Generic I0 at $\aleph_\omega$

Vincenzo Dimonte

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The second is a motivation for the *thesis*.

Definition (Chang's Conjecture, 1963)

Every model of type $(\aleph_2, \aleph_1)$ (i.e., the universe has cardinality $\aleph_2$ and there is a predicate of cardinality $\aleph_1$) for a countable language has an elementary submodel of type $(\aleph_1, \aleph_0)$.

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Theorem (Donder, 1979)

Chang's Conjecture $\to$ $\aleph_1$ is ω_1-Erdös in the core model.

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Theorem (Schindler)

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Let κ be a cardinal. Then κ is *ω-huge* or *Reinhardt* iff there is a $j : V \prec M$ with $\mathrm{crt}(j) = \kappa_0$, ${}^{\lambda}M \subseteq M$, with $\lambda = \sup_{n \in \omega} \kappa_n$.

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We say that κ is a generically measurable cardinal.

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Con(huge cardinal)$\to$Con($\aleph_1$ is generic huge cardinal and $j(\aleph_2)=\aleph_3$).

Proposition

If $j: V \prec M \subseteq V[G]$, M closed under $\aleph_3$-sequences, $\mathrm{crt}(j)=\aleph_2$ and $j(\aleph_2)=\aleph_3$, then $(\aleph_3,\aleph_2)\twoheadrightarrow(\aleph_2,\aleph_1)$.

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In the same way,

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If $j : V \prec M \subseteq V[G]$, M closed under $\aleph_{n+1}$-sequences, $\mathrm{crt}(j) = \aleph_1$ and $j(\aleph_1) = \aleph_2$, $j(\aleph_2) = \aleph_3, \dots,$ then $(\aleph_{n+1}, \dots, \aleph_2, \aleph_1) \twoheadrightarrow (\aleph_n, \dots, \aleph_1, \aleph_0)$.

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What about Con($\aleph_\omega$ is Jónsson)?

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There is no ω-huge (and Shelah proved there is no generic ω-huge)! What can we do?

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- I0 For some λ there exists a $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$, with $\mathrm{crt}(j) < \lambda$

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With the "right" forcing, generic I* implies $\aleph_\omega$ is Jónsson.

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What about Con($\aleph_1$ is generic 3-huge cardinal and ...)?

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Generic I0 at $\aleph_\omega$ is true if there exists a forcing notion $\mathbb{P}$ such that for any generic G there exists $j : L(\mathcal{P}(\aleph_\omega)) \prec L(\mathcal{P}(\aleph_\omega))^{V[G]}$ and $\mathbb{P}$ is reasonable.

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Examples: $\mathbb{P} = \mathrm{Coll}(\aleph_3, \aleph_2)$, $\mathbb{P} =$ product of $\mathbb{P}_n$, where $\mathbb{P}_n = \mathrm{Coll}(\aleph_{n+1}, \aleph_n)$.

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If $\aleph_\omega$ is strong limit, then $2^{\aleph_0} < \aleph_{\omega_4}$

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(From now on, let's suppose $\mathrm{crt}(j) = \aleph_2$ and $j(\aleph_2) = \aleph_3$).

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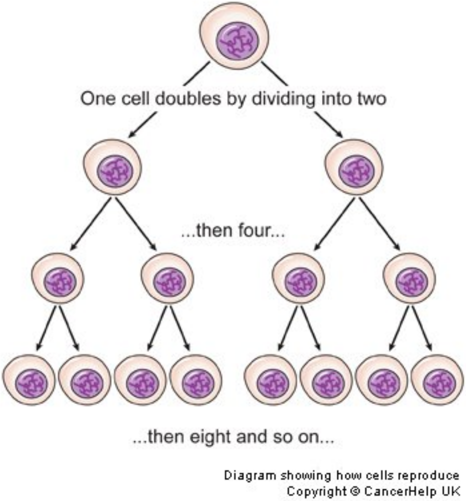
Proof of (1)

It is practically the same proof as Kunen's Theorem. Suppose $\langle S_\xi : \xi < \aleph_2 \rangle$ is an ω-stationary partition. Now, $j \upharpoonright L_\alpha(\mathcal{P}) \in L(\mathcal{P}(\aleph_\omega))[G]$, so $C = \{\alpha < \aleph_{\omega+1} : j(\alpha) = \alpha\} \in L(\mathcal{P}(\aleph_\omega))[G]$. As before, then there exists $\alpha \in T_\xi \cap T_{\aleph_2}$.
In $L(\mathcal{P}(\aleph_\omega))$ we have some choice, namely $\mathrm{DC}_{\aleph_\omega}$...


One cell doubles by dividing into two
...then four...
...then eight and so on...
Diagram showing how cells reproduce
Copyright © CancerHelp UK

For points (2) and (3) we need more choice than $DC_{\aleph_\omega}$

For points (2) and (3) we need more choice than $\mathsf{DC}_{\aleph_\omega}$:

Coding Lemma

$\forall\eta < \Theta\,\forall\rho : \mathcal{P}(\aleph_\omega) \twoheadrightarrow \eta\ \exists\gamma < \Theta\ \forall A \subseteq \mathcal{P}(\aleph_\omega)\ \exists B \subseteq \mathcal{P}(\aleph_\omega)\ B \in L_\gamma(\mathcal{P}(\aleph_\omega))\ B \subseteq A$ and $\{\rho(a) : a \in B\} = \{\rho(a) : a \in A\}$.

Proof of (2)

One has to prove that if there exists $\rho : \mathcal{P}(\aleph_\omega) \twoheadrightarrow \alpha$, then there exists $\pi : \mathcal{P}(\aleph_\omega) \twoheadrightarrow \mathcal{P}(\alpha)$

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Therefore, the Theorem proves that if we have generic I0 at $\aleph_\omega$, then $D(\aleph_\omega)$.

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$L(\mathbb{R}) \models \mathsf{AD} \rightarrow L(\mathbb{R}) \models D(\omega)$

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What is the consistency strength of $D(\lambda)$ with λ uncountable?

Thanks for your attention.