(\prod A/D) = \lambda$. So we have $\lambda \in pcf(A)$, which implies $\lambda \le \max\ pcf(A)$. To prove that $\lambda \ge \max\ pcf(A)$, let D be any ultrafilter on A. We claim that $\lambda \ge cf(\prod A/D)$. If $D \cap J_{<\lambda} \ne \emptyset$, then, by definition, $\lambda > cf(\prod A/D)$. But if $D \cap J_{<\lambda} = \emptyset$, then (like above) $\lambda = cf(\prod A/J_{<\lambda}) \ge cf(\prod A/D)$. This completes the proof. □

Theorem 4.17. *If A is a progressive set of regular cardinals, then*

$$cf(\textstyle\prod A, <) = max\ pcf(A),$$

where $<$ refers to the everywhere dominance relation $<_{\{\emptyset\}}$. Hence $cf(\prod A, <)$ is a regular cardinal.

Proof. We only give a scetch of the proof. It follows easily that $cf(\prod A, <) \ge \max\ pcf(A)$. Let $\lambda = \max\ pcf(A)$, and let D be an ultrafilter on A such that $\lambda = cf(\prod A/D)$. Then $<_D$ extends $<$, and thus (by Remark 3.2(5)), we have $cf(\prod A, <) \ge cf(\prod A, <_D) = \lambda$.

The converse, $cf(\prod A, <) \le \max\ pcf(A)$, is proved by finding a cofinal subset of $(\prod A, <)$ of cardinality $\max\ pcf(A) = \lambda$. Fix for every $\mu \in pcf(A)$ a universal sequence $f^\mu = \langle f^\mu_i : i < \mu\rangle$ for μ. Let F be the set of all functions of the form

$$\sup\{f^{\mu_1}_{i_1}, f^{\mu_2}_{i_2}, ..., f^{\mu_n}_{i_n}\},$$

where $\mu_1, \mu_2, ..., \mu_n$ is any finite sequence of cardinals in $pcf(A)$, and $i_k < \mu_k$ are arbitrary indices. Then F is a cofinal subset of $(\prod A, <)$ of cardinality λ (for details see the proof of Theorem 4.26). □

In order to prove the existence of generating sets $B_\lambda[A]$, we first make the following characterization.

Lemma 4.18. *Suppose that A is a progressive set of regular cardinals. Then for any set $B \subseteq A$,*

$$J_{<\lambda^+}[A] = J_{<\lambda}[A] + B \tag{4.4}$$

if and only if

$$B \in J_{<\lambda^+}[A] \tag{4.5}$$

and

whenever D is an ultrafilter on A with $cf(\prod A/D) = \lambda$, then $B \in D$. (4.6)

Proof. Assume first that (4.4) holds. Then (4.5) is obvious. We prove (4.6). Suppose that D is an ultrafilter on A with $cf(\prod A/D) = \lambda$. Then $D \cap J_{<\lambda^+} \neq \emptyset$. Let $X \in D \cap J_{<\lambda^+}$. By (4.4), $X \backslash B \in J_{<\lambda}$. Since $D \cap J_{<\lambda} = \emptyset$, it follows that $B \in D$. [For if $A \backslash B \in D$, then $(A \backslash B) \cap X = X \backslash B \in D \cap J_{<\lambda}$.]

Now assume that (4.5) and (4.6) hold. We prove (4.4). Since $B \in J_{<\lambda^+}$, we have $J_{<\lambda^+} \supseteq J_{<\lambda} + B$. To prove $J_{<\lambda^+} \subseteq J_{<\lambda} + B$, assume that $X \in J_{<\lambda^+}$ and show that $X \backslash B \in J_{<\lambda}$ as follows. Let D be an ultrafilter on A such that $X \backslash B \in D$. We claim that $cf(\prod A/D) < \lambda$. Since $X \in J_{<\lambda^+} \cap D$, $cf(\prod A/D) < \lambda^+$. But $cf(\prod A/D) = \lambda$ is impossible, because $B \notin D$. Hence $cf(\prod A/D) < \lambda$. □

Theorem 4.19. [Normality] *Suppose that A is a progressive set of regular cardinals. Then for every cardinal $\lambda \in pcf(A)$ there is a set $B \subseteq A$ such that*

$$J_{<\lambda^+}[A] = J_{<\lambda}[A] + B.$$

Proof. Let $\lambda \in pcf(A)$. The case $\lambda \in \{|A|^+, |A|^{++}, |A|^{+++}\}$ is rather trivial:

$$\begin{aligned} J_{<|A|^+} &= \{\emptyset\}; \\ J_{<|A|^{++}} &= \{\emptyset\} + \{|A|^+\} = \{\emptyset, \{|A|^+\}\}; \\ J_{<|A|^{+++}} &= \{\emptyset, \{|A|^+\}\} + \{|A|^{++}\} = \{\emptyset, \{|A|^+\}, \{|A|^{++}\}, \{|A|^+, |A|^{++}\}\}. \end{aligned}$$

Suppose now that $|A|^{+3} < \lambda$. Then $\{a \in A : a < |A|^{+3}\} = \{|A|^+, |A|^{++}\} \in J_{<|A|^{+3}} \subseteq J_{<\lambda}$. Hence we can apply Theorem 3.13 to A, $J_{<\lambda}$ and λ ($\kappa = |A|^+$): there exists a $<_{J_{<\lambda}}$-increasing sequence $f = \langle f_\xi : \xi < \lambda \rangle$ of functions in $\prod A/J_{<\lambda}$ with an exact upper bound h. Since the identity function id on A is a $<$-upper bound of f, we have $h \leq_{J_{<\lambda}} id$. By changing h on a null-set, we can assume that $h(a) \leq a$, for every $a \in A$.

Moreover, by Remark 3.14, we can assume that f dominates some universal sequence for λ. It follows that f is a universal sequence for λ as well.

By verifying (4.5) and (4.6), we show that

$$B := \{a \in A : h(a) = a\}$$

generates $J_{<\lambda^+}$ over $J_{<\lambda}$. In order to prove that $B \in J_{<\lambda^+}$, let D be any ultrafilter on A such that $B \in D$. We need to show that $cf(\prod A/D) < \lambda^+$. First, if $D \cap J_{<\lambda} \neq \emptyset$, then by definition $cf(\prod A/D) < \lambda$. Suppose now that $D \cap J_{<\lambda} = \emptyset$. Then $<_D$ extends $<_{J_{<\lambda}}$. It follows that f is a scale for $\prod h/D$ (because it is a scale for $\prod h/J_{<\lambda}$). But, since $B \in D$, $\prod h/D$ is equivalent to $\prod A/D$ (modulo D). Thus, $cf(\prod A/D) = cf(\prod h/D) = \lambda$.

We prove (4.6) indirectly. Suppose that $B \notin D$. Then $\{a \in A : h(a) < a\} \in D$, and thus, $h \in \prod A/D$. Assume that $D \cap J_{<\lambda} = \emptyset$. [If $D \cap J_{<\lambda} \neq \emptyset$, then $cf(\prod A/D) < \lambda$, and we are done.] It follows that $<_D$ extends $<_{J_{<\lambda}}$, thus, $f_\xi <_D h$ for every $\xi < \lambda$ (because $f_\xi <_{J_{<\lambda}} h$). This means that f has an upper bound in $\prod A/D$. Since f is a universal sequence for λ, we have $cf(\prod A/D) \neq \lambda$ (otherwise, f would be cofinal in $\prod A/D$). □

Generating sets are not uniquely determined. But if B_1 and B_2 are both generators for $J_{<\lambda^+}$, then they both satisfy (4.4), hence $B_1 =_{J_{<\lambda}} B_2$. Thus, by a generating set $B_\lambda[A]$ (or $B_\lambda[A]$ set) we mean any set B satisfying (4.4). In particular, for $\lambda = \max pcf(A)$ we can choose $B_\lambda[A] = A$, since A obviously satisfies (4.5) and (4.6).

We have the following analogue of Proposition 4.4, which will be useful later on.

Proposition 4.20. *If $A_0 \subseteq A$ and $\lambda \in pcf(A_0)$, then the restriction to A_0 of a generator for $J_{<\lambda^+}[A]$ is a generator for $J_{<\lambda^+}[A_0]$, i.e.*

$$B_\lambda[A_0] =_{J_{<\lambda}[A_0]} A_0 \cap B_\lambda[A].$$

(Hence we can write B_λ instead of $B_\lambda[A_0]$ and $B_\lambda[A]$.)

Proof. We need to verify (4.5) and (4.6) for $A_0 \cap B_\lambda[A]$. Since $B_\lambda[A] \in J_{<\lambda^+}[A]$, also $A_0 \cap B_\lambda[A] \in J_{<\lambda^+}[A]$. Proposition 4.4 implies that $A_0 \cap B_\lambda[A] \in J_{<\lambda^+}[A_0]$.

To verify (4.6), let D_0 be any ultrafilter on A_0 such that $cf(\prod A_0/D_0) = \lambda$. We need to show that $A_0 \cap B_\lambda[A] \in D_0$. We argue indirectly. Suppose that $A_0 \cap B_\lambda[A] \notin D_0$. Then $A_0 \setminus B_\lambda[A] \in D_0$. Extend D_0 canonically to an ultrafilter D_0' on A. Then $cf(\prod A/D_0') = cf(\prod A_0/D_0) = \lambda$ (see the proof of Proposition 4.4), and $B_\lambda[A] \notin D_0'$ (because $A_0 \setminus B_\lambda[A] \in D_0'$). This is in contradiction with (4.6) (for $B_\lambda[A]$). □

We have the following fundamental relation between generators and universal sequences.

Theorem 4.21. *Suppose that A is a progressive set of regular cardinals. Let $\lambda \in pcf(A)$, and let $f = \langle f_\xi : \xi < \lambda \rangle$ be a $<_{J_{<\lambda}}$-increasing sequence of functions in $\prod A$. Then*

$$f \text{ is universal for } \lambda \quad \text{iff} \quad f \restriction B_\lambda \text{ is cofinal in } \prod B_\lambda / J_{<\lambda}.$$

Proof. The case $\lambda \in \{|A|^+, |A|^{++}, |A|^{+3}\}$ is trivial (see the proof of Theorem 4.19). So suppose that $\lambda > |A|^{+3}$.

We first show, by an indirect argument, that if f is universal for λ, then $f \restriction B_\lambda$ is cofinal in $\prod B_\lambda / J_{<\lambda}$.[7] So, suppose that $f \restriction B_\lambda$ is not cofinal, i.e. there is some $h \in \prod B_\lambda / J_{<\lambda}$ such that $h \not\leq_{J_{<\lambda}} f_\xi \restriction B_\lambda$, for every $\xi < \lambda$. Then we have $\{a \in B_\lambda : f_\xi(a) < h(a)\} \notin J_{<\lambda}$, for every $\xi < \lambda$. Moreover, since f is $<_{J_{<\lambda}}$-increasing, we have for $\xi_1 < \xi_2 < \lambda$,

$$\{a \in B_\lambda : f_{\xi_1}(a) < h(a)\} \supseteq_{J_{<\lambda}} \{a \in B_\lambda : f_{\xi_2}(a) < h(a)\}.$$

Hence there is a filter on B_λ extending the dual filter of $J_{<\lambda}$ and containing the set $\{a \in B_\lambda : f_\xi(a) < h(a)\}$, for every $\xi < \lambda$. Extend this filter to an ultrafilter D on A. Then $cf(\prod A/D) = \lambda$, because $B_\lambda \in D$ and $D \cap J_{<\lambda} = \emptyset$. But f is not a scale for $\prod A/D$, since $f_\xi <_D h$ for every $\xi < \lambda$. Hence f is not universal for λ.

Conversely, suppose that $f \restriction B_\lambda$ is cofinal in $\prod B_\lambda / J_{<\lambda}$. We claim that f is universal for λ. Let D be any ultrafilter on A such that $cf(\prod A/D) = \lambda$. By (4.6), $B_\lambda \in D$. It follows that $D' := \{a \subseteq B_\lambda : a \in D\}$ is an ultrafilter on B_λ. We have $D' \cap J_{<\lambda} = D \cap J_{<\lambda} = \emptyset$. It means that $<_{D'}$ extends $<_{J_{<\lambda}}$, thus, $f \restriction B_\lambda$ is also cofinal in $\prod B_\lambda / D'$. Now, since $B_\lambda \in D$, $[g] \mapsto [g \restriction B_\lambda]$ is an isomorphism between $\prod A/D$ and $\prod B_\lambda / D'$ (i.e a bijection preserving the ordering relation). Hence f is cofinal in $\prod A/D$. $\square$

Corollary 4.22. Since there is always a universal sequence for λ, Theorem 4.21 implies that

$$\lambda = tcf(\prod B_\lambda / J_{<\lambda}).$$

Corollary 4.23. By the previous corollary and Lemma 4.16, we have that

$$\lambda = \max pcf(B_\lambda).$$

Now we prove the *covering property* which we mentioned at the beginning of the section.

Theorem 4.24. *Suppose that A is a progressive set of regular cardinals and $\langle B_\lambda : \lambda \in pcf(A) \rangle$ is a generating sequence for A. Then for every $X \subseteq A$,*

[7] This is a generalization of the proof of the implication $1 \Rightarrow 2$, on page 32. For then we considered only $\lambda = \max pcf(A)$ and $B_\lambda = A$.

$$X \subseteq B_{\lambda_1} \cup \cdots \cup B_{\lambda_n},$$

for some finite set $\{\lambda_1, ..., \lambda_n\} \subseteq pcf(X)$.

Proof. By induction on $\lambda = \max pcf(X)$. If $\lambda = \min A = \max pcf(X)$, then $X = \{\min A\} = B_\lambda$. Suppose that the theorem is true for all $\gamma < \lambda$, $\gamma \in pcf(A)$, and let $X \subseteq A$ such that $\lambda = \max pcf(X)$. Then $X \backslash B_\lambda \in J_{<\lambda}$. [If D is any ultrafilter on X such that $X \backslash B_\lambda \in D$, then $cf(\prod X/D) \neq \lambda$, because $B_\lambda \notin D$. Thus, $cf(\prod X/D) < \lambda = \max pcf(X)$.] It follows that $\max pcf(X \backslash B_\lambda) < \lambda$.

By induction hypothesis, $X \backslash B_\lambda \subseteq B_{\lambda_1} \cup \cdots \cup B_{\lambda_n}$, for some finite set $\{\lambda_1, ..., \lambda_n\} \subseteq pcf(X \backslash B_\lambda)$. Hence $X \subseteq B_{\lambda_1} \cup \cdots \cup B_{\lambda_n} \cup B_\lambda$. □

Towards the end of this chapter we state a few useful facts in terms of filters.

Lemma 4.25. *Suppose that A is a progressive set of regular cardinals and F is a filter on A. Then the following are equivalent for every cardinal λ:*

1. $tcf(\prod A/F) = \lambda$
2. $cf(\prod A/D) = \lambda$ *for every ultrafilter D on A with $F \subseteq D$*
3. $B_\lambda \in F$ *and F extends the dual filter of $J_{<\lambda}$*

Proof. $1 \Rightarrow 2$. See Remark 3.2(5).

$2 \Rightarrow 3$. We argue indirectly. Suppose first that $B_\lambda \notin F$. Then, by Proposition 2.10(3), there exists an ultrafilter $D \supseteq F$ on A such that $B_\lambda \notin D$. It follows by (4.6) that $cf(\prod A/D) \neq \lambda$.

Similarly, if F does not extend the dual filter of $J_{<\lambda}$, then there is an ultrafilter $D \supseteq F$ on A which does not extend the dual filter of $J_{<\lambda}$. It follows that $D \cap J_{<\lambda} \neq \emptyset$. Hence $cf(\prod A/D) < \lambda$.

$3 \Rightarrow 1$. By Corollary 4.22, we have $\lambda = tcf(\prod B_\lambda/J_{<\lambda})$. Note that the restrtiction $F' := \{a \subseteq B_\lambda : a \in F\}$ of F is a filter on B_λ. Since F extends the dual filter of $J_{<\lambda}$, F' extends the dual filter of $J_{<\lambda}[B_\lambda]$. Thus, $tcf(\prod B_\lambda/F') = tcf(\prod B_\lambda/J_{<\lambda}) = \lambda$. It follows that $tcf(\prod A/F) = \lambda$, because $\prod A/F$ and $\prod B_\lambda/F'$ are isomorphic (since $B_\lambda \in F$). □

Theorem 4.26. *Suppose that A is a progressive set of regular cardinals and F is a filter on A. Then $cf(\prod A/F)$ is a regular cardinal.*

Proof. We argue as follows. Define $pcf_F(A) := \{cf(\prod A/D) : D \supseteq F\}$. We first prove that $pcf_F(A)$ has a maximal element, and then deduce that $cf(\prod A/F) = \max pcf_F(A)$.

Let λ be the minimal cardinal for which $F \cap J_{\leq\lambda} \neq \emptyset$. We show that $\lambda = \max pcf_F(A)$. For any ultrafilter $D \supseteq F$ we have $F \cap J_{\leq\lambda} \subseteq D \cap J_{\leq\lambda} \neq \emptyset$, and hence $cf(\prod A/D) \leq \lambda$. Thus, $\sup pcf_F(A) \leq \lambda$.

Conversely, we find an ultrafilter $D \supseteq F$ such that $cf(\prod A/D) = \lambda$. [Then it follows that $\lambda = \max pcf_F(A)$.] We have that $I := \bigcup_{\gamma<\lambda} J_{\leq\gamma}$ is an ideal on A (since $J_{\leq\gamma}$'s are $\subseteq$-increasing). It follows that $F \cap I = \emptyset$, because $F \cap J_{\leq\gamma} = \emptyset$, for every $\gamma < \lambda$. Extend F to an ultrafilter D such that $D \cap I = \emptyset$. Then $cf(\prod A/D) \geq \lambda$, since $D \cap J_{\leq\gamma} = \emptyset$, for every $\gamma < \lambda$. Hence $cf(\prod A/D) = \lambda$.

Now we show that $cf(\prod A/F) = \lambda$. Since D extends F, it follows that $cf(\prod A/F) \geq cf(\prod A/D) = \lambda$.

To prove the converse inequality, we find a cofinal subset of $\prod A/F$ of cardinality λ. Fix for every cardinal $\mu \in pcf_F(A)$ a universal sequence $f^\mu = \langle f_i^\mu : i < \mu\rangle$ for μ, and let E be the set of all functions of the form

$$\sup\{f_{i_1}^{\mu_1}, ..., f_{i_n}^{\mu_n}\},$$

where $\mu_1, \mu_2, ..., \mu_n$ is any finite sequence of cardinals in $pcf_F(A)$, and $i_k < \mu_k$ are arbitrary indices. Clearly $|E| = \lambda$. It remains to prove the following claim.

Claim. E is cofinal in $\prod A/F$.

Proof of the claim. Let $g \in \prod A$ be any function. Consider the collection

$$I := \{(f > g) : f \in E\}$$

of subsets of A, where $(f > g) = \{a \in A : f(a) > g(a)\}$.

If $F \cap I \neq \emptyset$, then $(f > g) \in F$ for some $f \in E$, i.e. $f >_F g$ for some $f \in E$, as desired.

Otherwise we get the following contradiction. Suppose that $F \cap I = \emptyset$. We can extend I to an ideal, since it is closed under unions, namely,

$$(f_1 > g) \cup (f_2 > g) = (\sup\{f_1, f_2\} > g).$$

Moreover, we can extend I to a maximal ideal J on A such that $F \cap J = \emptyset$. Then $\mu := cf(\prod A/J) \in pcf_F(A)$ (because the dual filter of J is an ultrafilter extending F). It follows that the universal sequence f^μ for μ is cofinal in $\prod A/J$. But we have $(f_i^\mu > g) \in I \subseteq J$, that is, $f_i^\mu \leq_J g$, for every $i < \mu$. Contradiction. $\square$

We finish this chapter by proving another representation theorem.[8]

[8] Compare it with Theorem 3.16. Recall that if I and J are ideals such that $I \subset J$ and $tcf(\prod A/I)$ exists, then $tcf(\prod A/J) = tcf(\prod A/I)$ exists as well. But the converse is false. Hence, the following theorem is stronger than Theorem 3.16, since $J_{<\mu} = J_{<\mu^+} \subseteq J^{bd}$.

Theorem 4.27. *Suppose that μ is a singular cardinal of uncountable cofinality. Then there exists a closed unbounded set (of limit cardinals) $C \subseteq \mu$ such that $|C| < minC$ and*

$$\mu^+ = tcf(\prod C^{(+)}/J_{<\mu^+}).$$

Proof. By Theorem 3.16 there is a closed unbounded set of limit cardinals $C_0 \subseteq \mu$ such that $|C_0| < \min C_0$ and

$$\mu^+ = tcf(\prod C_0^{(+)}/J^{bd}),$$

where J^{bd} is the ideal of bounded subsets of $C_0^{(+)}$.

We claim that the set $C_0^{(+)} \backslash B_{\mu^+}$ is bounded in $C_0^{(+)}$. Let F^{cbd} be the filter of cobounded subsets of $C_0^{(+)}$, i.e. the dual filter of J^{bd}. Then, clearly,

$$\mu^+ = tcf(\prod C_0^{(+)}/J^{bd}) = tcf(\prod C_0^{(+)}/F^{cbd}).$$

By Lemma 4.25, we have $B_{\mu^+} \in F^{cbd}$. Thus, $C_0^{(+)} \backslash B_{\mu^+}$ is bounded in $C_0^{(+)}$. Define

$$C := C_0 \backslash \sup\{\alpha \in C_0 : \alpha^+ \in C_0^{(+)} \backslash B_{\mu^+}\}.$$

Note that C is also a closed unbounded set with $|C| < \min C$. It follows that

$$tcf(\prod C^{(+)}/J^{bd}) = tcf(\prod C_0^{(+)}/J^{bd}) = \mu^+,$$

since $C_0^{(+)} \backslash C^{(+)} \in J^{bd}$ is a null-set. So $\mu^+ \in pcf(C^{(+)})$. By Corollary 4.22,

$$\mu^+ = tcf(\prod B_{\mu^+}[C^{(+)}]/J_{<\mu^+}).$$

We complete the proof by showing that $B_{\mu^+}[C^{(+)}] =_{J_{<\mu^+}} C^{(+)}$, but this follows immediately by Proposition 4.20, applied to $C^{(+)} \subseteq C_0^{(+)}$ and $\mu^+ \in pcf(B_{\mu^+}[C^{(+)}])$:

$$B_{\mu^+}[C^{(+)}] =_{J_{<\mu^+[C^{(+)}]}} C^{(+)} \cap B_{\mu^+}[C_0^{(+)}] = C^{(+)}.$$

$\square$

Corollary 4.28. By Lemma 4.16, we have $\mu^+ = \max pcf(C^{(+)})$.

Chapter 5

Cardinal arithmetic

In this chapter we apply pcf theory to cardinal arithmetic. Our aim is to give a clear insight into the (somewhat long) proof of Shelah's famous theorem $\aleph_\omega^{\aleph_0} \leq 2^{\aleph_0} + \aleph_{\omega_4}$.

5.1 Cofinality of $([\mu]^\kappa, \subseteq)$

Let κ and μ be any cardinals with $\kappa \leq \mu$. The collection $\{X \subseteq \mu : |X| = \kappa\}$ of all subsets of μ of cardinality κ is denoted by $[\mu]^\kappa$. One can show by a short argument that $|[\mu]^\kappa| = \mu^\kappa$ (for a proof see [3, Lemma 5.7]). Note that the inclusion relation $\subseteq$ is a quasi ordering of $[\mu]^\kappa$.

There is the following relationship between the cardinality and the cofinality of $[\mu]^\kappa$:

$$|[\mu]^\kappa| = cf([\mu]^\kappa, \subseteq) \cdot 2^\kappa. \tag{5.1}$$

The proof is quite simple. Clearly $|[\mu]^\kappa| \geq cf([\mu]^\kappa, \subseteq) \cdot 2^\kappa$. We show that $\leq$ holds as well. Suppose that $cf([\mu]^\kappa, \subseteq) = \lambda$ and let $Y = \{Y_i : i < \lambda\}$ be a cofinal subset of $[\mu]^\kappa$. Define a one-to-one map from $[\mu]^\kappa$ to $Y \times 2^\kappa$ as follows. For every $E \in [\mu]^\kappa$ find some $Y_i \in Y$ such that $E \subseteq Y_i$. Since Y_i is isomorphic to κ, E is isomorphic to some subset S of κ. Map E to (Y_i, S).

One can prove by induction that for every $n \in \omega$, $cf([\aleph_n]^{\aleph_0}, \subseteq) = \aleph_n$, but it is hard to determine $cf([\mu]^\kappa, \subseteq)$ in general. However, by the means of pcf theory we are going to prove the following crucial theorem.

Theorem 5.1. *Suppose that μ is a singular cardinal, and $\kappa < \mu$ is an infinite regular cardinal such that the interval A of regular cardinals in (κ, μ) has size $\leq \kappa$. Then*

$$cf([\mu]^\kappa, \subseteq) = max\ pcf(A). \tag{5.2}$$

Using Theorem 5.1, and some results from Chapter 4, we can prove the following:

Theorem 5.2. $\aleph_\omega^{\aleph_0} < \aleph_{(2^{\aleph_0})^+}$.

Proof. Consider the interval $A = \{\aleph_n : 0 < n < \omega\}$ of regular cardinals. By Theorem 4.10, $pcf(A)$ is also an interval of regular cardinals, containing all regular cardinals from $\aleph_1$ to $\max pcf(A)$. Moreover, by Corollary 4.8,

$$|pcf(A)| \leq |P(A)| = 2^{\aleph_0}.$$

It follows that

$$\max pcf(A) < \aleph_{(2^{\aleph_0})^+}.$$

Applying (5.1) and Theorem 5.1 (for $\kappa = \aleph_0$ and $\mu = \aleph_\omega$), we get

$$|[\aleph_\omega]^{\aleph_0}| = cf([\aleph_\omega]^{\aleph_0}, \subseteq) \cdot 2^{\aleph_0} = \max pcf(A) \cdot 2^{\aleph_0} < \aleph_{(2^{\aleph_0})^+}.$$

□

Remark 5.3. (1) If $\aleph_\omega$ is a strong limit cardinal, i.e. $2^{\aleph_n} < \aleph_\omega$ for every $n < \omega$, then $2^{\aleph_\omega} = \aleph_\omega^{\aleph_0}$, and hence, $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$.

(2) It follows from the proof of Theorem 5.2 that we can get a smaller upper bound of $\aleph_\omega^{\aleph_0}$ by limiting the size of $pcf(A)$. Indeed, one can show that $|pcf(A)| \leq |A|^{+3}$ (whenever A is a progressive interval of regular cardinals), and hence get $\aleph_\omega^{\aleph_0} \leq 2^{\aleph_0} + \aleph_{\omega_4}$ (see the next section). [1]

The proof of Theorem 5.1.

We first prove the easier inequality $cf([\mu]^\kappa, \subseteq) \geq \max pcf(A)$. Note that A is a progresive set of regular cardinals, since $|A| \leq \kappa$. Let $\lambda = cf([\mu]^\kappa, \subseteq)$, and let $\{X_i : i < \lambda\}$ be cofinal in $([\mu]^\kappa, \subseteq)$. Define for each $i < \lambda$ a function $h_i \in \prod A$ by $h_i(a) := \sup a \cap X_i$. Then $\{h_i : i < \lambda\}$ is cofinal in $(\prod A, <)$. [If $f \in \prod A$, then the range of f is an element of $[\mu]^{<\kappa}$, and hence, it is covered by some X_i. Thus, $f \leq h_i$.] So $\lambda \geq cf(\prod A, <) = \max pcf(A)$ (the last equality by Theorem 4.17).

The proof of the other inequality, $cf([\mu]^\kappa, \subseteq) \leq \max pcf(A)$, is more complicated. We first assume that $|A| < \kappa$. The case $|A| = \kappa$ is then obtained by applying the first case to $|A| = \kappa^+$ and using

$$cf([\mu]^\kappa, \subseteq) \leq cf([\mu]^{\kappa^+}, \subseteq) \cdot \kappa^+. \tag{5.3}$$

For a cardinal γ, let H_γ be the $\in$-structure whose universe is the collection of all sets which have transitive closure of size $< \gamma$. Fix some large θ such that H_θ contains all sets that were discussed so far. We also add to the structure H_θ a well-ordering $<^*$ of its universe. It allows us to assume that the objects we talk about are uniquely determined.

[1] It is still an open question if actually $|pcf(A)| = |A|$ holds.

Definition 5.4. An elementary substructure $M \prec H_\theta$ is *κ-presentable* if there is a sequence $\langle M_i : i < \kappa \rangle$ of elementary substructures of H_θ such that

1. if $i < \kappa$, then $M_i \subset M_{i+1}$ and $M_i \in M_{i+1}$,
2. for limit ordinals $\delta < \kappa$, $M_\delta = \bigcup_{i<\delta} M_i$, and $M = \bigcup_{i<\kappa} M_i$,
3. M has cardinality κ and $\kappa + 1 \subset M$.

Let $\mathcal{M}$ be the collection of all κ-presentable substructures $M \prec H_\theta$ such that $A \in M$. Define

$$F = \{M \cap \mu : M \in \mathcal{M}\}.$$

We are going to show that F is cofinal in $([\mu]^\kappa, \subseteq)$ and of cardinality at most $\max pcf(A)$. This will complete our proof.

It follows easily that F is cofinal, because if $X \in [\mu]^\kappa$, then there is a κ-presentable substructure M such that $A \in M$ and $X \subset M$. [To construct such an M use Löwenheim-Skolem theorem. Define the approaching substructures as follows. Start with an arbitrary $M_0 \prec H_\theta$ of cardinality κ such that $A \in M_0$, $X \subset M$ and $\kappa + 1 \subset M$. For each $i < \kappa$, M_i is an element of H_θ, and thus, can be incorporated in $M_{i+1} \prec H_\theta$.]

It remains to show that F has cardinality at most $\max pcf(A)$. For any structure N, define the 'characteristic function' Ch_N of N by

$$Ch_N(\gamma) = \sup N \cap \gamma, \text{ for regular cardinals } \gamma > |N|.$$

Note that if M is a κ-presentable substructure, then $Ch_M \restriction A$ is an element of $\prod A$, because $|M| < \min A$.

We argue as follows. We first show (in the next lemma) that for $M \in \mathcal{M}$, $Ch_M \restriction A$ determines $M \cap \mu$, i.e. $Ch_M \restriction A = Ch_{M'} \restriction A$ implies $M \cap \mu = M' \cap \mu$, and then prove that $|\{Ch_M \restriction A : M \in \mathcal{M}\}| \leq \max pcf(A)$.

Note that whenever $X \in M$ ($M \in \mathcal{M}$) such that $|X| \leq \kappa$, then $X \subset M$. In particular, $A \subset M$. [Since $|X| \leq \kappa$, there is a function in H_θ (and hence in M) from κ onto X. It follows that $X \subset M$, because $\kappa \subset M$.]

Lemma 5.5. *Suppose that M is κ-presentable. Then $Ch_M \restriction A$ determines $M \cap \mu$.*

Proof. Suppose that M and M' are two κ-presentable substructures of H_θ such that $Ch_M \restriction A = Ch_{M'} \restriction A$. We show by induction that $M \cap \gamma = M' \cap \gamma$ for every cardinal $\gamma \leq \mu$.

Clearly, $M \cap \gamma = M' \cap \gamma = \gamma$, for every cardinal $\gamma \leq \kappa$. If $\gamma \leq \mu$ is a limit cardinal, then $M \cap \gamma = \bigcup_{\gamma' < \gamma} M \cap \gamma'$, and hence $M \cap \gamma = M' \cap \gamma$ follows by the induction hypothesis.

Assume now that $M \cap \gamma = M' \cap \gamma$ for some γ with $\kappa < \gamma < \mu$. We need to show that also $M \cap \gamma^+ = M' \cap \gamma^+$. Observe that there is a closed unbounded subset E of $Ch_M(\gamma^+) = \sup\ M \cap \gamma^+$ of order-type κ such that $E \subseteq M$. [Proof. For each $i < k$, we have $M_i, \gamma^+ \in M$, and thus, $\sup\ M_i \cap \gamma^+ \in M$ and $E = \{\sup\ M_i \cap \gamma^+ : i < k\} \subseteq M$. E is closed, since for limit ordinals $\delta < \kappa$, $M_\delta = \bigcup_{i<\delta} M_i$, and it is cofinal in $M \cap \gamma^+$, because $M = \bigcup_{i<\kappa} M_i$.] Similarly, there is a closed unbounded subset E' of $Ch_{M'}(\gamma^+) = \sup\ M' \cap \gamma^+$ of order-type κ such that $E' \subseteq M'$.

Since κ is uncountable, $E \cap E' \subseteq M \cap M'$ is a closed unbounded subset of $Ch_M(\gamma^+) = Ch_{M'}(\gamma^+)$. In particular, $M \cap M' \cap \gamma^+$ is cofinal in both $M \cap \gamma^+$ and $M' \cap \gamma^+$.

Let $\alpha \in M \cap M' \cap \gamma^+ \backslash \gamma$ be any ordinal. There is a bijection $f : \gamma \to \alpha$ (in H_θ). Since $M, M' \prec H_\theta$, the $<^*$-least such f is in both M and M'. Hence, we have $M \cap \alpha = f''(M \cap \gamma) = f''(M' \cap \gamma) = M' \cap \alpha$. It follows that $M \cap \gamma^+ = M' \cap \gamma^+$. $\square$

To prove $|\{Ch_M \restriction A : M \in \mathcal{M}\}| \leq \max pcf(A)$, we define a special type of universal sequence.

Suppose that $\lambda \in pcf(A)$ and $f = \langle f_\xi : \xi < \lambda \rangle$ is a sequence of functions in $\prod A$. Let $\delta < \lambda$ be a limit cardinal with $cf(\delta) = \kappa$. For every closed unbounded set $E \subseteq \delta$ of order-type $cf(\delta)$ let

$$h_E := \sup\{f_\xi : \xi \in E\}.$$

There is a closed unbounded set $C \subseteq \delta$ such that $h_C \leq h_E$, for every closed unbounded set $E \subseteq \delta$. [Proof. Otherwise, we can contruct a decreasing sequence $\langle E_\alpha : \alpha < |A|^+ \rangle$ of closed unbounded sets of δ such that for every $\alpha < |A|^+$, $h_{E_\alpha} \not\leq h_{E_{\alpha+1}}$ (since $|A| < cf(\delta)$, at limit stages we can take intersections of the sets so far constructed). It follows that there is a single $a \in A$ such that $h_{E_\alpha}(a) > h_{E_{\alpha+1}}(a)$ for infinitely many α's. Contradiction.] The function h_C is called a *minimal club-obedient bound of* $f = \langle f_\xi : \xi < \delta \rangle$. The sequence $f = \langle f_\xi : \xi < \lambda \rangle$ is said to be *minimally obedient* (*at cofinality* κ) if for every $\delta < \lambda$ with $cf(\delta) = \kappa$, f_δ is a minimal club-obedient bound of $\langle f_\xi : \xi < \delta \rangle$.

We can construct a universal sequence $\langle f_\xi : \xi < \lambda \rangle$ for λ, which is minimally obedient, as follows. Let $\langle f^0_\xi : \xi < \lambda \rangle$ be any universal sequence for λ. Define $\langle f_\xi : \xi < \lambda \rangle$ by induction on $\xi < \lambda$ such that

1. $f_0 = f^0_0$, and $f_{\xi+1} > \max\{f_\xi, f^0_\xi\}$,

2. at limit stages $\delta < \lambda$ with $cf(\delta) = \kappa$, f_δ is a minimal club-obedient bound of $\langle f_\xi : \xi < \delta \rangle$,

3. at limit stages $\delta < \lambda$ with $cf(\delta) \neq \kappa$, f_δ is any $J_{<\lambda}$-upper bound of $\langle f_\xi : \xi < \delta\rangle$, guaranteed by the λ-directedness of $\prod A/J_{<\lambda}$.

Note that the sequence $\langle f_\xi : \xi < \lambda\rangle$ is $J_{<\lambda}$-increasing. It is by construction minimally obedient, and it is universal, since $f_{\xi+1} > f^0_\xi$ for every $\xi < \lambda$.

Fix for every cardinal $\lambda \in pcf(A)$ a minimally obedient universal sequence $f^\lambda = \langle f^\lambda_\xi : \xi < \lambda\rangle$ for λ, which is least in the well-ordering $<^*$ of H_θ (and hence, by elementarity, contained in each $M \in \mathcal{M}$ with $\lambda \in M$).

Definition 5.6. A sequence $f = \langle f_\xi : \xi < \lambda\rangle$ of functions in $\prod A$ is said to be *persistently cofinal for* λ if for every $h \in \prod A$ there exists $\xi_0 < \lambda$ such that for every ξ, with $\xi_0 \leq \xi < \lambda$,

$$h \restriction B_\lambda <_{J_{<\lambda}} f_\xi \restriction B_\lambda.$$

The minimally obedient universal sequences f^λ are persistently cofinal for λ, because they are $J_{<\lambda}$-increasing, and hence cofinal in $B_\lambda/J_{<\lambda}$ (see Theorem 4.21).

The following lemma is the crucial observation, which will also be used in the next section.

For any structure N, let $\bar{N}$ denote the ordinal closure of N, that is, $\gamma \in \bar{N}$ iff $\gamma \in N \cap Ord$ or γ is a limit of ordinals in N.

Lemma 5.7. *Suppose that A is a progressive set of regular cardinals, $\lambda \in pcf(A)$, and $f = \langle f_\xi : \xi < \lambda\rangle$ is a sequence of functions in $\prod A$. Let κ be a regular cardinal with $|A| < \kappa < minA$, and let $M \prec H_\theta$ be a κ-presentable substructure such that $f, A \in M$ (and hence $\lambda \in M$). Let $\gamma = Ch_M(\lambda)$. Then the following hold.*

1. *If f is persistently cofinal for λ, then*

$$\{a \in A : Ch_M(a) \leq f_\gamma(a)\} \text{ is a } B_\lambda[A] \text{ set.} \tag{5.4}$$

2. *If f is a minimally obedient universal sequence for λ, then for every limit ordinal $\gamma' \in (\bar{M}\cap\lambda)\setminus M$ there is a closed unbounded set $C \subseteq \gamma' \cap M$ (of order-type κ) such that $f_{\gamma'} = sup\{f_\xi : \xi \in C\}$, and thus*

$$f_{\gamma'}(a) \in \bar{M} \cap a, \text{ for every } a \in A.$$

In particular, $f_\gamma(a) \in \bar{M} \cap a$, for every $a \in A$, and hence

$$f_\gamma \leq Ch_M \tag{5.5}$$

and

$$\begin{aligned}&\text{for every } h \in M \cap \prod A, \text{ there is some } d \in M \cap \prod A\\ &\text{such that } h \restriction B_\lambda <_{J_{<\lambda}} d \restriction B_\lambda \text{ and } d \leq f_\gamma.\end{aligned} \tag{5.6}$$

For a proof of (1) and (2) see [1, Lemma 5.4] and [1, Lemma 5.7], respectively.

Corollary 5.8. Suppose that A is a progressive set of regular cardinals, $\lambda \in pcf(A)$, and $f = \langle f_\xi : \xi < \lambda \rangle$ is a sequence of functions in $\prod A$. Let κ be a regular cardinal with $|A| < \kappa < \min A$, and let $M \prec H_\theta$ be a κ-presentable substructure such that $f, A \in M$ (and hence $\lambda \in M$). Let $\gamma = Ch_M(\lambda)$. Suppose that (5.4), (5.5) and (5.6) hold[2]. Then

$$b_\lambda := \{a \in A : Ch_M(a) = f_\gamma(a)\}$$

is a $B_\lambda[A]$ set. Moreover, there is a subset b'_λ of b_λ which is a generating set, as well, and which is in M.

Proof. [3] It follows immediately from (5.4) and (5.5) that b_λ is a $B_\lambda[A]$ set.

In order to define a subset of b_λ which is a generating set, and which is in M, we modify the definition of b_λ, substituting M and γ by parameters from M.

If $a \in A$ and $f_\gamma(a) < Ch_M(a)$, then there exists some $i < \kappa$ such that $f_\gamma(a) < Ch_{M_i}(a)$, because $M = \bigcup_{i<\kappa} M_i$. Since $|A| < \kappa$, there is a single $i < \kappa$ such that

$$f_\gamma(a) < Ch_M(a) \ \text{ iff } \ f_\gamma(a) < Ch_{M_i}(a),$$

for every $a \in A$. By negating both sides, we get

$$a \in b_\lambda \ \text{ iff } \ Ch_{M_i}(a) \le f_\gamma(a).$$

Hence, we have replaced the parameter M by M_i in the definition of b_λ. To substitute γ, we use the property (5.6) of f (for $h = Ch_{M_i}$): there exists a function $d \in M \cap \prod A$ such that

1. $Ch_{M_i} \restriction B_\lambda <_{J_{<\lambda}} d \restriction B_\lambda$, and
2. $d \le f_\gamma$.

We replace f_γ in the definition of b_λ by the function d, and define

$$b'_\lambda := \{a \in A : Ch_{M_i}(a) \le d(a)\}.$$

Since all parameters in the definition of b'_λ are in M, we have $b'_\lambda \in M$. Properties 1 and 2 above imply that

$$B_\lambda \subseteq_{J_{<\lambda}} \{a \in A : Ch_{M_i}(a) < d(a)\} \subseteq b'_\lambda \subseteq b_\lambda.$$

Thus, $b'_\lambda \subseteq b_\lambda$ is also a $B_\lambda[A]$ set. $\square$

[2] In the next section we will apply this corollary to sequences which are not neccessarily universal and minimally obedient, but satisfy (5.4), (5.5) and (5.6).

[3] This proof is taken from [1, page 53].

If we (first) fix a κ-presentable substructure $M \prec H_\theta$ with $A \in M$ (i.e. $M \in \mathcal{M}$), and then consider all cardinals $\lambda \in pcf(A) \cap M$, we get

Corollary 5.9. Suppose that A is a progressive set of regular cardinals, κ is a regular cardinal such that $|A| < \kappa < \min A$, and $M \prec H_\theta$ with $A \in M$ is a κ-presentable substructure. Suppose that M contains for every $\lambda \in pcf(A) \cap M$ a sequence $f^\lambda = \langle f^\lambda_\xi : \xi < \lambda \rangle$ that satisfies properties (5.4), (5.5) and (5.6). Then there are cardinals $\lambda_0 > \lambda_1 \cdots > \lambda_n$ in $pcf(A) \cap M$ such that

$$Ch_M \restriction A = \sup\{f^{\lambda_0}_{\gamma_0}, \ldots, f^{\lambda_n}_{\gamma_n}\}, \tag{5.7}$$

where $\gamma_i = Ch_M(\lambda_i)$.

Proof. By Corollary 5.8, for every $\lambda \in pcf(A) \cap M$ there is a $B_\lambda[A]$ set $b'_\lambda \in M$, such that

$$b'_\lambda \subseteq \{a \in A : Ch_M(a) = f^\lambda_{Ch_M(\lambda)}(a)\}. \tag{5.8}$$

We claim that there exist cardinals $\lambda_0 > \cdots > \lambda_n$ in $pcf(A) \cap M$ such that

$$A = b'_{\lambda_0} \cup \cdots \cup b'_{\lambda_n}, \tag{5.9}$$

i.e. the 'covering cardinals' can be found in M (compare with Theorem 4.24). To prove this, we inductively construct a descending sequence $\lambda_0 > \cdots > \lambda_i$ of cardinals in $pcf(A) \cap M$ as follows:

1. let $\lambda_0 = \max\, pcf(A)$,
2. if $A_{i+1} := A \backslash (b'_{\lambda_0} \cup \cdots \cup b'_{\lambda_i}) \neq \emptyset$, then let $\lambda_{i+1} = \max\, pcf(A_{i+1})$.

Since $b'_{\lambda_0}, \ldots, b'_{\lambda_i} \in M$, we have $A_{i+1} \in M$ ($A_{i+1} \neq \emptyset$), and hence $\lambda_{i+1} \in M$. Obviously, $\lambda_i \geq \lambda_{i+1}$, because $A_i \supseteq A_{i+1}$. But $\lambda_i = \lambda_{i+1}$ is impossible, since $b'_{\lambda_i} \cap A_{i+1} = \emptyset$ (and thus $\lambda_i \notin pcf(A_{i+1})$). Hence $\lambda_i > \lambda_{i+1}$. It follows that the sequence terminates. That is, for some i, $A_{i+1} = \emptyset$. Then $A = b'_{\lambda_0} \cup \cdots \cup b'_{\lambda_i}$.

By (5.5), $f^\lambda_{Ch_M(\lambda)} \leq Ch_M$ holds for every $\lambda \in pcf(A) \cap M$. Therefore, (5.8) and (5.9) imply that (5.7) holds. □

It follows from Corollary 5.9 that $|\{Ch_M \restriction A : M \in \mathcal{M}\}| \leq \max\, pcf(A)$. Namely, there are only $\max\, pcf(A)$ many sequences $f^{\lambda_0}_{Ch_M(\lambda_0)}, \ldots, f^{\lambda_n}_{Ch_M(\lambda_n)}$, where $M \in \mathcal{M}$ and $\lambda_0, \ldots, \lambda_n \in pcf(A) \cap M$. Thus, we have completed the proof of Theorem 5.1.

Remark 5.10. Since (5.1) holds and its proof is quite short, one could ask: why is it not simply incorporated in Theorem 5.1, such that (5.2) becomes $|\mu^\kappa| = \max\, pcf(A) \cdot 2^\kappa$. The answer is - we want to stress the importance of studying cofinalities; according to Shelah, this approach is *the* key to new results. Another reason for working with cofinalities is the fact that cofinalities are more immune to forcing methods then cardinalities.

Remark 5.11. (1)If we additionally assume (in Theorem 5.1) that $2^{|A|} \leq \kappa$, then we have $pcf(A) \leq 2^{|A|} \leq \kappa$, and hence $pcf(A) \subseteq M$, since $\kappa + 1 \subseteq M$. In this case, the proof of Corollary 5.9 is very short, bacause we can apply Theorem 4.24. Actually, Theorem 5.1, as it is, follows easily from this special case (where $2^{|A|} \leq \kappa$); for a proof see [2, 5.4].

We have the following straightforward generalization of Theorem 5.2.

Theorem 5.12. *Suppose that $\aleph_\delta$ is a singular cardinal such that $\delta < \aleph_\delta$. Then*

$$\aleph_\delta^{|\delta|} < \aleph_{(2^{|\delta|})^+}. \tag{5.10}$$

Proof. Consider the progressive interval A of regular cardinals in $(|\delta|^+, \aleph_\delta)$. By Theorem 4.10, $pcf(A)$ is also an interval of regular cardinals, containing all regular cardinals from $|\delta|^{++}$ to $\max pcf(A)$, and by Corollary 4.8,

$$|pcf(A)| \leq |P(A)| \leq 2^{|\delta|}.$$

It follows that

$$\max pcf(A) < \aleph_{(2^{|\delta|})^+}.$$

Therefore, applying (5.1), (5.3) and Theorem 5.1 to $|\delta|^+, \aleph_\delta$, we get

$$|[\aleph_\delta]^{|\delta|}| = cf([\aleph_\delta]^{|\delta|}, \subseteq)\cdot 2^{|\delta|} \leq cf([\aleph_\delta]^{|\delta|^+}, \subseteq)\cdot 2^{|\delta|} = \max pcf(A)\cdot 2^{|\delta|} < \aleph_{(2^{|\delta|})^+}.$$

$\square$

We now mention a tighter bound then (5.10). Let μ and $\tau \leq \mu$ be cardinals. A *cover* for $[\mu]^{<\tau}$ is a collection $\mathcal{C}$ of subsets of μ such that for every $X \in [\mu]^{<\tau}$ there exists $Y \in \mathcal{C}$ with $X \subseteq Y$. If θ is a cardinal such that $\mu \geq \theta \geq \tau$, then $cov(\mu, \theta, \tau)$ denotes the least cardinality of a cover for $[\mu]^{<\tau}$ consisting of sets taken from $[\mu]^{<\theta}$. [Note that $cf([\mu]^\kappa, \subseteq) = cov(\mu, \kappa^+, \kappa^+)$.]

Theorem 5.13. *Suppose that μ is a singular cardinal, and $\kappa < \mu$ is a regular cardinal such that the interval A of regular cardinals in (κ^+, μ) has size $\leq \kappa$. Then*

$$cov(\mu, \kappa^+, cf(\mu)^+) = sup\ pcf_{cf(\mu)}(A).$$ [4]

Corollary 5.14. Suppose that δ is a limit ordinal such that $\delta < \aleph_\delta$. Then

$$\aleph_\delta^{cf(\delta)} < \aleph_{(|\delta|^{cf(\delta)})^+}.$$

For proofs see [1, page 57]. Here we only prove the following.

[4] See Definition 4.11 for $pcf_{cf(\mu)}(A)$.

Theorem 5.15. *Suppose that μ is a singular cardinal, and $\kappa < \mu$ is a regular cardinal such that the interval A of regular cardinals in (κ^+, μ) has size $\leq \kappa$. Then*

$$cov(\mu, \kappa^+, \aleph_1) = sup\ pcf_{\aleph_0}(A). \tag{5.11}$$

Proof. By induction on μ, i.e. for a fixed regular cardinal κ we show by induction on μ that whenever $\mu > \kappa$ is a singular cardinal such that the interval A of regular cardinals in (κ^+, μ) has size $\leq \kappa$, then (5.11) holds.

Let $\kappa = \aleph_\alpha$. If $\mu = \aleph_{\alpha+\omega}$, then $cf(\mu) = \aleph_0$, and hence (5.11) follows from Theorem 5.13.

Assume now that $\mu > \aleph_{\alpha+\omega}$ and $cov(\nu, \kappa^+, \aleph_1) = \sup\ pcf_{\aleph_0}(A)$ holds for every singular cardinal ν such that $\aleph_{\alpha+\omega} < \nu < \mu$.

If $cf(\mu) = \aleph_0$, then we can use Theorem 5.13 again. So assume that $cf(\mu) > \aleph_0$. It follows that there is a cofinal subset $\{\nu_i > \kappa^{++} : i < cf(\mu)\}$ of μ consisting of singular cardinals.

Let A_i be the interval of regular cardinals in (κ^+, ν_i). Then, by induction hypothesis, $cov(\nu_i, \kappa^+, \aleph_1) = \sup\ pcf_{\aleph_0}(A_i)$, for every $i < cf(\mu)$. Hence,

$$cov(\mu, \kappa^+, \aleph_1) = \sup_{i<cf(\mu)}\big(cov(\nu_i, \kappa^+, \aleph_1)\big) = \sup_{i<cf(\mu)}\big(\sup\ pcf_{\aleph_0}(A_i)\big).$$

We complete the proof by showing that

$$\sup_{i<cf(\mu)}\big(\sup\ pcf_{\aleph_0}(A_i)\big) = \sup\ pcf_{\aleph_0}(A).$$

The $\leq$ inequality is obvious, since $\sup\ pcf_{\aleph_0}(A_i) \leq \sup\ pcf_{\aleph_0}(A)$, for every $i < cf(\mu)$. Conversely, if $\lambda \in pcf_{\aleph_0}(A)$, then for some $i < cf(\mu)$, $\lambda \in pcf_{\aleph_0}(A_i)$, because $cf(\mu) > \aleph_0$. Thus, $\sup_{i<cf(\mu)}\big(\sup\ pcf_{\aleph_0}(A_i)\big) \geq \sup\ pcf_{\aleph_0}(A)$. $\square$

Corollary 5.16. Suppose that δ is a limit ordinal such that $\delta < \aleph_\delta$. Then

$$\aleph_\delta^{\aleph_0} < \aleph_{(|\delta|^{\aleph_0})^+}.$$

Proof. Consider the interval A of regular cardinals in $(|\delta|^+, \aleph_\delta)$. By Corollary 4.8, we have

$$|pcf_{\aleph_0}(A)| \leq |[A]^{\aleph_0}| \cdot 2^{\aleph_0} \leq |\delta|^{\aleph_0}. \tag{5.12}$$

Since $pcf_{\aleph_0}(A)$ is also an interval of regular cardinals (see Theorem 4.12), (5.12) means that

$$|\big(|\delta|^+, \sup\ pcf_{\aleph_0}(A)\big) \cap Reg| \leq |\delta|^{\aleph_0}. \tag{5.13}$$

It follows now from (5.13) that

$$\sup\ pcf_{\aleph_0}(A) < \aleph_{(|\delta|^{\aleph_0})^+},$$

because
$$|\big(|\delta|^+, \aleph_{(|\delta|^{\aleph_0})^+}\big) \cap Reg| = (|\delta|^{\aleph_0})^+.$$
By Theorem 5.15, $cov(\aleph_\delta, |\delta|^+, \aleph_1) = \sup pcf_{\aleph_0}(A)$, therefore, we have
$$cov(\aleph_\delta, |\delta|^+, \aleph_1) < \aleph_{(|\delta|^{\aleph_0})^+},$$
which implies
$$\aleph_\delta^{\aleph_0} = |[\aleph_\delta]^{\aleph_0}| \leq |\delta|^{\aleph_0} \cdot cov(\aleph_\delta, |\delta|^+, \aleph_1) < \aleph_{(|\delta|^{\aleph_0})^+}.$$
□

Corollary 5.17. Suppose that δ is a cardinal such that for every cardinal $\mu < \delta$, $\mu^{\aleph_0} < \delta$. Then $\aleph_\delta$ has the same property, namely, for every $\mu < \aleph_\delta$, $\mu^{\aleph_0} < \aleph_\delta$.

Proof. By induction on μ. If $\mu < \delta$, then, by assumption, $\mu^{\aleph_0} < \delta$, and hence $\mu^{\aleph_0} < \aleph_\delta$.

Assume now that $\delta \leq \mu < \aleph_\delta$, and for every cardinal $\gamma < \mu$, $\gamma^{\aleph_0} < \aleph_\delta$. If μ is a successor cardinal, i.e. $\mu = \aleph_{\alpha+1}$, for some ordinal α, then, by induction hypothesis (and Proposition 2.7),
$$\aleph_{\alpha+1}^{\aleph_0} = \aleph_\alpha^{\aleph_0} \cdot \aleph_{\alpha+1} < \aleph_\delta.$$
If $\mu = \aleph_\alpha$ is a limit cardinal, then $\alpha < \aleph_\alpha$, and thus, by the previous corollary,
$$\aleph_\alpha^{\aleph_0} < \aleph_{(|\alpha|^{\aleph_0})^+} \leq \aleph_\delta.$$
The last inequality holds because $\alpha < \delta$, and thus $(|\alpha|^{\aleph_0})^+ \leq \delta$. □

5.2 Improving the upper bound on $|pcf(A)|$

As we mentioned in Remark 5.3, one can show that $|pcf(A)| < |A|^{+4}$, for a progressive set A of regular uncountable cardinals. It follows then easily by the proof of Theorem 5.12 that for limit ordinals δ with $|\delta|^{cf(\delta)} < \aleph_\delta$, we have $\aleph_\delta^{cf(\delta)} < \aleph_{|\delta|^{+4}}$ (see Theorem 5.22). In partiular, $\aleph_\omega^{\aleph_0} \leq 2^{\aleph_0} + \aleph_{\omega_4}$.

We give an outline of the proof of $|pcf(A)| < |A|^{+4}$. Recall first some facts from the previous section. We showed that there is a κ-presentable elementary substructure $M \prec H_\theta$ ($|A| < \kappa < \min A$), and proved the existence of minimally obedient universal sequences f^λ for $\lambda \in pcf(A)$. In Lemma 5.7 we proved that for each $\lambda \in pcf(A) \cap M$, f^λ satisfies conditions (5.4), (5.5) and (5.6). It followed then by Corollary 5.8 that there exists a special generating sequence $\langle b_\lambda : \lambda \in pcf(A) \cap M\rangle$. [5]

Modifying the minimally obedient universal sequences f^λ, we define the *elevated* sequences of functions in $\prod A$, which also satisfy conditions (5.4), (5.5) and (5.6), but moreover, the corresponding generating sequence $\langle b_\lambda : \lambda \in pcf(A) \cap M\rangle$ is *transitive* (definitions follow).

Definition 5.18. A generating sequence $\langle B_\lambda : \lambda \in pcf(A) \cap M\rangle$ is said to be *transitive* (or *smooth*) if for every $\lambda \in pcf(A) \cap M$, $\theta \in B_\lambda$ implies $B_\theta \subseteq B_\lambda$.

For $\lambda_0 \in pcf(A)$, we define the *elevated* sequence $F^{\lambda_0} = \langle F^{\lambda_0}_\gamma : \gamma < \lambda_0\rangle$ of functions in $\prod A$ as follows.

For every sequence $\lambda_1, \dots, \lambda_n \in A$, such that $\lambda_0 > \lambda_1 > \dots > \lambda_n$, and for every ordinal $\gamma_0 \in \lambda_0$, inductively define a sequence $\gamma_1 \in \lambda_1, \dots, \gamma_n \in \lambda_n$ by

$$\gamma_{i+1} := f^{\lambda_i}_{\gamma_i}(\lambda_{i+1}). \tag{5.14}$$

So $\gamma_1 = f^{\lambda_0}_{\gamma_0}(\lambda_1), \gamma_2 = f^{\lambda_1}_{\gamma_1}(\lambda_2), \dots, \gamma_n = f^{\lambda_{n-1}}_{\gamma_{n-1}}(\lambda_n)$. The *elevation* function $El_{\lambda_0,\dots,\lambda_n}$ on λ_0 is given by

$$El_{\lambda_0,\dots,\lambda_n}(\gamma_0) := \gamma_n.$$

We first define F^{λ_0} on $A \cap \lambda_0$. Given $\lambda \in A \cap \lambda_0$, let $F_{\lambda_0,\lambda}$ be the set of all sequences $\lambda_1, \lambda_2, \dots, \lambda_n \in A$, such that $\lambda_0 > \lambda_1 > \dots > \lambda_n = \lambda$. For every $\gamma_0 \in \lambda_0$ we ask whether there is a maximal value in

$$\{El_{\lambda_0,\dots,\lambda_n}(\gamma_0) : \langle \lambda_0, \lambda_1, \dots, \lambda_n\rangle \in F_{\lambda_0,\lambda}\}.$$

If this set contains a maximum, let $F^{\lambda_0}_{\gamma_0}(\lambda)$ be that maximum, and otherwise let $F^{\lambda_0}_{\gamma_0}(\lambda) := f^{\lambda_0}_{\gamma_0}(\lambda)$. For $\lambda \in A \backslash \lambda_0$, let $F^{\lambda_0}_{\gamma_0}(\lambda) := \gamma_0$, for each $\gamma_0 \in \lambda_0$.

Lemma 5.19. *For each $\lambda \in pcf(A) \cap M$, the elevated sequence F^λ satisfies conditions (5.4), (5.5) and (5.6), and the generating sequence $\langle b_\lambda : \lambda \in pcf(A) \cap M\rangle$, where $b_\lambda := \{a \in A : Ch_M(a) = F^\lambda_{Ch_M(\lambda)}(a)\}$, is transitive.*

For a proof see [1, pages 61,62].

The transitive generators can be used to prove the following *localization* property of A.

Theorem 5.20. *If $B \subseteq pcf(A)$ is progressive, then $pcf(B) = pcf_{|A|}(B)$. That is, if $B \subseteq pcf(A)$ is progressive, then for every $\lambda \in pcf(B)$ there exists $B_0 \subseteq B$ with $|B_0| \le |A|$, such that $\lambda \in pcf(B_0)$.*

[5] In this section we assume in addition that M also contains the array $\langle f^\lambda : \lambda \in pcf(A)\rangle$.

For a proof see [1, Theorem 6.6].

The simplest case of localization (in which $|B| = |A|^+$) implies that $|pcf(A)| < |A|^{+4}$:

Theorem 5.21. *Suppose that A is a progressive interval of regular cardinals. Then*

$$|pcf(A)| < |A|^{+4}.$$

For a proof see [1, Theorem 7.1].

Theorem 5.22. *Suppose that δ is a limit ordinal such that $|\delta|^{cf(\delta)} < \aleph_\delta$. Then*

$$\aleph_\delta^{cf(\delta)} < \aleph_{|\delta|^{+4}}.$$

Proof. Consider the progressive interval A of regular cardinals in $(|\delta|^+, \aleph_\delta)$. We basically repeat the proof of Theorem 5.12. Since $pcf(A)$ is also an interval of regular cardinals, containing all regular cardinals from $|\delta|^{++}$ to $\max pcf(A)$, and since, by Theorem 5.21,

$$|pcf(A)| < |A|^{+4} = |\delta|^{+4},$$

it follows that

$$\max pcf(A) < \aleph_{|\delta|^{+4}}.$$

Therefore,

$$|[\aleph_\delta]^{cf(\delta)}| = cf([\aleph_\delta]^{|\delta|}, \subseteq) \cdot |\delta|^{cf(\delta)} = \left(cf([\aleph_\delta]^{|\delta|^+}, \subseteq) \cdot |\delta|^+\right) \cdot |\delta|^{cf(\delta)} =$$

$$cf([\aleph_\delta]^{|\delta|^+}, \subseteq) \cdot |\delta|^{cf(\delta)} \leq \max pcf(A) \cdot \aleph_\delta < \aleph_{|\delta|^{+4}}.$$

□

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