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# THE TREE PROPERTY AT $\aleph_{\omega+2}$

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**Abstract.** Assuming the existence of a weakly compact hypermeasurable cardinal we prove that in some forcing extension $\aleph_\omega$ is a strong limit cardinal and $\aleph_{\omega+2}$ has the tree property. This improves a result of Matthew Foreman (see [2]).

**§1. Introduction.** For an infinite cardinal $\kappa$, a *$\kappa$-tree* is a tree $T$ of height $\kappa$ such that every level of $T$ has size less than $\kappa$. A tree $T$ is a *$\kappa$-Aronszajn tree* if $T$ is a $\kappa$-tree which has no cofinal branches. We say that *the tree property holds at $\kappa$*, or $\mathrm{TP}(\kappa)$ *holds*, if every $\kappa$-tree has a cofinal branch, i.e. a branch of length $\kappa$ through it. Thus, $\mathrm{TP}(\kappa)$ holds iff there is no $\kappa$-Aronszajn tree. $\mathrm{TP}(\aleph_0)$ holds in ZFC, and it is actually exactly the statement of the well-known König's lemma. Aronszajn showed also in ZFC that there is an $\aleph_1$-Aronszajn tree. Hence, $\mathrm{TP}(\aleph_1)$ fails in ZFC.

Large cardinals are needed once we consider trees of height greater than $\aleph_1$. Silver proved that for $\kappa > \aleph_1$ $\mathrm{TP}(\kappa)$ implies $\kappa$ is weakly compact in $L$. Mitchell proved that given a weakly compact cardinal $\lambda$ above a regular cardinal $\kappa$, one can make $\lambda$ into $\kappa^+$ so that in the extension, $\kappa^+$ has the tree property. Thus, $\mathrm{TP}(\aleph_2)$ is equiconsistent with the existence of a weakly compact cardinal.

For more of the relevant literature on the tree property we refer the reader to the following: Abraham [1], Cummings and Foreman [2], and Foreman, Magidor and Schindler [4] have done work on the tree property at two or more successive cardinals; Magidor and Shelah [7] have worked on the tree property at successors of singular cardinals.

Natasha Dobrinen and Sy-D. Friedman [3] used a generalization of Sacks forcing to reduce the large cardinal strength required to obtain the tree property at the double successor of a measurable cardinal from a supercompact to a weakly compact hypermeasurable cardinal (see Definition 3).

In this paper we extend the method of [3] to obtain improved upper bounds on the consistency strength of the tree property at the double successor of singular cardinals.

**§2. The tree property at $\kappa^{++}$.**

DEFINITION 1. Let $\rho$ be a strongly inaccessible cardinal. Then $\mathrm{Sacks}(\rho)$ denotes the following forcing notion. A condition $p$ is a subset of $2^{<\rho}$ such that:

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1. $s \in p$, $t \subseteq s \to t \in p$.
2. Each $s \in p$ has a proper extension in $p$.
3. For any $\alpha < \rho$, if $\langle s_\beta : \beta < \alpha \rangle$ is a sequence of elements of $p$ such that $\beta < \beta' < \alpha \to s_\beta \subseteq s_{\beta'}$, then $\bigcup \{s_\beta : \beta < \alpha\} \in p$.
4. Let $\mathrm{Split}(p)$ denote the set of $s \in p$ such that both $s^\frown 0$ and $s^\frown 1$ are in $p$. Then for some club denoted $C(p) \subseteq \rho$, $\mathrm{Split}(p) = \{s \in p : \mathrm{length}(s) \in C(p)\}$.

The conditions are ordered as follows: $q \leq p$ iff $q \subseteq p$, where $q \leq p$ means that $q$ is stronger than $p$.

Given $p \in \mathrm{Sacks}(\rho)$, let $\langle \gamma_\alpha : \alpha < \rho \rangle$ be the increasing enumeration of $C(p)$. For $\alpha < \rho$, the $\alpha$-th splitting level of $p$, $\mathrm{Split}_\alpha(p)$, is the set of $s \in p$ of length $\gamma_\alpha$. For $\alpha < \rho$ we write $q \leq_\alpha p$ iff $q \leq p$ and $\mathrm{Split}_\beta(q) = \mathrm{Split}_\beta(p)$ for all $\beta < \alpha$.

$\mathrm{Sacks}(\rho)$ satisfies the following *$\rho$-fusion* property: Every decraesing sequence $\langle p_\alpha : \alpha < \rho \rangle$ of elements in $\mathrm{Sacks}(\rho)$ such that for each $\alpha < \rho$, $p_{\alpha+1} \leq_\alpha p_\alpha$, has a lower bound, namely $\bigcap_{\alpha<\rho} p_\alpha \in \mathrm{Sacks}(\rho)$.

The forcing notion $\mathrm{Sacks}(\rho)$ is also $< \rho$ -closed, satisfies the $\rho^{++}$-c.c., and preserves $\rho^+$. For a proof see [6] or [3].

Definition 2. Let $\rho$ be a strongly inaccessible cardinal and let $\lambda > \rho$ be a regular cardinal. $\mathrm{Sacks}(\rho, \lambda)$ denotes the $\lambda$-length iteration of $\mathrm{Sacks}(\rho)$ with supports of size $\leq \rho$.

$\mathrm{Sacks}(\rho, \lambda)$ satisfies the *generalized $\rho$-fusion* property which we describe next: For $\alpha < \rho$, $X \subseteq \rho$ of size less than $\rho$, and $p, q \in \mathrm{Sacks}(\rho, \lambda)$, we write $q \leq_{\alpha, X} p$ iff $q \leq p$ (i.e. $q \restriction i \Vdash q(i) \leq p(i)$ for each $i < \lambda$) and in addition, for each $i \in X$, $q \restriction i \Vdash q(i) \leq_\alpha p(i)$. Every decraesing sequence $\langle p_\alpha : \alpha < \rho \rangle$ of elements in $\mathrm{Sacks}(\rho, \lambda)$ such that for each $\alpha < \rho$, $p_{\alpha+1} \leq_{\alpha, X_\alpha} p_\alpha$, where the $X_\alpha$'s form an increasing sequence of subsets of $\lambda$ each of size less than $\rho$ whose union is the union of the supports of the $p_\alpha$'s, has a lower bound. [The lower bound is $q$ where $q(0) = \bigcap_{\alpha<\rho} p_\alpha(0)$, $q(1)$ is a name s.t. $q(0) \Vdash q(1) = \bigcap_{\alpha<\rho} p_\alpha(1)$, etc.]

Assuming $2^\rho = \rho^+$, $\mathrm{Sacks}(\rho, \lambda)$ is $< \rho$ -closed, satisfies the $\lambda$-c.c., preserves $\rho^+$, collapses $\lambda$ to $\rho^{++}$ and blows up $2^\rho$ to $\rho^{++}$. For a proof see [6] or [3].

Definition 3. We say that $\kappa$ is *weakly compact hypermeasurable* if there is weakly compact cardinal $\lambda > \kappa$ and an elementary embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ such that $H(\lambda)^V = H(\lambda)^M$.

Let $\kappa$ be a weakly compact hypermeasurable cardinal. Define a forcing notion $P$ as follows. Let $\rho_0$ be the first inaccessible cardinal and let $\lambda_0$ be the least weakly compact cardinal above $\rho_0$. For $k < \kappa$, given $\lambda_k$, let $\rho_{k+1}$ be the least inaccessible cardinal above $\lambda_k$ and let $\lambda_{k+1}$ be the least weakly compact cardinal above $\rho_{k+1}$. For limit ordinals $k < \kappa$, let $\rho_k$ be the least inaccessible cardinal greater than or equal to $\sup_{l<k} \lambda_l$ and let $\lambda_k$ be the least weakly compact cardinal above $\rho_k$. Note that $\rho_\kappa = \kappa$ and $\lambda_\kappa$ is the least weakly compact cardinal above $\kappa$.

Let $P_0 = \{1_0\}$. For $i < \kappa$, if $i = \rho_k$ for some $k < \kappa$, let $\dot{Q}_i$ be a $P_i$-name for the direct sum $\bigoplus_{\eta \leq \lambda_k} \mathrm{Sacks}(\rho_k, \eta) := \{\langle \mathrm{Sacks}(\rho_k, \eta), p \rangle : \eta \text{ is an inaccessible } \leq \lambda_k \text{ and } p \in \mathrm{Sacks}(\rho_k, \eta)\}$, where $\langle \mathrm{Sacks}(\rho_k, \eta), p \rangle \leq \langle \mathrm{Sacks}(\rho_k, \eta'), p' \rangle$ iff $\eta = \eta'$ and $p \leq_{\mathrm{Sacks}(\rho_k, \eta)} p'$. Otherwise let $\dot{Q}_i$ be a $P_i$-name for the trivial forcing. Let

$P_{i+1} = P_i * \dot{Q}_i$. Let $P_\kappa$ be the iteration $\langle \langle P_i, \dot{Q}_i \rangle : i < \kappa \rangle$ with reverse Easton support.

THEOREM 1 (N. Dobrinen, S. Friedman). *Assume that $V$ is a model of ZFC in which GCH holds and $\kappa$ is a weakly compact hypermeasurable cardinal in $V$. Let $\lambda > \kappa$ be a weakly compact cardinal and let $j : V \to M$ be an elementary embedding with $crit(j) = \kappa$, $j(\kappa) > \lambda$ and $H(\lambda)^V = H(\lambda)^M$, witnessing the weakly compact hypermeasurability of $\kappa$. Let $G * g$ be a generic subset of $P = P_\kappa * \dot{Sacks}(\kappa, \lambda)$ over $V$. Then in $V[G][g]$, $2^\kappa = \kappa^{++}$, $\kappa^{++}$ has the tree property, and $\kappa$ is still measurable, i.e. the embedding $j : V \to M$ can be lifted to an elementary embedding $j : V[G][g] \to M[G][g][H][h]$, where $G * g * H * h$ is a generic subset of $j(P)$ over $M$.*

For a proof see [3].

## §3. The tree property at the double successor of a singular cardinal.

THEOREM 2. *Assume that $V$ is a model of ZFC and $\kappa$ is a weakly compact hypermeasurable cardinal in $V$. Then there exists a forcing extension of $V$ in which $cof(\kappa) = \omega$ and $\kappa^{++}$ has the tree property.*

PROOF. Let $\lambda > \kappa$ be a weakly compact cardinal and let $j : V \to M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $H(\lambda)^V = H(\lambda)^M$. We may assume that $M$ is of the form $M = \{j(f)(\alpha) : \alpha < \lambda, f : \kappa \to V, f \in V\}$. First force as in Theorem 1 with $P = P_\kappa * \dot{\mathrm{Sacks}}(\kappa, \lambda)$ over $V$ to get a model $V[G][g]$ in which $2^\kappa = \kappa^{++}$, $\kappa^{++}$ has the tree property, and $\kappa$ is still measurable, i.e. there is an elementary embedding $j : V[G][g] \to M[G][g][H][h]$, where $G * g * H * h$ is a generic subset of $j(P)$ over $M$.

Now force with the usual Prikry forcing which we will denote by $R := \{(s, A) : s \in [\kappa]^{<\omega}, A \in U\}$, where $U$ is the normal measure on $\kappa$ derived from $j$. We say that $s$ is *the lower part* of $(s, A)$. A condition $(t, B)$ is stronger than a condition $(s, A)$ iff $s$ is an initial segment of $t$, $B \subseteq A$, and $t - s \subset A$. The Prikry forcing preserves cardinals and introduces an $\omega$-sequence of ordinals which is cofinal in $\kappa$. It remains to show that it also preserves the tree property on $\kappa^{++} = \lambda$.

In order to get a contradiction suppose that there is a $\kappa^{++}$-Aronszajn tree in some $R$-extension of $V[G][g]$. Then in $V[G]$ there is a $\mathrm{Sacks}(\kappa, \lambda) * \dot{R}$ - name $\dot{T}$ of size $\lambda$ (because $\mathrm{Sacks}(\kappa, \lambda) * \dot{R}$ satisfies $\lambda$-c.c.) and a condition $(p, \dot{r}) \in \mathrm{Sacks}(\kappa, \lambda) * \dot{R}$ which forces $\dot{T}$ to be a $\kappa^{++}$-Aronszajn tree. Recall that $\lambda$ is a weakly compact cardinal in $V[G]$. Therefore, there exist in $V[G]$ transitive $ZF^-$-models $N_0, N_1$ of size $\lambda$ and an elementary embedding $k : N_0 \to N_1$ with critical point $\lambda$, such that $N_0 \supseteq H(\lambda)^{V[G]}$ and $G, \dot{T} \in N_0$.

Since $g$ is also $\mathrm{Sacks}(\kappa, \lambda)$-generic over $N_0$ and the critical point of $k$ is $\lambda$, $k$ can be lifted to $k^* : N_0[g] \to N_1[g][K]$, where $K$ is any $N_1[g]$-generic subset of $\mathrm{Sacks}(\kappa, [\lambda, k(\lambda)))$ in some larger universe (and where $\mathrm{Sacks}(\kappa, [\lambda, k(\lambda)))$ is the quotient $\mathrm{Sacks}(\kappa, k(\lambda))/\mathrm{Sacks}(\kappa, \lambda)$, i.e. the iteration of $\mathrm{Sacks}(\kappa)$ indexed by ordinals between $\lambda$ and $k(\lambda)$). Consider the forcing $R^* := k^*(\dot{R}^g)$ in $N_1[g][K]$ and choose any generic $C^*$ for it such that $k^*(r) \in C^*$, where $r = \dot{r}^g$. Let $C := (k^*)^{-1}[C^*]$ be the pullback of $C^*$ under $k^*$. Then $C$ is an $N_0[g]$-generic subset of $R$, because if $\Delta \in N_0[g]$ is a maximal antichain of $R$ then $k^*(\Delta) = k^*[\Delta]$ (since $\mathrm{crit}(k) = \lambda$ and $R$ has the $\kappa^+$-c.c.) and by elementarity $k^*(\Delta)$ is maximal in $k^*(R) = R^*$, so $k^*[\Delta]$

meets $C^*$ and hence $\Delta$ meets $C$. It follows that there is an elementary embedding $k^{**} : N_0[g][C] \to N_1[g][K][C^*]$ extending $k^*$.

We have $r \in C$. So it follows that the evaluation $T$ of $\dot{T}$ in $N_0[g][C]$ is a $\lambda$-Aronszajn tree. By elementarity $k^{**}(T)$ is a $k^{**}(\lambda)$-Aronszajn tree in $N_1[g][K][C^*]$ which coincides with $T$ up to level $\lambda$. Hence $T$ has a cofinal branch $b$ in $N_1[g][K][C^*]$. We will show that $b$ has to belong to $N_1[g][C]$ (i.e. the quotient $Q$ of the natural projection $\pi : \mathrm{Sacks}(\kappa, k(\lambda)) * \dot{R}^* \to RO(\mathrm{Sacks}(\kappa, \lambda) * \dot{R})$ can not add a new branch), and thereby reach the desired contradiction!

Let us first analyse the quotient $Q$ of the projection above. In $N_1[g][C]$ we have $Q = \{(p^*, (s^*, \dot{A}^*)) \in \mathrm{Sacks}(\kappa, k(\lambda)) * \dot{R}^* \mid \text{for all } (p, (s, \dot{A})) \in g * C,\ (p, (s, \dot{A}))$ does not force that $(p^*, (s^*, \dot{A}^*))$ is not a condition in the quotient$\}$. Observe that $(p, (s, \dot{A}))$ forces that $(p^*, (s^*, \dot{A}^*))$ is not a condition in $Q$ iff the two conditions are incompatible, which is the case iff one of the following holds:

1. $p^* \restriction \lambda$ is incompatible with $p$.
2. $s^* \subsetneq s$ and $s \subsetneq s^*$.
3. $p^* \restriction \lambda$ is compatible with $p$, $s^* \subseteq s$, and $p^* \cup p$ forces that $s - s^* \subsetneq \dot{A}^*$.
4. $p^* \restriction \lambda$ is compatible with $p$, $s \subseteq s^*$, and $p^* \restriction \lambda \cup p$ forces that $s^* - s \subsetneq \dot{A}$.

It follows that $Q = \{(p^*, (s^*, \dot{A}^*)) \in \mathrm{Sacks}(\kappa, k(\lambda)) * \dot{R}^* \mid (p^*, (s^*, \dot{A}^*))$ is compatible with all $(p, (s, \dot{A})) \in g * C\}$, i.e. $Q$ is the set of all $(p^*, (s^*, \dot{A}^*)) \in \mathrm{Sacks}(\kappa, k(\lambda)) * \dot{R}^*$ such that for all $(p, (s, \dot{A})) \in g * C$ either

1. $p^* \restriction \lambda$ is compatible with $p$, $s^* \subseteq s$, and $p^* \cup p$ does not force that $s - s^* \subsetneq \dot{A}^*$, or
2. $p^* \restriction \lambda$ is compatible with $p$, $s \subseteq s^*$, and $p^* \restriction \lambda \cup p$ does not force that $s^* - s \subsetneq \dot{A}\}$.

Equivalently, $Q$ is the set of all $(p^*, (s^*, \dot{A}^*)) \in \mathrm{Sacks}(\kappa, [\lambda, k(\lambda))) * \dot{R}^*$ such that

1. $p^* \in \mathrm{Sacks}(\kappa, [\lambda, k(\lambda)))$,
2. $s^*$ is an initial segment of $S(C)$ (the Prikry $\omega$-sequence arising from $C$)
3. $p^*$ forces that $\dot{A}^*$ is in $\dot{U}^*$, and
4. for any finite subset $x$ of $S(C)$, some extension $q$ of $p^*$ forces $x$ to be a subset of $s^* \cup \dot{A}^*$.

We now again argue indirectly. Assume that $b$ is not in $N_1[g][C]$, and let $\dot{b}$ in $N_1[g]$ be an $R * \dot{Q}$ - name for $b$. Identify $k(\dot{T})$ with the $R * \dot{Q}$ - name defined by interpreting the $\mathrm{Sacks}(\kappa, k(\lambda)) * \dot{R}^*$ - name $k(\dot{T})$ in $N_1$ as an $R * \dot{Q}$ - name in $N_1[g]$. Let $((s_0, A_0), (p_0, (t_0, \dot{A}_0)))$ be an $R * \dot{Q}$ - condition forcing that the Prikry-name $\dot{T}$ is a $\lambda$-tree and that $\dot{b}$ is a branch through $\dot{T}$ not belonging to $N_1[g][\dot{C}]$.

Let us take a closer look at the condition $((s_0, A_0), (p_0, (t_0, \dot{A}_0)))$. Note that the forcing $Q$ lives in $N_1[g][C]$, but its elements are in $N_1[g]$, so we can assume that $(p_0, (t_0, \dot{A}_0))$ is a real object and not just a Prikry-name. The Prikry condition $(s_0, A_0)$ forces that $p_0$ is an element of $\mathrm{Sacks}(\kappa, [\lambda, k(\lambda)))$, that $t_0$ is an initial segment of $S(\dot{C})$, and that for all finite subsets $x$ of $S(\dot{C})$, some extension of $p_0$ forces $x$ to be a subset of $t_0 \cup \dot{A}_0$. This simply means that $t_0$ is an initial segment of $s_0$ and for every finite subset $x$ of $s_0 \cup A_0$, some extension of $p_0$ forces $x$ to be a subset of $t_0 \cup \dot{A}_0$.

Moreover, we can assume that $s_0$ equals $t_0$. Namely, from the next claim follows that the set of conditions of the form $((s, A), (p, (s, \dot{A})))$ is dense in $R * \dot{Q}$.

CLAIM. Suppose that $p$ is an element of $\mathrm{Sacks}(\kappa, [\lambda, k(\lambda)))$ which forces that $\dot{A}$ is in $\dot{U}^*$. Then there is $A(p) \in U$ such that whenever $x$ is a finite subset of $A(p)$, there is $q \leq p$ forcing $x$ to be contained in $\dot{A}$.

PROOF OF THE CLAIM. Define the function $f : [\kappa]^{<\omega} \to 2$ by

$$f(x) = \begin{cases} 1 & \text{if } \exists q \leq p \; q \Vdash x \subseteq \dot{A} \\ 0 & \text{otherwise.} \end{cases}$$

By normality $f$ has a homogeneous set $A(p) \in U$. It follows that for each $n \in \omega$, $f \restriction [A(p)]^n$ has the constant value 1: Assume on the contrary that there is some $n \in \omega$ such that $f \restriction [A(p)]^n$ has the constant value 0. Then $p \Vdash x \not\subseteq \dot{A}$ for every $x \in [A(p)]^n$, but this is in contradiction with the facts that the measure $U^*$ extends $U$, $p \Vdash \dot{A} \in U^*$, and $A(p) \in U$.

It now follows easily that the set of conditions of the form $((s, A), (p, (s, \dot{A})))$ is dense in $R * \dot{Q}$. Assume that $((s, A), (p, (t, \dot{A})))$ is an arbitrary condition in $R * \dot{Q}$. We have $t \subseteq s$. There is some $q \leq p$ which forces that $x := s - t$ is contained in $\dot{A}$. Now by shrinking $A$ to $A(q)$ we get that $((s, A(q)), (q, (s, \dot{A})))$ is a condition which is below $((s, A), (p, (t, \dot{A})))$. We will from now on work with this dense subset of $R * \dot{Q}$.

Now in $N_1[g]$ build a $\kappa$-tree $E$ of conditions in $\mathrm{Sacks}(\kappa, [\lambda, k(\lambda)))$, whose branches will be fusion sequences, together with a sequence of ordinals $\langle \lambda_\beta : \beta < \kappa \rangle$, each $\lambda_\beta < \lambda$, as follows:

Consider an enumeration $\langle s_\beta : \beta < \kappa \rangle$ of all possible lower parts of conditions in $R$, i.e. all finite increasing sequences of ordinals less than $\kappa$, in which every lower part appears cofinally often. Start building the tree $E$ below the condition $p_0$ ($p_0$ was chosen such that $((s_0, A_0), (p_0, (s_0, \dot{A}_0)))$ forces $\dot{b}$ to be a bad branch). Assume that the tree $E$ is built up to level $\beta$. Then, at stage $\beta$ of the construction of the tree, at each node $v$ (a condition in $\mathrm{Sacks}(\kappa, [\lambda, k(\lambda)))$), is associated an $X_v \subset [\lambda, k(\lambda)), |X_v| < \kappa$; we will find stronger (incompatible) conditions $v_0$ and $v_1$ which on all indices in $X_v$ equal $v$ below level $\beta$ (for purposes of fusion), i.e. $v_0, v_1 \leq_{\beta, X_v} v$. (The sets $X_v$ can be chosen in different ways, the only condition they have to satisfy is that at the end of the construction of the tree $E$ for every branch through the tree the union of the supports of the conditions (nodes) on the branch is equal to the union of the corresponding $X$'s.) Before we start the construction of the level $\beta + 1$ of the tree $E$ we need to set some notation. Given $i \in [\lambda, k(\lambda))$, let $S_i$ denote $\mathrm{Sacks}(\kappa, [\lambda, i))$. For a node $v$ on level $\beta$, let $\delta_v = o.t.(X_v)$ and $d_v = |{}^{\delta_v}({}^{\beta+1}2)|$. Let $\langle i^v_\varepsilon : \varepsilon < \delta_v \rangle$ be the strictly increasing enumeration of $X_v$ and let $i_{\delta_v} = \sup\{i^v_\varepsilon : \varepsilon < \delta_v\}$. For each $\varepsilon < \delta_v$ there are $S_{i^v_\varepsilon}$- names $\dot{s}^v_{\varepsilon,\zeta}$ ($\zeta \in {}^{\beta+1}2$) such that $S_{i^v_\varepsilon} \Vdash (\dot{s}^v_{\varepsilon,\zeta}$ is the $\zeta$-th node of $\mathrm{Split}_{\beta+1}(v(i^v_\varepsilon)))$, where the nodes of $\mathrm{Split}_{\beta+1}(v(i^v_\varepsilon)))$ are ordered canonically lexicographically (by choosing an $S_{i^v_\varepsilon}$- name for an isomorphism between $v(i^v_\varepsilon)$ and ${}^{<\kappa}2$). Let $\langle u^v_l : l < d_v \rangle$ enumerate ${}^{\delta_v}({}^{\beta+1}2)$ (the $\delta_v$-length sequences whose entries are elements of ${}^{\beta+1}2$) so that $u^v_l = \langle u^v_l(\varepsilon) : \varepsilon < \delta_v \rangle$, where each $u^v_l(\varepsilon) \in {}^{\beta+1}2$. We now need the following two facts:

FACT 1. Suppose that $v$ is a node and $l < d_v$. We can construct a condition $r \leq v$ called $v$ *thinned through* $u_l$, denoted by $(v)^{u_l}$, in the following manner:

$r \restriction i^v_0 = v \restriction i^v_0$, for each $\varepsilon < \delta_v$, $r(i^v_\varepsilon) = v(i^v_\varepsilon) \restriction \dot{s}^v_{\varepsilon, u^v_l(\varepsilon)}$, $r \restriction (i^v_\varepsilon, i^v_{\varepsilon+1}) = v \restriction (i^v_\varepsilon, i^v_{\varepsilon+1})$ and $r \restriction (i_{\delta_v}, k(\lambda)) = v \restriction (i_{\delta_v}, k(\lambda))$, where $v(i^v_\varepsilon) \restriction \dot{s}^v_{\varepsilon, u^v_l(\varepsilon)}$ is the subtree of $v(i^v_\varepsilon)$ whose branches go through $\dot{s}^v_{\varepsilon, u^v_l(\varepsilon)}$.

FACT 2. Suppose that $v$ and $r$ are conditions in $\mathrm{Sacks}(\kappa, [\lambda, k(\lambda)))$ with $r \leq (v)^{u_l}$. Then there is a condition $v'$ such that $v' \leq_{\beta, X_v} v$ and $(v')^{u_l} \sim r$ (i.e. $(v')^{u_l} \leq r$ and $r \leq (v')^{u_l}$). We say that $v'$ is *$v$ refined through $u_l$* to $r$.

Let $\langle v_j : j < 2^{\beta+1} \rangle$ be an enumeration of level $\beta$ of the tree $E$ and let $\langle u_m \rangle_{m < \sum_{j<2^{\beta+1}} d_{v_j}}$ be an enumeration of $Y := \bigcup_{j<2^{\beta+1}} \{u^{v_j}_l : l < d_{v_j}\}$. In order to construct the next level of the tree we will first thin out all the nodes on level $\beta$ (by considering all the pairs in $Y$) and then split each of them into two incompatible nodes. The thinning out is done as follows: Consider $u_0$ and $u_1$. If they belong to the same node, i.e. if there is $j < 2^{\beta+1}$ and $l_0, l_1 < d_{v_j}$ s.t. $u_0 = u^{v_j}_{l_0}$ and $u_1 = u^{v_j}_{l_1}$, then no thinning takes place. So assume that $u_0$ and $u_1$ belong to different nodes, say $v_{j_0}$ and $v_{j_1}$, respectively. Use Fact 1 to construct conditions $r_{01} = (v_{j_0})^{u_0}$ and $r_{10} = (v_{j_1})^{u_1}$, i.e. thin $v_{j_0}$ and $v_{j_1}$ through $u_0$ and $u_1$ to $r_{01}$ and $r_{10}$, respectively. Now ask whether there exist extensions $r'_{01}$ and $r'_{10}$ of $r_{01}$ and $r_{10}$, respectively, such that for some $\gamma_{01} < \lambda$ and some $A_{01}$, $A_{10}$, $\dot{A}_{01}$, $\dot{A}_{10}$, $((s_\beta, A_{01}), (r'_{01}, (s_\beta, \dot{A}_{01})))$ and $((s_\beta, A_{10}), (r'_{10}, (s_\beta, \dot{A}_{10})))$ force different nodes on level $\gamma_{01}$ of $\dot{T}$ to lie on $\dot{b}$. If the answer is 'yes', use Fact 2 to refine $v_{j_0}$ and $v_{j_1}$ through $r'_{01}$ and $r'_{10}$, respectively, and continue with the next pair: $u_0$, $u_2$. And if the answer is 'no', go to the pair $u_0$, $u_2$ without refining $v_{j_0}$ and $v_{j_1}$. The next pairs are $u_1, u_2$; $u_0, u_3$ and so on, i.e. all pairs of the form $u_\delta$, $u_\eta$, for $\eta < \sum_{j<2^{\beta+1}} d_{v_j}$ and $\delta < \eta$. At the limit stages take lower bounds, they exist since the forcing is $\kappa$-closed. Let $\lambda_\beta$ be the supremum of (the increasing sequence of) $\gamma_{\delta\eta}$'s. Now extend each node $v$ on level $\beta$ (after thinning out the whole level) to two incompatible conditions $v_o$ and $v_1$, such that $v_0, v_1 \leq_{\beta, X_v} v$.

Let $\alpha$ be the supremum of $\lambda_\beta$'s. Note that $\alpha < \lambda$, because $\lambda = (\kappa^{++})^{N_1[g]}$. Let $p$ be the result of a fusion along a branch through $E$. By the claim we can choose $A_0(p) \subseteq A_0$ in $U$ such that $((s_0, A_0(p)), (p, (s_0, \dot{A}_0)))$ is a condition. Extend this condition to some $((s_1(p), A_1(p)), (p^*, (s_1(p), \dot{A}_1(p))))$ which decides $\dot{b}(\alpha)$, say it forces $\dot{b}(\alpha) = x_p$.

As level $\alpha$ of $\dot{T}$ has size $< \lambda$, there exist limits $p, q$ of $\kappa$-fusion sequences arising from distinct $\kappa$-branches through $E$ for which $x_p$ equals $x_q$ and $s_1(p)$ equals $s_1(q)$. Moreover, we can intersect $A_1(p)$ and $A_1(q)$ to get a common $A_1$. Say, $((s_1, A_1), (p^*, (s_1, \dot{A}_1(p))))$ and $((s_1, A_1), (q^*, (s_1, \dot{A}_1(q))))$ force $\dot{b}(\alpha) = x$.

Now choose a Prikry generic $C$ containing $(s_1, A_1)$ (and therefore containing $(s_0, A_0)$). As $\dot{b}$ is forced by $((s_0, A_0), (p_0, (s_0, \dot{A}_0)))$ to not belong to $N_1[g][\dot{C}]$ and $((s_1, A_1), (p^*, (s_1, \dot{A}_1(p))))$ extends $((s_0, A_0), (p_0, (s_0, \dot{A}_0)))$, we can extend $((s_1, A_1), (p^*, (s_1, \dot{A}_1(p))))$ to incompat. conditions $((s_{2_0}, A_{2_0}), (p^{**}_0, (s_{2_0}, \dot{A}_{2_0})))$, $((s_{2_1}, A_{2_1}), (p^{**}_1, (s_{2_1}, \dot{A}_{2_1})))$, with $(s_{2_0}, A_{2_0}), (s_{2_1}, A_{2_1}) \in C$ and $p^{**}_0, p^{**}_1 \leq p^*$, which force a disagreement about $\dot{b}$ at some level $\gamma$ above $\alpha$.

Now extend $((s_1, A_1), (q^*, (s_1, \dot{A}_1(q))))$ to some $((s_3, A_3), (q^{**}, (s_3, \dot{A}_3)))$ deciding $\dot{b}(\gamma)$ with $(s_3, A_3)$ in $C$. Suppose w.l.o.g. that $((s_3, A_3), (q^{**}, (s_3, \dot{A}_3)))$ and $((s_{2_0}, A_{2_0}), (p^{**}_0, (s_{2_0}, \dot{A}_{2_0})))$ disagree about $\dot{b}(\gamma)$. Also w.l.o.g. we can assume that $s_3 \supseteq s_{2_0}$.

Using the claim extend $((s_{2_0}, A_{2_0}), (p_0^{**}, (s_{2_0}, \dot{A}_{2_0})))$ to some $((s_3, A_3'), (p^{***}, (s_3, \dot{A}_{2_0})))$ with $A_3' \in U$ and $p^{***} \leq p_0^{**}$.

Now, for some $\beta < \kappa$ we have $s_3 = s_\beta$ where $s_\beta$ is the $\beta$th element of the enumeration of the lower parts ($s_3$ is not the third element!). Since $s_\beta$ appears cofinally often in the construction of the tree $E$, we can assume that the branches which fuse to $p$ and $q$ split in $E$ at some node below level $\beta$ and go through some nodes $v_{j_0}$ and $v_{j_1}$ at level $\beta$. It follows that for some $l < d_{v_{j_0}}$ and $k < d_{v_{j_1}}$,

$$r_1 := ((s_3, A_3'((p^{***})^{u_l^{v_{j_0}}})), ((p^{***})^{u_l^{v_{j_0}}}, (s_3, \dot{A}_{2_0})))$$

and

$$r_2 := ((s_3, A_3((q^{**})^{u_k^{v_{j_1}}})), ((q^{**})^{u_k^{v_{j_1}}}, (s_3, \dot{A}_3)))$$

force different nodes to lie on $\dot{b}$ at level $\gamma > \alpha$. By construction, this means that for some $\eta < \sum_{j<2^{\beta+1}} d_{v_j}$ and $\delta < \eta$,

$$r_3 := ((s_\beta, A_{\delta\eta}), (r'_{\delta\eta}, (s_\beta, \dot{A}_{\delta\eta})))$$

and

$$r_4 := ((s_\beta, A_{\eta\delta}), (r'_{\eta\delta}, (s_\beta, \dot{A}_{\eta\delta})))$$

force different nodes on level $\gamma_{\delta\eta} (< \alpha)$ of $\dot{T}$ to lie on $\dot{b}$. Say, $\dot{b}(\gamma_{\delta\eta}) = y_0$ and $\dot{b}(\gamma_{\delta\eta}) = y_1$, respectively.

On the other side, $r_1$ and $r_2$ extend $((s_1, A_1), (p^*, (s_1, \dot{A}_1(p))))$ and $((s_1, A_1), (q^*, (s_1, \dot{A}_1(q))))$, respectively. Therefore we have that $r_1$ and $r_2$ also force $\dot{b}(\alpha) = x$.

Note that $(p^{***})^{u_l^{v_{j_0}}} \leq r'_{\delta\eta}$ and $(q^{**})^{u_k^{v_{j_1}}} \leq r'_{\eta\delta}$. Since any two $R * \dot{Q}$ conditions with the same lower part and compatible Sacks conditions are compatible, we have that $r_1 \parallel r_3$ and $r_2 \parallel r_4$. Let $((s_3, B'), (\bar{p}, (s_3, \dot{B}')))$ be a common lower bound of $r_1$ and $r_3$, and let $((s_3, B''), (\bar{q}, (s_3, \dot{B}'')))$ be a common lower bound of $r_2$ and $r_4$. The first condition forces $\dot{b}(\gamma_{\delta\eta}) = y_0$ and $\dot{b}(\alpha) = x$, and the second condition forces $\dot{b}(\gamma_{\delta\eta}) = y_1$ and $\dot{b}(\alpha) = x$.

Finally, let $\bar{B} := B' \cap B''$. Then $(s_3, \bar{B})$ forces that $y_0, y_1 <_{\dot{T}} x$ in the ordering of the tree $\dot{T}$, because $\dot{T}$ is a Prikry-name, i.e. all the relations between the nodes of $\dot{T}$ are determined by the Prikry parts of the conditions above. Contradiction. ⊣

**§4. The tree property at $\aleph_{\omega+2}$.** Using a forcing notion which makes $\kappa$ into $\aleph_\omega$ instead of Prikry forcing in the proof of Theorem 2 one can get from the same assumptions the tree property at $\aleph_{\omega+2}$, $\aleph_\omega$ strong limit.

THEOREM 3. *Assume that $V$ is a model of ZFC and $\kappa$ is a weakly compact hypermeasurable cardinal in $V$. Then there exists a forcing extension of $V$ in which $\aleph_{\omega+2}$ has the tree property.*

PROOF. Let $\lambda > \kappa$ be a weakly compact cardinal and let $j : V \to M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $H(\lambda)^V = H(\lambda)^M$. We may assume that $M$ is of the form $M = \{j(f)(\alpha) : \alpha < \lambda, f : \kappa \to V, f \in V\}$. First force as in Theorem 1 with $P = P_\kappa * \dot{\mathrm{Sacks}}(\kappa, \lambda)$ over $V$ to get a model $V[G][g]$ in which $2^\kappa = \kappa^{++}$, $\kappa^{++}$ has the tree property, and $\kappa$ is still measurable, i.e. there is an elementary embedding $j : V[G][g] \to M[G][g][H][h]$, where $G * g * H * h$ is a generic subset of $j(P)$ over $M$. Let $M^* := M[G][g][H][h]$. Note that $M^*$ is the ultrapower of $V[G][g]$ (by the normal measure $U$ induced by $j$), i.e. every

element in $M^*$ is of the form $j(f)(\kappa)$ for some $f : \kappa \to V[G][g], f \in V[G][g]$. This is because every element in $M^*$ is of the form $j(f)(\alpha)$ for some $\alpha < \lambda$, $f : \kappa \to V[G][g], f \in V[G][g]$, and every $\alpha < \lambda$ is of the form $j(g)(\kappa)$ for some $g : \kappa \to V[G][g], g \in V[G][g]$.

CLAIM. *Define* $Q' := Coll((\kappa^{+++})^{M^*}, j(\kappa))^{M^*}$, *the forcing that collapses each ordinal less than* $j(\kappa)$ *to* $(\kappa^{+++})^{M^*}$ *using conditions of size* $\leq (\kappa^{++})^{M^*}$. *There exists* $G'$ *in* $V[G][g]$, *a generic subset of* $Q'$ *over* $M^*$.

PROOF OF THE CLAIM. Every maximal antichain $\Delta \subset Q'$ in $M^*$ is actually in $M[G][g][H]$, and thus of the form $\sigma^{G*g*H}$ for some $j(P_\kappa)$-name $\sigma$ in $M$. It follows that $\Delta$ is of the form $j(f)(\alpha)^{G*g*H}$ for some $\alpha < \lambda = (\kappa^{++})^{M^*}$, and some $f : \kappa \to V, f \in V$. Since we can assume that $\sigma = j(f)(\alpha)$ is in $V_{j(\kappa)}$ (because $|j(P_\kappa)| = j(\kappa)$ and $j(P_\kappa)$ has $j(\kappa)$-c.c.), it follows that we can assume that $f : \kappa \to V_\kappa$.

For a fixed $f : \kappa \to V_\kappa$ we have that $F_f := \{\Delta \subset Q' \text{ — } \Delta \text{ maximal antichain}, \Delta \in M[G][g][H], \text{ and } j(f)(\alpha)^{G*g*H} = \Delta \text{ for some } \alpha < (\kappa^{++})^{M^*}\}$ is an element of $M[G][g][H]$. Therefore, since $Q'$ is $(\kappa^{+++})^{M^*}$-distributive in $M[G][g][H]$, there exists a single condition $p_f \in Q'$ which lies below every antichain in $F_f$.

Now, there are $2^\kappa = \kappa^+$ functions $f : \kappa \to V_\kappa$ in $V$. Enumerate them as $f_1, f_2, f_3...$ We can find conditions $q_\gamma \in Q'$ for $\gamma < \kappa^+$ such that $q_\gamma$ is a lower bound of $(p_{f_\beta})_{\beta<\gamma}$, because $M[G][g][H]^\kappa \cap V[G][g] \subseteq M[G][g][H]$ and $Q'$ is $(\kappa^+)^V$-closed in $M[G][g][H]$. The sequence $\{q_\gamma \text{ — } \gamma < \kappa^+\}$ generates a filter $G'$ for $Q'$ in $V[G][g]$, which is generic over $M[G][g][H]$. Here ends the proof of the claim. ⊣

We now define in $V[G][g]$ a $\kappa^+$-c.c. forcing notion $R(G', U)$, or just $R$, called *Collapse Prikry*, which makes $\kappa$ into $\aleph_\omega$ and preserves the tree property on $\kappa^{++}$ as follows: An element $p$ of $R$ is of the form $(\aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{n-1}, f_{n-1}, A, F)$ where

1. $\aleph_0 < \alpha_1 < \cdots < \alpha_{n-1} < \kappa$ are inaccessibles
2. $f_i \in \text{Coll}(\alpha_i^{+++}, \alpha_{i+1})$ for $i < n-1$ and $f_{n-1} \in \text{Coll}(\alpha_{n-1}^{+++}, \kappa)$
3. $A \in U, \min A > \alpha_{n-1}$
4. $F$ is a function on $A$ such that $F(\alpha) \in \text{Coll}(\alpha^{+++}, \kappa)$
5. $[F]_U$, which is an element of $\text{Coll}((\kappa^{+++})^{M^*}, j(\kappa))^{M^*}$, belongs to $G'$.

The conditions in $R$ are ordered as follows:
$(\aleph_0, g_0, \beta_1, g_1, ..., \beta_{m-1}, g_{m-1}, B, H) \leq (\aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{n-1}, f_{n-1}, A, F)$ iff

1. $m \geq n$
2. $\forall i < n\ \beta_i = \alpha_i, g_i \supseteq f_i$
3. $B \subseteq A$
4. $\forall i \geq n\ \beta_i \in A, g_i \supseteq F(\beta_i)$
5. $\forall \alpha \in B\ H(\alpha) \supseteq F(\alpha)$.

We often abbreviate the lower part of a condition by a single letter and write $(s, A, F)$ instead of $(\aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{n-1}, f_{n-1}, A, F)$ where $|s| = n$ denotes the length of the lower part. Let $S$ denote the 'generic sequence', i.e. the Prikry sequence together with the generic collapsing functions.

CLAIM. $R$ satisfies $\kappa^+$-c.c.

PROOF OF THE CLAIM. There are only $\kappa$ lower parts and any two conditions with the same lower part are compatible, so no antichain has size bigger than $\kappa$. $\dashv$

CLAIM. Let $(s, A, F) \in R$ and let $\sigma$ be a statement of the forcing language. There exists a stronger condition $(s', A^*, F^*)$ with $|s| = |s'|$ which decides $\sigma$.

For a proof see [5].

CLAIM. Let $C$ be a $V[G][g]$-generic subset of $R$ and let $\langle \aleph_0, \alpha_1, ..., \alpha_n, ... \rangle$ be the Prikry sequence in $\kappa$ introduced by $R$. For $j \in \omega$, define $R \restriction j := \mathrm{Coll}(\aleph_0^{+++}, \alpha_1) \times \mathrm{Coll}(\alpha_1^{+++}, \alpha_2) \times ... \times \mathrm{Coll}(\alpha_{j-1}^{+++}, \alpha_j)$. Then $V[G][g][C]$ and $V[G][g][C \restriction j]$ have the same cardinal structure below $\alpha_j + 1$, namely $\aleph_1, \aleph_2, \aleph_3, \alpha_1, \alpha_1^+, \alpha_1^{++}, \alpha_1^{+++}, \ldots, \alpha_{j-1}, \alpha_{j-1}^+, \alpha_{j-1}^{++}, \alpha_{j-1}^{+++}, \alpha_j$, where $C \restriction j$ is the restriction of $C$ to $R \restriction j$.

PROOF OF THE CLAIM. Write $R$ as $R \restriction j * R/(\dot{R} \restriction j)$, where the quotient $R/(\dot{R} \restriction j)$ is defined in the same way as $R$ (using only inaccessibles between $\alpha_j$ and $\kappa$). We need to show that $R/(R \restriction j)$ does not add bounded subsets of $\alpha_j$, but this follows immediately from the last claim. $\dashv$

So we proved that $R$ makes $\kappa$ into $\aleph_\omega$. It remains to show that it also preserves the tree property on $\kappa^{++} = \lambda$.

In order to get a contradiction suppose that there is a $\kappa^{++}$-Aronszajn tree in some $R$-extension of $V[G][g]$. Then in $V[G]$ there is a $\mathrm{Sacks}(\kappa, \lambda) * \dot{R}$ - name $\dot{T}$ of size $\lambda$ (because $\mathrm{Sacks}(\kappa, \lambda) * \dot{R}$ satisfies $\lambda$-c.c.) and a condition $(p, \dot{r}) \in \mathrm{Sacks}(\kappa, \lambda) * \dot{R}$ which forces $\dot{T}$ to be a $\kappa^{++}$-Aronszajn tree. Let $\dot{G}'$ be a $\mathrm{Sacks}(\kappa, \lambda)$-name in $V[G]$ for $G'$ of size $\lambda$ (there is such a name because $\mathrm{Sacks}(\kappa, \lambda)$ has the $\lambda$-c.c. and $|Q'| = \lambda$). We can assume w.l.o.g. that $p$ forces $\dot{G}'$ to be generic over $Q'$. Recall that $\lambda$ is a weakly compact cardinal in $V[G]$. Therefore, there exist in $V[G]$ transitive $ZF^-$-models $N_0, N_1$ of size $\lambda$ and an elementary embedding $k : N_0 \to N_1$ with critical point $\lambda$, such that $N_0 \supseteq H(\lambda)^{V[G]}$ and $G, \dot{T}, \dot{G}' \in N_0$.

Since $g$ is also $\mathrm{Sacks}(\kappa, \lambda)$-generic over $N_0$ and the critical point of $k$ is $\lambda$, $k$ can be lifted to $k^* : N_0[g] \to N_1[g][K]$, where $K$ is any $N_1[g]$-generic subset of $\mathrm{Sacks}(\kappa, [\lambda, k(\lambda)))$ in some larger universe (and where $\mathrm{Sacks}(\kappa, [\lambda, k(\lambda)))$ is the quotient $\mathrm{Sacks}(\kappa, k(\lambda))/\mathrm{Sacks}(\kappa, \lambda)$, i.e. the iteration of $\mathrm{Sacks}(\kappa)$ indexed by ordinals between $\lambda$ and $k(\lambda)$). Consider the forcing $R^* := k^*(R) = R(k(G'), k(U))$ in $N_1[g][K]$ and choose any generic $C^*$ for it such that $k^*(r) \in C^*$, where $r = \dot{r}^g, R = \dot{R}^g, G' = \dot{G}'^g$. Let $C := (k^*)^{-1}[C^*]$ be the pullback of $C^*$ under $k^*$. Then $C$ is an $N_0[g]$-generic subset of $R$ because $\mathrm{crit}(k) = \lambda$ and $R$ has the $\kappa^+$-c.c. It follows that there is an elementary embedding $k^{**} : N_0[g][C] \to N_1[g][K][C^*]$ extending $k^*$.

We have $r \in C$. So it follows that the evaluation $T$ of $\dot{T}$ in $N_0[g][C]$ is a $\lambda$-Aronszajn tree. By elementarity $k^{**}(T)$ is a $k^{**}(\lambda)$-Aronszajn tree in $N_1[g][K][C^*]$ which coincides with $T$ up to level $\lambda$. Hence $T$ has a cofinal branch $b$ in $N_1[g][K][C^*]$. We will show that $b$ has to belong to $N_1[g][C]$ and thereby reach the desired contradiction!

Let us first analyse the quotient $Q$ arising from the natural projection $\pi : \mathrm{Sacks}(\kappa, k(\lambda)) * \dot{R}^* \to RO(\mathrm{Sacks}(\kappa, \lambda) * \dot{R})$. As in the previous section, $Q$ is the set of all $(p^*, (\aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{n-1}, f_{n-1}, \dot{A}^*, \dot{F}^*)) \in \mathrm{Sacks}(\kappa, k(\lambda)) * \dot{R}^*$ which are compatible with each $(p, (\aleph_0, g_0, \beta_1, g_1, ..., \beta_{m-1}, g_{m-1}, \dot{A}, \dot{F})) \in g * C$, that is, either

1. $p^* \restriction \lambda$ is compatible with $p$,
2. $n < m$,
3. for all $i < n$ $\alpha_i = \beta_i \wedge f_i \parallel g_i$,
4. there is $q \leq p \cup p^*$ such that $q \Vdash$ "$\beta_n, ..., \beta_{m-1} \subset \dot{A}^*$ and $\dot{F}^*(\beta_i) \parallel g_i$ for $n \leq i < m$",

or

1. $p^* \restriction \lambda$ is compatible with $p$,
2. $n \geq m$,
3. for all $i < m$ $\alpha_i = \beta_i \wedge f_i \parallel g_i$,
4. there is $q \leq p \cup p^*$ such that $q \Vdash$ "$\alpha_m, ..., \alpha_{n-1} \subset \dot{A}$ and $\dot{F}(\alpha_i) \parallel f_i$ for $m \leq i < n$".

[Note that in both cases the condition $q$ also forces $\dot{F}$ and $\dot{F}^*$ to be compatible on a measure one set. This is because the weaker condition $p$ (by definition) forces $j(\dot{F})(\kappa)$ to be in $\dot{G}'$, and therefore, by elementarity, also forces $k(j)(k(\dot{F}))(\kappa)$ to be in $k(\dot{G}')$, but $k(j)(k(\dot{F}))(\kappa)$ is the same as $k(j)(\dot{F})(\kappa) = [\dot{F}]_{U^*}$, since the trivial condition forces $k(\dot{F}) = \dot{F}$.]

Equivalently, $Q$ is the set of conditions $(p^*, (\aleph_0, f_0, ..., \alpha_{n-1}, f_{n-1}, \dot{A}^*, \dot{F}^*))$ in $\text{Sacks}(\kappa, [\lambda, k(\lambda))) * \dot{R}^*$ such that

1. $p^* \in \text{Sacks}(\kappa, [\lambda, k(\lambda)))$,
2. $\langle \aleph_0, \alpha_1 ..., \alpha_{n-1} \rangle$ is an initial segment of $S(C)$ (the Prikry sequence arising from $C$),
3. the collapsing function $\bar{g}_i : \alpha_i^{+++} \to \alpha_{i+1}$ arising from $C$ extends $f_i$, $i < n$,
4. $p^*$ forces that $\dot{A}^*$ is in $\dot{U}^*$, and that $\dot{F}^*$ is a function on $\dot{A}^*$ such that $\dot{F}^*(\alpha) \in \text{Coll}(\alpha^{+++}, \kappa)$ for each $\alpha \in \dot{A}^*$,
5. for every finite subset $x = \langle \beta_n, ..., \beta_{m-1} \rangle$ of $S(C)$ and every sequence of functions $\langle g_n, ..., g_{m-1} \rangle$ with $g_i \subseteq \bar{g}_i$, $n \leq i < m$, there is some extension $q$ of $p^*$ which forces that $x$ is a subset of $\{\aleph_0, \alpha_1, ..., \alpha_{n-1}\} \cup \dot{A}^*$ and that $\dot{F}^*(\beta_i) \parallel g_i$ for $n \leq i < m$.

We now again argue indirectly. Assume that $b$ is not in $N_1[g][C]$, and let $\dot{b}$ in $N_1[g]$ be an $R * \dot{Q}$ - name for $b$. Identify $k(\dot{T})$ with the $R * \dot{Q}$ - name defined by interpreting the $\text{Sacks}(\kappa, k(\lambda)) * \dot{R}^*$ - name $k(\dot{T})$ in $N_1$ as an $R * \dot{Q}$ - name in $N_1[g]$. Let $((s_0, A_0, F_0), (p_0, (t_0, \dot{A}_0, \dot{F}_0)))$ be an $R * \dot{Q}$ - condition forcing that the Prikry-name $\dot{T}$ is a $\lambda$-tree and that $\dot{b}$ is a branch through $\dot{T}$ not belonging to $N_1[g][\dot{C}]$.

Let us take a closer look at the condition $((s_0, A_0, F_0), (p_0, (t_0, \dot{A}_0, \dot{F}_0)))$. Say, $s_0 = \langle \aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{n-1}, f_{n-1} \rangle$ and $t_0 = \langle \aleph_0, g_0, \beta_1, g_1, ..., \beta_{m-1}, g_{m-1} \rangle$. Note that the forcing $Q$ lives in $N_1[g][C]$, but its elements are in $N_1[g]$, so we can assume that $(p_0, (t_0, \dot{A}_0, \dot{F}_0))$ is a real object and not just an $R$-name. The condition $(s_0, A_0, F_0)$ forces $(p_0, (t_0, \dot{A}_0, \dot{F}_0))$ to be an element of $\dot{Q}$. But this simply means that:

1. $p_0$ is an element of $\text{Sacks}(\kappa, [\lambda, k(\lambda)))$,
2. $\langle \aleph_0, \beta_1, ..., \beta_{m-1} \rangle$ is an initial segment of $\langle \aleph_0, \alpha_1, ..., \alpha_{n-1} \rangle$,
3. $g_i \subseteq f_i$ for $i < m$, and
4. for every finite subset $x = \langle \delta_1, ..., \delta_l \rangle$ of $\{\aleph_0, \alpha_1, ..., \alpha_{n-1}\} \cup A_0$ and every sequence of functions $\langle g_{\delta_1}, ..., g_{\delta_l} \rangle$ with $g_{\delta_i} \supseteq F_0(\delta_i)$ if $\delta_i > \alpha_{n-1}$, and $g_{\delta_i} \supseteq f_i$

if $\delta_i = \alpha_i$ (for some $i < n$), some extension of $p_0$ forces that $x$ is a subset of $\{\aleph_0, \beta_1, ..., \beta_{m-1}\} \cup \dot{A}_0$ and that $\dot{F}_0(\delta_i) \parallel g_{\delta_i}$ for $i < l$.

Moreover, we can assume that $s_0 = t_0$. Namely, the following claim gives us a nice dense subset of $R * \dot{Q}$ on which we will work from now on.

CLAIM. *Let $((s, A, F), (p, (t, \dot{A}, \dot{F})))$ be an arbitrary condition in $R * \dot{Q}$. There is a stronger condition $((s', A', F'), (p', (s', \dot{A}, \dot{F})))$ with the property that for each $\alpha \in A'$ $p' \Vdash F'(\alpha) \leq \dot{F}(\alpha)$.*

PROOF OF THE CLAIM. Say, $s$ is of the form $\langle \aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{n-1}, f_{n-1}\rangle$ and $t$ is of the form $\langle \aleph_0, g_0, \beta_1, g_1, ..., \beta_{m-1}, g_{m-1}\rangle$. Let $q$ be an extension of $p$ which forces that $\{\alpha_m, ..., \alpha_{n-1}\}$ is a subset of $\dot{A}$ and that $f_i \parallel \dot{F}(\alpha_i)$ for $m \leq i < n$. Extend $q$ further to $q'$ to decide $\dot{F}(\alpha_i)$ and let $f'_i := f_i \cup \dot{F}(\alpha_i)$. Define $s'$ to be $\langle \aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{m-1}, f_{m-1}, \alpha_m, f'_m, ..., \alpha_{n-1}, f'_{n-1}\rangle$.

Using the fusion property of Sacks$(\kappa, [\lambda, k(\lambda)))$ we can find a condition $q'' \leq q'$ and a ground model function $F^*$ on $A$ with $|F^*(\alpha)| \leq \alpha^{++}$ for each $\alpha$ such that $q'' \Vdash \dot{F}(\alpha) \in \mathrm{Coll}(\alpha^{+++}, \kappa) \cap F^*(\alpha)$. It follows that $q''$ forces that in $\mathrm{Ult}(N_1[g], U)$, the ultrapower of $N_1[g]$ by $U$, $j_U(\dot{F})(\kappa) \in \mathrm{Coll}(\kappa^{+++}, j_U(\kappa)) \cap j_U(F^*)(\kappa)$, where $|j_U(F^*)(\kappa)| \leq \kappa^{++}$, that is, $q''$ forces that there are fewer than $\kappa^{+++}$ possibilities for $j_U(\dot{F})(\kappa)$. Note that $\mathrm{Coll}(\kappa^{+++}, j_U(\kappa))$ of $\mathrm{Ult}(N_1[g], U)$ is the same as $\mathrm{Coll}(\kappa^{+++}, j_U(\kappa))$ of $\mathrm{Ult}(N_0[g], U)$, because these two ultrapowers agree below $j_U(\kappa)$.

Since $\mathrm{Coll}(\kappa^{+++}, j_U(\kappa))$ is $\kappa^{+++}$-closed we can densely often find conditions in $\mathrm{Coll}(\kappa^{+++}, j_U(\kappa))$ which are either stronger than or incompatible with all elements in $j_U(F^*)(\kappa)$. Therefore we can choose some $j_U(F')(\kappa) \leq j_U(F)(\kappa)$ in $G'$ with this property, i.e. $q'' \Vdash j_U(F')(\kappa) \leq j_U(\dot{F})(\kappa) \vee j_U(F')(\kappa) \perp j_U(\dot{F})(\kappa)$. But actually we have $q'' \Vdash j_U(F')(\kappa) \leq j_U(\dot{F})(\kappa)$, because for any generic $K$ below $q''$, $j_U(F')(\kappa)$ and $j_U(\dot{F}^K)(\kappa)$ can not be incompatible as $k(j_U(F')(\kappa))$ and $k(j_U(\dot{F}^K)(\kappa)) = j_{k(U)}(\dot{F}^K)(\kappa)$ both belong to the guiding generic $k(G')$.

It follows that $q''$ forces that for some $B \in U, B \subseteq A$, for each $\alpha \in B$, $q'' \Vdash F'(\alpha) \leq \dot{F}(\alpha)$. Extend $q''$ to some $p'$ deciding $B$.

Finally, using the claim from the previous section, shrink $B$ to some $A'$ such that every finite subset of $A'$ is forced by some extension of $p'$ to belong to $\dot{A}$. Then we have $((s', A', F'), (p', (s', \dot{A}, \dot{F}))) \leq ((s, A, F), (p, (t, \dot{A}, \dot{F})))$ such that for each $\alpha \in A'$ $p' \Vdash F'(\alpha) \leq \dot{F}(\alpha)$. This proves the claim.

Now in $N_1[g]$ build a $\kappa$-tree $E$ of conditions in Sacks$(\kappa, [\lambda, k(\lambda)))$, whose branches will be fusion sequences, together with a sequence of ordinals $\langle \lambda_\beta : \beta < \kappa\rangle$, each $\lambda_\beta < \lambda$, in the same way as in the last section (using the same notation, Fact 1 and Fact 2):

Let $\langle v_j : j < 2^{\beta+1}\rangle$ be an enumeration of level $\beta$ of the tree $E$ and let $\langle u_m\rangle_{m<\sum_{j<2^{\beta+1}} d_{v_j}}$ be an enumeration of $Y := \bigcup_{j<2^{\beta+1}} \{u_l^{v_j} : l < d_{v_j}\}$. In order to construct the next level of the tree we will first thin out all the nodes on level $\beta$ (by considering all the pairs in $Y$) and then split each of them into two incompatible nodes. The thinning out is done as follows: Consider $u_0$ and $u_1$. If they belong to the same node, i.e. if there is $j < 2^{\beta+1}$ and $l_0, l_1 < d_{v_j}$ s.t. $u_0 = u_{l_0}^{v_j}$ and $u_1 = u_{l_1}^{v_j}$, then no thinning takes place. So assume that $u_0$ and $u_1$ belong to different nodes, say $v_{j_0}$ and $v_{j_1}$, respectively. Use Fact 1 to construct conditions

$r_{01} = (v_{j_0})^{u_0}$ and $r_{10} = (v_{j_1})^{u_1}$, i.e. thin $v_{j_0}$ and $v_{j_1}$ through $u_0$ and $u_1$ to $r_{01}$ and $r_{10}$, respectively. Now ask whether there exist extensions $r'_{01}$ and $r'_{10}$ of $r_{01}$ and $r_{10}$, respectively, such that for some $\gamma_{01} < \lambda$ and some $A_{01}$, $A_{10}$, $F_{01}$, $F_{10}$, $\dot{A}_{01}$, $\dot{A}_{10}$, $\dot{F}_{01}$, $\dot{F}_{10}$, $((s_\beta, A_{01}, F_{01}), (r'_{01}, (s_\beta, \dot{A}_{01}, \dot{F}_{01})))$ and $((s_\beta, A_{10}, F_{10}), (r'_{10}, (s_\beta, \dot{A}_{10}, \dot{F}_{10})))$ force different nodes on level $\gamma_{01}$ of $\dot{T}$ to lie on $\dot{b}$. If the answer is 'yes', use Fact 2 to refine $v_{j_0}$ and $v_{j_1}$ through $r'_{01}$ and $r'_{10}$, respectively, and continue with the next pair: $u_0$, $u_2$. And if the answer is 'no', go to the pair $u_0$, $u_2$ without refining $v_{j_0}$ and $v_{j_1}$. The next pairs are $u_1$, $u_2$; $u_0$, $u_3$ and so on, i.e. all pairs of the form $u_\delta$, $u_\eta$, for $\eta < \sum_{j<2^{\beta+1}} d_{v_j}$ and $\delta < \eta$. At the limit stages take lower bounds, they exist since the forcing is $\kappa$-closed. Let $\lambda_\beta$ be the supremum of (the increasing sequence of) $\gamma_{\delta\eta}$'s. Now extend each node $v$ on level $\beta$ (after thinning out the whole level) to two incompatible conditions $v_o$ and $v_1$, such that $v_0, v_1 \leq_{\beta, X_v} v$.

Let $\alpha$ be the supremum of $\lambda_\beta$'s. Note that $\alpha < \lambda$, because $\lambda = (\kappa^{++})^{N_1[g]}$. Let $p$ be the result of a fusion along a branch through $E$. As before we can choose $A_0(p) \subseteq A_0$ in $U$ such that $((s_0, A_0(p), F_0), (p, (s_0, \dot{A}_0, \dot{F}_0)))$ is a condition. Extend this condition to some $((s_1(p), A_1(p), F_1(p)), (p^*, (s_1(p), \dot{A}_1(p), \dot{F}_1(p))))$ which decides $\dot{b}(\alpha)$, say it forces $\dot{b}(\alpha) = x_p$.

As level $\alpha$ of $\dot{T}$ has size $< \lambda$, there exist limits $p, q$ of $\kappa$-fusion sequences arising from distinct $\kappa$-branches through $E$ for which $x_p$ equals $x_q$ and $s_1(p)$ equals $s_1(q)$. Moreover, we can extend $(s_1(p), A_1(p), F_1(p))$ and $(s_1(q), A_1(q), F_1(q))$ to get a common $(s_1, A_1, F_1)$. Say, $((s_1, A_1, F_1), (p^*, (s_1, \dot{A}_1(p), \dot{F}_1(p))))$ and $((s_1, A_1, F_1), (q^*, (s_1, \dot{A}_1(q), \dot{F}_1(q))))$ force $\dot{b}(\alpha) = x$.

Now choose a Collapse Prikry generic $C$ containing $(s_1, A_1, F_1)$ (and hence containing $(s_0, A_0, F_0)$). As $((s_0, A_0, F_0), (p_0, (s_0, \dot{A}_0, \dot{F}_0))) \Vdash \dot{b} \notin N_1[g][\dot{C}]$ and $((s_1, A_1, F_1), (p^*, (s_1, \dot{A}_1(p), \dot{F}_1(p))))$ extends $((s_0, A_0, F_0), (p_0, (s_0, \dot{A}_0, \dot{F}_0)))$, we can extend $((s_1, A_1, F_1), (p^*, (s_1, \dot{A}_1(p), \dot{F}_1(p))))$ to two incompatible conditions, $((s_{2_0}, A_{2_0}, F_{2_0}), (p_0^{**}, (s_{2_0}, \dot{A}_{2_0}, \dot{F}_{2_0})))$ and $((s_{2_1}, A_{2_1}, F_{2_1}), (p_1^{**}, (s_{2_1}, \dot{A}_{2_1}, \dot{F}_{2_1})))$, with $(s_{2_0}, A_{2_0}, F_{2_0})$, $(s_{2_1}, A_{2_1}, F_{2_1}) \in C$ and $p_0^{**}, p_1^{**} \leq p^*$, which force a disagreement about $\dot{b}$ at some level $\gamma$ above $\alpha$.

Now extend $((s_1, A_1, F_1), (q^*, (s_1, \dot{A}_1(q), \dot{F}_1(q))))$ to some stronger condition $((s_3, A_3, F_3), (q^{**}, (s_3, \dot{A}_3, \dot{F}_3)))$ which decides $\dot{b}(\gamma)$ with $(s_3, A_3, F_3)$ in $C$. Say, $((s_3, A_3, F_3), (q^{**}, (s_3, \dot{A}_3, \dot{F}_3)))$ and $((s_{2_0}, A_{2_0}, F_{2_0}), (p_0^{**}, (s_{2_0}, \dot{A}_{2_0}, \dot{F}_{2_0})))$ do not agree about $\dot{b}(\gamma)$, and say, $s_3$ is of the form $\langle \aleph_0, f_0, \alpha_1, f_1, ..., \alpha_{n-1}, f_{n-1} \rangle$, and $s_{2_0}$ is of the form $\langle \aleph_0, g_0, \beta_1, g_1, ..., \beta_{m-1}, g_{m-1} \rangle$.

We can assume w.l.o.g. that $m < n$. As both $(s_3, A_3, F_3)$ and $(s_{2_0}, A_{2_0}, F_{2_0})$ are in $C$, we have that $\langle \aleph_0, \beta_1, ..., \beta_{m-1} \rangle$ is an initial segment of $\langle \aleph_0, \alpha_1, ..., \alpha_{n-1} \rangle$, $g_i \parallel f_i$ for $i < m$, $\{\alpha_m, ..., \alpha_{n-1}\} \subset A_{2_0}$, and $F_{2_0}(\alpha_i) \parallel f_i$ for $m \leq i < n$. Let $f'_i := f_i \cup g_i$ for $i < m$, and $f'_i := f_i \cup F_{2_0}(\alpha_i)$ for $m \leq i < n$. Define $s'_3$ to be $\langle \aleph_0, f'_0, \alpha_1, f'_1, ..., \alpha_{n-1}, f'_{n-1} \rangle$.

Note that $((s'_3, A_3, F_3), (q^{**}, (s'_3, \dot{A}_3, \dot{F}_3))) \leq ((s_3, A_3, F_3), (q^{**}, (s_3, \dot{A}_3, \dot{F}_3)))$ is also a condition.

Since $\{\alpha_m, ..., \alpha_{n-1}\} \subset A_{2_0}$, there exists some $p^{***} \leq p_0^{**}$ which forces that $\{\alpha_m, ..., \alpha_{n-1}\} \subset \dot{A}_{2_0}$. It follows that there is also some $A'_3 \in U$ such that $((s'_3, A'_3, F_{2_0}), (p^{***}, (s'_3, \dot{A}_{2_0}, \dot{F}_{2_0}))) \leq ((s_{2_0}, A_{2_0}, F_{2_0}), (p_0^{**}, (s_{2_0}, \dot{A}_{2_0}, \dot{F}_{2_0})))$.

Now, for some $\beta < \kappa$ we have $s'_3 = s_\beta$ where $s_\beta$ is the $\beta$th element of the enumeration of the lower parts. Since $s_\beta$ appears cofinally often in the construction

of the tree $E$, we can assume that the branches which fuse to $p$ and $q$ split in $E$ at some node below level $\beta$ and go through some nodes $v_{j_0}$ and $v_{j_1}$ at level $\beta$. It follows that for some $l < d_{v_{j_0}}$ and $k < d_{v_{j_1}}$,

$$r_1 := \big((s_3', A_3'((p^{***})^{u_l^{v_{j_0}}}), F_{2_0}),\ ((p^{***})^{u_l^{v_{j_0}}}, (s_3', \dot{A}_{2_0}, \dot{F}_{2_0}))\big)$$

and

$$r_2 := \big((s_3', A_3((q^{**})^{u_k^{v_{j_1}}}), F_3),\ ((q^{**})^{u_k^{v_{j_1}}}, (s_3', \dot{A}_3, \dot{F}_3))\big)$$

force different nodes to lie on $\dot{b}$ at level $\gamma > \alpha$. By construction, this means that for some $\eta < \sum_{j<2^{\beta+1}} d_{v_j}$ and $\delta < \eta$,

$$r_3 := ((s_\beta, A_{\delta\eta}, F_{\delta\eta}), (r'_{\delta\eta}, (s_\beta, \dot{A}_{\delta\eta}, \dot{F}_{\delta\eta})))$$

and

$$r_4 := ((s_\beta, A_{\eta\delta}, F_{\eta\delta}),\ (r'_{\eta\delta}, (s_\beta, \dot{A}_{\eta\delta}, \dot{F}_{\eta\delta})))$$

force different nodes on level $\gamma_{\delta\eta} (< \alpha)$ of $\dot{T}$ to lie on $\dot{b}$. Say, $\dot{b}(\gamma_{\delta\eta}) = y_0$ and $\dot{b}(\gamma_{\delta\eta}) = y_1$, respectively.

On the other side, $r_1$ and $r_2$ extend $((s_1, A_1, F_1), (p^*, (s_1, \dot{A}_1(p), \dot{F}_1(p))))$ and $((s_1, A_1, F_1), (q^*, (s_1, \dot{A}_1(q), \dot{F}_1(q))))$, respectively. Therefore we have that $r_1$ and $r_2$ also force $\dot{b}(\alpha) = x$.

Note that $(p^{***})^{u_l^{v_{j_0}}} \leq r'_{\delta\eta}$ and $(q^{**})^{u_k^{v_{j_1}}} \leq r'_{\eta\delta}$. Since any two $R * \dot{Q}$ conditions with the same lower part and compatible Sacks conditions are compatible (this follows by the same arguments used in the proof of the last claim), we have that $r_1 \parallel r_3$ and $r_2 \parallel r_4$. Let $((s_3', B', H'), (\bar{p}, (s_3', \dot{B}', \dot{H}')))$ be a common lower bound of $r_1$ and $r_3$, and let $((s_3', B'', H''), (\bar{q}, (s_3', \dot{B}'', \dot{H}'')))$ be a common lower bound of $r_2$ and $r_4$. The first condition forces $\dot{b}(\gamma_{\delta\eta}) = y_0$ and $\dot{b}(\alpha) = x$, and the second condition forces $\dot{b}(\gamma_{\delta\eta}) = y_1$ and $\dot{b}(\alpha) = x$.

Finally, let $\bar{B} := B' \cap B''$ and $\bar{H} := H' \cap H''$. Then $(s_3', \bar{B}, \bar{H})$ forces that $y_0, y_1 <_{\dot{T}} x$ in the ordering of the tree $\dot{T}$, because $\dot{T}$ is a Collapse Prikry-name, i.e. all the relations between the nodes of $\dot{T}$ are determined by the Collapse Prikry parts of the conditions above. Contradiction. ⊣

**Open questions.**

1. What is the consistency strength of $\aleph_\omega$ strong limit with the tree property at $\aleph_{\omega+2}$? [The best known lower bound is a weakly compact $\lambda$ such that for each $n < \omega$ there exists $\kappa < \lambda$ with $o(\kappa) = \kappa^{+n}$.]

2. What is the consistency strength of the tree property at every even successor cardinal?

3. Is it consistent with ZFC to have the tree property at each $\aleph_n$, $1 < n < \omega$, and $\aleph_{\omega+2}$?

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