Generic I0 at $\aleph_\omega$

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TAKE
ACTION

Let's introduce a new definition

- Chapter I: The Importance of Being Generic Rank-into Rank

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Definition (Chang's Conjecture, 1963)

Every model of type $(\aleph_2, \aleph_1)$ (i.e., the universe has cardinality $\aleph_2$ and there is a predicate of cardinality $\aleph_1$) for a countable language has an elementary submodel of type $(\aleph_1, \aleph_0)$.

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What about $(\aleph_3, \aleph_2) \twoheadrightarrow (\aleph_2, \aleph_1)$? Or $(\aleph_3, \aleph_2, \aleph_1) \twoheadrightarrow (\aleph_2, \aleph_1, \aleph_0)$?

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Definition (Kunen, 1972)

Let κ be a cardinal. Then κ is *huge* iff there is a $j : V \prec M$ with $\mathrm{crt}(j) = \kappa$, ${}^{j(\kappa)}M \subseteq M$.

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Let $j : V \prec M$ with $\text{crt}(j) = \kappa$. We define the critical sequence $\langle \kappa_0, \kappa_1, \dots \rangle$ as $\kappa_0 = \kappa$ and $j(\kappa_n) = \kappa_{n+1}$.

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Let κ be a cardinal. Then κ is *ω-huge* or *Reinhardt* iff there is a $j : V \prec M$ with $\mathrm{crt}(j) = \kappa_0$, ${}^{\lambda}M \subseteq M$, with $\lambda = \sup_{n \in \omega} \kappa_n$.

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Theorem (Kunen, 1971)

There is no Reinhardt cardinal.

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One can extend the definition to all the large cardinals above: generic γ-supercompact, generic huge, generic n-huge

Theorem (Laver)

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Proposition

If $j : V \prec M \subseteq V[G]$, M closed under $\aleph_3$-sequences, $\mathrm{crt}(j) = \aleph_2$ and $j(\aleph_2) = \aleph_3$, then $(\aleph_3, \aleph_2) \twoheadrightarrow (\aleph_2, \aleph_1)$.

In the same way

Proposition

If $j : V \prec M \subseteq V[G]$, M closed under $\aleph_{n+1}$-sequences, $\mathrm{crt}(j) = \aleph_1$ and $j(\aleph_1) = \aleph_2$, $j(\aleph_2) = \aleph_3, \ldots,$ then $(\aleph_{n+1}, \ldots, \aleph_2, \aleph_1) \twoheadrightarrow (\aleph_n, \ldots, \aleph_1, \aleph_0)$.

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Theorem (Silver)

$\aleph_\omega$ is Jónsson iff there are $k_n \in \{\aleph_m : m \in \omega\}$, strictly increasing, such that $(\ldots, \aleph_{k_2}, \aleph_{k_1}) \to (\ldots, \aleph_{k_1}, \aleph_{k_0})$

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There is no ω-huge (and Shelah proved there is no generic ω-huge)! What can we do?

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With the "right" forcing, generic I1 or I0 at $\aleph_\omega$ implies $\aleph_\omega$ is Jónsson:

Remark

If there exists $j : V \prec M \subseteq V[G]$, $j(\aleph_\omega) = \aleph_\omega$, $j"\aleph_\omega \in M$, then $\aleph_\omega$ is Jónsson.

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Open Problem

What about Con($\aleph_1$ is generic 3-huge cardinal and $\ldots$)?

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If I is precipitous, then there exists $j : V \prec M \subseteq V[G]$, with G.
But not always $M^{<\mathrm{crt}(j)} \subseteq M$, only when I is saturated.

Definition

Suppose GCH below $\aleph_\omega$. We say that generic I0 holds at $\aleph_\omega$ if there exists a forcing notion $\mathbb{P}$ and a generic G such that:

1. in $V[G]$ there exists $j : L(\mathcal{P}(\aleph_\omega)) \prec L(\mathcal{P}(\aleph_\omega))[G]$;
2. $\mathbb{P} \in L(\mathcal{P}(\aleph_\omega))$ and in $L(\mathcal{P}(\aleph_\omega))$ there exists $\pi : \mathcal{P}(\aleph_\omega) \twoheadrightarrow \mathbb{P}$;
3. $\aleph_\omega^V = \aleph_\omega^{V[G]}$;
4. every element of $\mathcal{P}(\aleph_\omega)^{V[G]}$ has a name (coded) in $\mathcal{P}(\aleph_\omega)$;
5. there is a $\mathbb{P}$-term for $H^{V^{\mathbb{P}}}(\aleph_\omega)$ and $j \restriction H(\aleph_\omega) : H(\aleph_\omega) \prec H^{V^{\mathbb{P}}}(\aleph_\omega)$

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Examples: $\mathbb{P} = \mathrm{Coll}(\aleph_3, \aleph_2)$, $\mathbb{P} =$ product of $\mathbb{P}_n$, where $\mathbb{P}_n = \mathrm{Coll}(\aleph_{n+3}, \aleph_{n+2})$.

Definition

$$\Theta = \sup\{\alpha : \exists \pi : \mathcal{P}(\aleph_\omega) \twoheadrightarrow \alpha,\ \pi \in L(\mathcal{P}(\aleph_\omega))\}$$

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1. $\aleph_{\omega+1}$ is measurable (in fact ω-strongly measurable);
2. Θ is weakly inaccessible;
3. Θ is limit of measurable cardinals.

Confront this with:

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So:

- Either generic I0 is consistent, and then pcf-theory without AC has some serious limits;
- or generic I0 is inconsistent, and that would put a shadow on the consistency of I0.

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Definition

Define $D(\lambda)$ as the following: in $L(\mathcal{P}(\lambda))$:

- λ^+ is measurable;
- Θ is a weakly inaccessible limit of measurable cardinals.

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Theorem (Woodin)

$I0(\lambda) \to D(\lambda)$.

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If λ has uncountable cofinality, then $L(\mathcal{P}(\lambda)) \vDash$ AC, therefore $\neg D(\lambda)$

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Theorem (Shelah, 1996)

If λ has uncountable cofinality, then $L(\mathcal{P}(\lambda)) \models$ AC, therefore $\neg D(\lambda)$.

Theorem

In the Mitchell-Steel core model, if λ is singular, then $L(\mathcal{P}(\lambda)) \models$ AC, therefore $\neg D(\lambda)$.

Conjecture (Woodin)

In Ultimate L, internal $I0(\lambda)$ iff $L(\mathcal{P}(\lambda)) \nvDash$ AC

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In Ultimate L, internal $I0(\lambda)$ iff $L(\mathcal{P}(\lambda)) \not\models$ AC.

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What is the consistency strength of $D(\lambda)$ with λ uncountable?

Thanks for your attention.