Non proper elementary embeddings beyond $L(V_{\lambda+1})$

Vincenzo Dimonte

19 June 2009

ACHTUNG!

The following seminar talk will not contain forcing.
We apologize for the inconvenience.

Chart of Cardinals

The arrows indicates direct implications or relative consistency implications, often both.

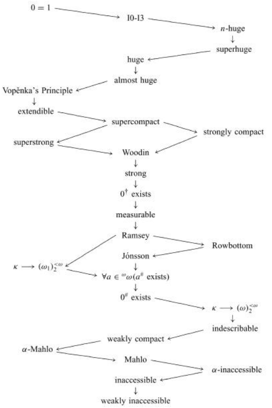
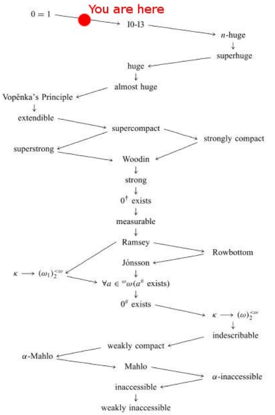


Chart of Cardinals

The arrows indicates direct implications or relative consistency implications, often both.



Reinhardt Hypothesis: there exists an elementary embedding $j : V \prec V$.

Reinhardt Hypothesis: there exists an elementary embedding $j : V \prec V$.

It's a natural strengthening of the hypotheses with a $j : V \prec M$.

Reinhardt Hypothesis: there exists an elementary embedding $j : V \prec V$.

It's a natural strengthening of the hypotheses with a $j : V \prec M$.

Theorem (Kunen, 1971)

If $j : V \prec M$, then $M \neq V$.

Reinhardt Hypothesis: there exists an elementary embedding $j : V \prec V$.

It's a natural strengthening of the hypotheses with a $j : V \prec M$.

Theorem (Kunen, 1971)

If $j : V \prec M$, then $M \neq V$.

The *critical sequence* has an important role in the proof:

Reinhardt Hypothesis: there exists an elementary embedding $j : V \prec V$.

It's a natural strengthening of the hypotheses with a $j : V \prec M$.

Theorem (Kunen, 1971)

If $j : V \prec M$, then $M \neq V$.

The *critical sequence* has an important role in the proof:

Definition

$\kappa_0 = crit(j)$, $\kappa_{n+1} = j(\kappa_n)$,

Reinhardt Hypothesis: there exists an elementary embedding $j : V \prec V$.

It's a natural strengthening of the hypotheses with a $j : V \prec M$.

Theorem (Kunen, 1971)

If $j : V \prec M$, then $M \neq V$.

The *critical sequence* has an important role in the proof:

Definition

$\kappa_0 = crit(j)$, $\kappa_{n+1} = j(\kappa_n)$, $\lambda = \sup_{n \in \omega} \kappa_n$.

Reinhardt Hypothesis: there exists an elementary embedding $j : V \prec V$.

It's a natural strengthening of the hypotheses with a $j : V \prec M$.

Theorem (Kunen, 1971)

If $j : V \prec M$, then $M \neq V$.

The *critical sequence* has an important role in the proof:

Definition

$\kappa_0 = crit(j)$, $\kappa_{n+1} = j(\kappa_n)$, $\lambda = \sup_{n\in\omega} \kappa_n$.

Kunen's proof uses a choice function that is in $V_{\lambda+2}$. So

Corollary

There is no $j : V_\eta \prec V_\eta$,

Reinhardt Hypothesis: there exists an elementary embedding $j : V \prec V$.

It's a natural strengthening of the hypotheses with a $j : V \prec M$.

Theorem (Kunen, 1971)

If $j : V \prec M$, then $M \neq V$.

The *critical sequence* has an important role in the proof:

Definition

$\kappa_0 = crit(j)$, $\kappa_{n+1} = j(\kappa_n)$, $\lambda = \sup_{n\in\omega} \kappa_n$.

Kunen's proof uses a choice function that is in $V_{\lambda+2}$. So

Corollary

There is no $j : V_\eta \prec V_\eta$, with $\eta \geq \lambda + 2$.

It's quite natural to define the following Axioms

It's quite natural to define the following Axioms

Definition

- I3: There exists an elementary embedding $j : V_\lambda \prec V_\lambda$.

It's quite natural to define the following Axioms

Definition

- I3: There exists an elementary embedding $j : V_\lambda \prec V_\lambda$.
- I1: There exists an elementary embedding $j : V_{\lambda+1} \prec V_{\lambda+1}$.

It's quite natural to define the following Axioms

Definition

- I3: There exists an elementary embedding $j : V_\lambda \prec V_\lambda$.
- I1: There exists an elementary embedding $j : V_{\lambda+1} \prec V_{\lambda+1}$.

Technical note: if $j, k : V_{\lambda+1} \prec V_{\lambda+1}$ and $j \upharpoonright V_\lambda = k \upharpoonright V_\lambda$, then $j = k$.

Woodin proposed an even stronger axiom:

Definition

I0: There exists an elementary embedding $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$.

Woodin proposed an even stronger axiom:

Definition

I0: There exists an elementary embedding $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\text{crt}(j) < \lambda$.

This axiom is more interesting, since it produces a structure on $L(V_{\lambda+1})$ that is strikingly similar to the structure of $L(\mathbb{R})$ under AD.

Woodin proposed an even stronger axiom:

Definition

I0: There exists an elementary embedding $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\text{crt}(j) < \lambda$.

This axiom is more interesting, since it produces a structure on $L(V_{\lambda+1})$ that is strikingly similar to the structure of $L(\mathbb{R})$ under AD.

Since λ has cofinality ω, V_λ is similar to V_ω, so $V_{\lambda+1}$ is similar to $\mathbb{R}$.

First degree analogies (without I0 and AD):

First degree analogies (without I0 and AD):

Let $\Theta^{L(V_{\lambda+1})}_{V_{\lambda+1}}$ be the supremum of the α's such that in $L(V_{\lambda+1})$ there exists a surjection $\pi : V_{\lambda+1} \twoheadrightarrow \alpha$.

First degree analogies (without I0 and AD):

Let $\Theta^{L(V_{\lambda+1})}_{V_{\lambda+1}}$ be the supremum of the α's such that in $L(V_{\lambda+1})$ there exists a surjection $\pi : V_{\lambda+1} \twoheadrightarrow \alpha$.

$L(\mathbb{R})$	$L(V_{\lambda+1})$

First degree analogies (without I0 and AD):

Let $\Theta^{L(V_{\lambda+1})}_{V_{\lambda+1}}$ be the supremum of the α's such that in $L(V_{\lambda+1})$ there exists a surjection $\pi : V_{\lambda+1} \twoheadrightarrow \alpha$.

$L(\mathbb{R})$	$L(V_{\lambda+1})$
Θ is regular	

First degree analogies (without I0 and AD):

Let $\Theta^{L(V_{\lambda+1})}_{V_{\lambda+1}}$ be the supremum of the α's such that in $L(V_{\lambda+1})$ there exists a surjection $\pi : V_{\lambda+1} \twoheadrightarrow \alpha$.

$L(\mathbb{R})$	$L(V_{\lambda+1})$
Θ is regular	$\Theta^{L(V_{\lambda+1})}_{V_{\lambda+1}}$ is regular

First degree analogies (without I0 and AD):

Let $\Theta^{L(V_{\lambda+1})}_{V_{\lambda+1}}$ be the supremum of the α's such that in $L(V_{\lambda+1})$ there exists a surjection $\pi : V_{\lambda+1} \twoheadrightarrow \alpha$.

$L(\mathbb{R})$	$L(V_{\lambda+1})$
Θ is regular	$\Theta^{L(V_{\lambda+1})}_{V_{\lambda+1}}$ is regular
DC holds	

First degree analogies (without I0 and AD):

Let $\Theta^{L(V_{\lambda+1})}_{V_{\lambda+1}}$ be the supremum of the α's such that in $L(V_{\lambda+1})$ there exists a surjection $\pi : V_{\lambda+1} \twoheadrightarrow \alpha$.

$L(\mathbb{R})$	$L(V_{\lambda+1})$
Θ is regular	$\Theta^{L(V_{\lambda+1})}_{V_{\lambda+1}}$ is regular
DC holds	DC_λ holds.

First degree analogies (without I0 and AD):

Let $\Theta^{L(V_{\lambda+1})}_{V_{\lambda+1}}$ be the supremum of the α's such that in $L(V_{\lambda+1})$ there exists a surjection $\pi : V_{\lambda+1} \twoheadrightarrow \alpha$.

$L(\mathbb{R})$	$L(V_{\lambda+1})$
Θ is regular	$\Theta^{L(V_{\lambda+1})}_{V_{\lambda+1}}$ is regular
DC holds	DC_λ holds.

In fact these analogies hold for every model of $\mathrm{HOD}_{V_{\lambda+1}}$.

Second degree analogies (under I0 and AD):

Second degree analogies (under I0 and AD):

$L(\mathbb{R})$ under AD	$L(V_{\lambda+1})$ under I0

Second degree analogies (under I0 and AD):

$L(\mathbb{R})$ under AD	$L(V_{\lambda+1})$ under I0
ω_1 is measurable	

Second degree analogies (under I0 and AD):

$L(\mathbb{R})$ under AD	$L(V_{\lambda+1})$ under I0
ω_1 is measurable	λ^+ is measurable

Second degree analogies (under I0 and AD):

$L(\mathbb{R})$ under AD	$L(V_{\lambda+1})$ under I0
ω_1 is measurable	λ^+ is measurable
the Coding Lemma holds	

Second degree analogies (under I0 and AD):

$L(\mathbb{R})$ under AD	$L(V_{\lambda+1})$ under I0
ω_1 is measurable	λ^+ is measurable
the Coding Lemma holds	the Coding Lemma holds.

Second degree analogies (under I0 and AD):

$L(\mathbb{R})$ under AD	$L(V_{\lambda+1})$ under I0
ω_1 is measurable	λ^+ is measurable
the Coding Lemma holds	the Coding Lemma holds.

The most immediate corollary for the Coding Lemma is: For every $\alpha < \Theta$

Second degree analogies (under I0 and AD):

$L(\mathbb{R})$ under AD	$L(V_{\lambda+1})$ under I0
ω_1 is measurable	λ^+ is measurable
the Coding Lemma holds	the Coding Lemma holds.

The most immediate corollary for the Coding Lemma is:
For every $\alpha < \Theta$ there exists a surjection $\pi : \mathbb{R} \twoheadrightarrow \mathcal{P}(\alpha)$.

Second degree analogies (under I0 and AD):

$L(\mathbb{R})$ under AD	$L(V_{\lambda+1})$ under I0
ω_1 is measurable	λ^+ is measurable
the Coding Lemma holds	the Coding Lemma holds.

The most immediate corollary for the Coding Lemma is:
For every $\alpha < \Theta$ there exists a surjection $\pi : \mathbb{R} \twoheadrightarrow \mathcal{P}(\alpha)$.

Bonus result: Let $S_\delta^{\lambda^+}$ be the set of the ordinals in λ^+ with cofinality δ.

Second degree analogies (under I0 and AD):

$L(\mathbb{R})$ under AD	$L(V_{\lambda+1})$ under I0
ω_1 is measurable	λ^+ is measurable
the Coding Lemma holds	the Coding Lemma holds.

The most immediate corollary for the Coding Lemma is:
For every $\alpha < \Theta$ there exists a surjection $\pi : \mathbb{R} \twoheadrightarrow \mathcal{P}(\alpha)$.

Bonus result: Let $S^{\lambda^+}_\delta$ be the set of the ordinals in λ^+ with cofinality δ. Then there exists a partition $\langle S_\alpha : \alpha < \eta \rangle$ of $S^{\lambda^+}_\delta$ in $\eta < \lambda$ stationary sets such that for every $\alpha < \eta$ the club filter of λ^+ on S_α is an ultrafilter.

Third degree analogy:

Theorem

Third degree analogy:

Theorem

Suppose that there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$.

Third degree analogy:

Theorem

Suppose that there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$. Then Θ is a limit of γ such that:

- γ is weakly inaccessible in $L(V_{\lambda+1})$;

Third degree analogy:

Theorem

Suppose that there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$. Then Θ is a limit of γ such that:

- γ is weakly inaccessible in $L(V_{\lambda+1})$;
- $\gamma = \Theta^{L_\gamma(V_{\lambda+1})}$ and $j(\gamma) = \gamma$;

Third degree analogy:

Theorem

Suppose that there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$. Then Θ is a limit of γ such that:

- γ is weakly inaccessible in $L(V_{\lambda+1})$;
- $\gamma = \Theta^{L_\gamma(V_{\lambda+1})}$ and $j(\gamma) = \gamma$;
- for all $\beta < \gamma$, $\mathcal{P}(\beta) \cap L(V_{\lambda+1}) \in L_\gamma(V_{\lambda+1})$;

Third degree analogy:

Theorem

Suppose that there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$. Then Θ is a limit of γ such that:

- γ is weakly inaccessible in $L(V_{\lambda+1})$;
- $\gamma = \Theta^{L_\gamma(V_{\lambda+1})}$ and $j(\gamma) = \gamma$;
- for all $\beta < \gamma$, $\mathcal{P}(\beta) \cap L(V_{\lambda+1}) \in L_\gamma(V_{\lambda+1})$;
- for cofinally $\kappa < \gamma$, κ is a measurable cardinal in $L(V_{\lambda+1})$ and this is witnessed by the club filter on a stationary set;

Third degree analogy:

Theorem

Suppose that there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$. Then Θ is a limit of γ such that:

- γ is weakly inaccessible in $L(V_{\lambda+1})$;
- $\gamma = \Theta^{L_\gamma(V_{\lambda+1})}$ and $j(\gamma) = \gamma$;
- for all $\beta < \gamma$, $\mathcal{P}(\beta) \cap L(V_{\lambda+1}) \in L_\gamma(V_{\lambda+1})$;
- for cofinally $\kappa < \gamma$, κ is a measurable cardinal in $L(V_{\lambda+1})$ and this is witnessed by the club filter on a stationary set;
- $L_\gamma(V_{\lambda+1}) \prec L_\Theta(V_{\lambda+1})$.

I0 is called Higher Determinacy Axiom, because it has consequences similar to Determinacy, but in a larger model.

I0 is called Higher Determinacy Axiom, because it has consequences similar to Determinacy, but in a larger model. Is it possible to find stronger Higher Determinacy Axioms?

I0 is called Higher Determinacy Axiom, because it has consequences similar to Determinacy, but in a larger model. Is it possible to find stronger Higher Determinacy Axioms? We will consider elementary embeddings between two kind of models:

- $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$, with $X \subset V_{\lambda+1}$;

I0 is called Higher Determinacy Axiom, because it has consequences similar to Determinacy, but in a larger model. Is it possible to find stronger Higher Determinacy Axioms? We will consider elementary embeddings between two kind of models:

- $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$, with $X \subset V_{\lambda+1}$;
- $j : L(N) \prec L(N)$, with $V_{\lambda+1} \subset N \subset V_{\lambda+2}$ and $N = L(N) \cap V_{\lambda+2}$.

In the first case, the first and second degree analogies hold.

In the first case, the first and second degree analogies hold. However, the third analogy resisted all attempts to be proved, without further hypotheses.

In the first case, the first and second degree analogies hold. However, the third analogy resisted all attempts to be proved, without further hypotheses.

Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$.

In the first case, the first and second degree analogies hold. However, the third analogy resisted all attempts to be proved, without further hypotheses.

Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$.

Then

$$U_j = \{Z \in L(X, V_{\lambda+1}) \cap V_{\lambda+2} : j \upharpoonright V_\lambda \in j(Z)\}$$

generates an elementary embedding j_U,

In the first case, the first and second degree analogies hold. However, the third analogy resisted all attempts to be proved, without further hypotheses.

Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$.

Then

$$U_j = \{Z \in L(X, V_{\lambda+1}) \cap V_{\lambda+2} : j \upharpoonright V_\lambda \in j(Z)\}$$

generates an elementary embedding j_U, and there exists a $k_U : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $\mathrm{crt}(k_U) > \Theta$ such that $j = k_U \circ j_U$.

In the first case, the first and second degree analogies hold. However, the third analogy resisted all attempts to be proved, without further hypotheses.

Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$.

Then

$$U_j = \{Z \in L(X, V_{\lambda+1}) \cap V_{\lambda+2} : j \upharpoonright V_\lambda \in j(Z)\}$$

generates an elementary embedding j_U, and there exists a $k_U : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $\mathrm{crt}(k_U) > \Theta$ such that $j = k_U \circ j_U$.

So the "important part" of j is under $L_\Theta(X, V_{\lambda+1})$.

We all know about the richness of life when we reach twenty. I think after that things we learn about life do not add up much to it, and I think that once we formed [...] a map of humanity in our mind, [...] understanding of humanity doesn't change much.

Orhan Pamuk

We all know about the richness of life when we reach twenty. I think after that things we learn about life do not add up much to it, and I think that once we formed [...] a map of humanity in our mind, [...] understanding of humanity doesn't change much.

Orhan Pamuk

Passé la puberté, tout le reste n'est qu'un épilogue. (From puberty onwards, life is just an epilogue)

Amélie Nothomb, Le sabotage amoureux

We all know about the richness of life when we reach twenty. I think after that things we learn about life do not add up much to it, and I think that once we formed [...] a map of humanity in our mind, [...] understanding of humanity doesn't change much.

Orhan Pamuk

Passé la puberté, tout le reste n'est qu'un épilogue. (From puberty onwards, life is just an epilogue)

Amélie Nothomb, Le sabotage amoureux (maybe)

We all know about the richness of life when we reach twenty. I think after that things we learn about life do not add up much to it, and I think that once we formed [. . .] a map of humanity in our mind, [. . .] understanding of humanity doesn't change much.

Orhan Pamuk

Passé la puberté, tout le reste n'est qu'un épilogue. (From puberty onwards, life is just an epilogue)

Amélie Nothomb, Le sabotage amoureux (maybe)

Proof by Woodin.

If $j, k : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ and $j \restriction V_\lambda = k \restriction V_\lambda$, then $j \restriction L_\Theta(V_{\lambda+1}) = k \restriction L_\Theta(V_{\lambda+1})$. □

Definition

Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$. Then j is *weakly proper* iff $j = j_U$.

Definition

Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$. Then j is *weakly proper* iff $j = j_U$.

Definition

Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$. Then j is *proper* if it is weakly proper

Definition

Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$. Then j is *weakly proper* iff $j = j_U$.

Definition

Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$. Then j is *proper* if it is weakly proper and $\langle X, j(X), j(j(X)), \dots \rangle \in L(X, V_{\lambda+1})$.

Definition

Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$. Then j is *weakly proper* iff $j = j_U$.

Definition

Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$. Then j is *proper* if it is weakly proper and $\langle X, j(X), j(j(X)), \dots \rangle \in L(X, V_{\lambda+1})$.

If j is proper, then the third degree analogies hold.

Definition

Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$. Then j is *weakly proper* iff $j = j_U$.

Definition

Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$. Then j is *proper* if it is weakly proper and $\langle X, j(X), j(j(X)), \dots \rangle \in L(X, V_{\lambda+1})$.

If j is proper, then the third degree analogies hold.

The second case is more complicated. It can be even that the first degree analogy doesn't hold.

Definition

Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$. Then j is *weakly proper* iff $j = j_U$.

Definition

Let $j : L(X, V_{\lambda+1}) \prec L(X, V_{\lambda+1})$ with $X \subseteq V_{\lambda+1}$. Then j is *proper* if it is weakly proper and $\langle X, j(X), j(j(X)), \dots \rangle \in L(X, V_{\lambda+1})$.

If j is proper, then the third degree analogies hold.
The second case is more complicated. It can be even that the first degree analogy doesn't hold.
But if we have that $L(N) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$, then the first and second degree analogy hold.

A similar ultrapower theorem exists, and we define similarly weakly proper embeddings

A similar ultrapower theorem exists, and we define similarly weakly proper embeddings.

Definition

Let $j : L(N) \prec L(N)$ with $V_{\lambda+1} \subset N \subset V_{\lambda+2}$ and $N = L(N) \cap V_{\lambda+2}$. Then j is *proper* if it is weakly proper and for every $X \in N$ $\langle X, j(X), j(j(X)), \ldots \rangle \in L(N)$.

A similar ultrapower theorem exists, and we define similarly weakly proper embeddings.

Definition

Let $j : L(N) \prec L(N)$ with $V_{\lambda+1} \subset N \subset V_{\lambda+2}$ and $N = L(N) \cap V_{\lambda+2}$. Then j is *proper* if it is weakly proper and for every $X \in N$ $\langle X, j(X), j(j(X)), \dots \rangle \in L(N)$.

And if j is proper, then the third degree analogy hold.

A similar ultrapower theorem exists, and we define similarly weakly proper embeddings.

Definition

Let $j : L(N) \prec L(N)$ with $V_{\lambda+1} \subset N \subset V_{\lambda+2}$ and $N = L(N) \cap V_{\lambda+2}$. Then j is *proper* if it is weakly proper and for every $X \in N$ $\langle X, j(X), j(j(X)), \dots \rangle \in L(N)$.

And if j is proper, then the third degree analogy hold.

Now we have to define new axioms of this kind, with the ultimate purpose of finding an analogous of $\mathrm{AD}_{\mathbb{R}}$.

A similar ultrapower theorem exists, and we define similarly weakly proper embeddings.

Definition

Let $j : L(N) \prec L(N)$ with $V_{\lambda+1} \subset N \subset V_{\lambda+2}$ and $N = L(N) \cap V_{\lambda+2}$. Then j is *proper* if it is weakly proper and for every $X \in N$ $\langle X, j(X), j(j(X)), \dots \rangle \in L(N)$.

And if j is proper, then the third degree analogy hold.

Now we have to define new axioms of this kind, with the ultimate purpose of finding an analogous of $AD_{\mathbb{R}}$.

There is no evident elementary embedding form...

A similar ultrapower theorem exists, and we define similarly weakly proper embeddings.

Definition

Let $j : L(N) \prec L(N)$ with $V_{\lambda+1} \subset N \subset V_{\lambda+2}$ and $N = L(N) \cap V_{\lambda+2}$. Then j is *proper* if it is weakly proper and for every $X \in N$ $\langle X, j(X), j(j(X)), \dots \rangle \in L(N)$.

And if j is proper, then the third degree analogy hold.

Now we have to define new axioms of this kind, with the ultimate purpose of finding an analogous of $AD_{\mathbb{R}}$.

There is no evident elementary embedding form... so the way chose by Woodin is defining an analogous of the minimum model of $AD_{\mathbb{R}}$.

Definition

Define a sequence of $\Gamma_\alpha \subseteq \mathcal{P}(\mathbb{R})$ by induction on α:

Definition

Define a sequence of $\Gamma_\alpha \subseteq \mathcal{P}(\mathbb{R})$ by induction on α:

- $\Gamma_0 = L(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$;

Definition

Define a sequence of $\Gamma_\alpha \subseteq \mathcal{P}(\mathbb{R})$ by induction on α:

- $\Gamma_0 = L(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$;
- If α is a limit ordinal then $\Gamma_\alpha = L((\bigcup_{\beta<\alpha} \Gamma_\beta)^\omega) \cap \mathcal{P}(\mathbb{R})$;

Definition

Define a sequence of $\Gamma_\alpha \subseteq \mathcal{P}(\mathbb{R})$ by induction on α:

- $\Gamma_0 = L(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$;
- If α is a limit ordinal then $\Gamma_\alpha = L((\bigcup_{\beta<\alpha} \Gamma_\beta)^\omega) \cap \mathcal{P}(\mathbb{R})$;
- If $\mathrm{cof}(\Theta^{L(\Gamma_\alpha)}) = \omega$, then $\Gamma_{\alpha+1} = L((\Gamma_\alpha)^\omega, \mathbb{R}) \cap \mathcal{P}(\mathbb{R})$,

Definition

Define a sequence of $\Gamma_\alpha \subseteq \mathcal{P}(\mathbb{R})$ by induction on α:

- $\Gamma_0 = L(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$;
- If α is a limit ordinal then $\Gamma_\alpha = L((\bigcup_{\beta<\alpha} \Gamma_\beta)^\omega) \cap \mathcal{P}(\mathbb{R})$;
- If $\mathrm{cof}(\Theta^{L(\Gamma_\alpha)}) = \omega$, then $\Gamma_{\alpha+1} = L((\Gamma_\alpha)^\omega, \mathbb{R}) \cap \mathcal{P}(\mathbb{R})$, otherwise $\Gamma_{\alpha+1} = L(\Gamma_\alpha)[\mathcal{F}] \cap \mathcal{P}(\mathbb{R})$, where $\mathcal{F}$ is the ω-club filter in $\Theta^{L(\Gamma_\alpha)}$.

Definition

Define a sequence of $\Gamma_\alpha \subseteq \mathcal{P}(\mathbb{R})$ by induction on α:

- $\Gamma_0 = L(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$;
- If α is a limit ordinal then $\Gamma_\alpha = L((\bigcup_{\beta<\alpha} \Gamma_\beta)^\omega) \cap \mathcal{P}(\mathbb{R})$;
- If $\operatorname{cof}(\Theta^{L(\Gamma_\alpha)}) = \omega$, then $\Gamma_{\alpha+1} = L((\Gamma_\alpha)^\omega, \mathbb{R}) \cap \mathcal{P}(\mathbb{R})$, otherwise $\Gamma_{\alpha+1} = L(\Gamma_\alpha)[\mathcal{F}] \cap \mathcal{P}(\mathbb{R})$, where $\mathcal{F}$ is the ω-club filter in $\Theta^{L(\Gamma_\alpha)}$.

The sequence stops when $L(\Gamma_\alpha) \not\models \mathsf{AD}$ or $\Gamma_\alpha = \Gamma_{\alpha+1}$

Definition

The sequence

$$\langle E^0_\alpha(V_{\lambda+1}) : \alpha < \Upsilon_{V_{\lambda+1}} \rangle$$

is defined as:

- $E^0_0(V_{\lambda+1}) = L(V_{\lambda+1}) \cap V_{\lambda+2}$;
- for α limit, $E^0_\alpha(V_{\lambda+1}) = L(\bigcup_{\beta<\alpha} E^0_\beta(V_{\lambda+1})) \cap V_{\lambda+2}$;
- for α limit,
  - if $(\mathrm{cof}(\Theta^{E^0_\alpha(V_{\lambda+1})}) < \lambda)^{L(E^0_\alpha(V_{\lambda+1}))}$ then
    $$E^0_{\alpha+1}(V_{\lambda+1}) = L((E^0_\alpha(V_{\lambda+1}))^\lambda) \cap V_{\lambda+2};$$
  - if $(\mathrm{cof}(\Theta^{E^0_\alpha(V_{\lambda+1})}))^{L(E^0_\alpha(V_{\lambda+1}))} > \lambda$ then
    $$E^0_{\alpha+1}(V_{\lambda+1}) = L(\mathcal{E}(E^0_\alpha(V_{\lambda+1}))) \cap V_{\lambda+2};$$

Definition

- for $\alpha = \beta + 2$, if there exists $X \subseteq V_{\lambda+1}$ such that $E^0_{\beta+1}(V_{\lambda+1}) = L(X, V_{\lambda+1}) \cap V_{\lambda+2}$ and $E^0_\beta(V_{\lambda+1}) < X$, then
$$E^0_{\beta+2}(V_{\lambda+1}) = L((X, V_{\lambda+1})^\sharp) \cap V_{\lambda+2}$$
otherwise we stop the sequence.
- $\forall\alpha < \Upsilon_{V_{\lambda+1}}$ $\exists X \subseteq V_{\lambda+1}$ such that $E^0_\alpha(V_{\lambda+1}) < X$ and $\exists j\colon L(X, V_{\lambda+1}) \to L(X, V_{\lambda+1})$ proper;
- $\forall\alpha$ limit $\alpha + 1 < \Upsilon_{V_{\lambda+1}}$ iff
$$(\mathrm{cof}(\Theta^{E^0_\alpha(V_{\lambda+1})}))^{L(E^0_\alpha(V_{\lambda+1}))} > \lambda \to$$
$$\exists Z \in E^0_\alpha(V_{\lambda+1})\ L(E^0_\alpha(V_{\lambda+1})) = (\mathrm{HOD}_{V_{\lambda+1}\cup\{Z\}})^{L(E^0_\alpha(V_{\lambda+1}))}.$$

Definition

Let $N = L(\bigcup\{E^0_\alpha(V_{\lambda+1}) : \alpha < \Upsilon_{V_{\lambda+1}}\}) \cap V_{\lambda+2}$. Suppose that

- $\mathrm{cof}(\Theta^N) > \lambda$;
- for all $Z \in N$ $L(N) \neq (\mathrm{HOD}_{V_{\lambda+1}\cup\{Z\}})^{L(N)}$;
- there is an elementary embedding $j : L(N) \prec L(N)$ with $\mathrm{crt}(j) < \lambda$.

Definition

Let $N = L(\bigcup\{E^0_\alpha(V_{\lambda+1}) : \alpha < \Upsilon_{V_{\lambda+1}}\}) \cap V_{\lambda+2}$. Suppose that

- $\mathrm{cof}(\Theta^N) > \lambda$;
- for all $Z \in N$ $L(N) \neq (\mathrm{HOD}_{V_{\lambda+1}\cup\{Z\}})^{L(N)}$;
- there is an elementary embedding $j : L(N) \prec L(N)$ with $\mathrm{crt}(j) < \lambda$.

Then $E^0_\infty(V_{\lambda+1})$ exists and $E^0_\infty(V_{\lambda+1}) = N$.

Three important facts:

Theorem

- If $\alpha < \beta < \Upsilon$, then $\Theta^{E^0_\alpha} < \Theta^{E^0_\beta}$.

Three important facts:

Theorem

- If $\alpha < \beta < \Upsilon$, then $\Theta^{E^0_\alpha} < \Theta^{E^0_\beta}$.
- The E^0_α sequence is absolute, i.e. for every M such that $L(M) \cap V_{\lambda+2}, V_{\lambda+1} \subseteq M$ for every $\alpha < \Upsilon^M$, $(\langle E^0_\beta : \beta < \alpha \rangle)^M = \langle E^0_\beta : \beta < \alpha \rangle$.

Three important facts:

Theorem

- If $\alpha < \beta < \Upsilon$, then $\Theta^{E^0_\alpha} < \Theta^{E^0_\beta}$.
- The E^0_α sequence is absolute, i.e. for every M such that $L(M) \cap V_{\lambda+2}, V_{\lambda+1} \subseteq M$ for every $\alpha < \Upsilon^M$, $(\langle E^0_\beta : \beta < \alpha \rangle)^M = \langle E^0_\beta : \beta < \alpha \rangle$.
- If $\alpha < \Upsilon$, then there exists an elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$.

We've seen that properness is quite important for establishing Determinacy results.

We've seen that properness is quite important for establishing Determinacy results.
But is it really a property?

We've seen that properness is quite important for establishing Determinacy results.
But is it really a property?

Theorem

Suppose $\alpha < \Upsilon$. If

- $\alpha = 0$,

then every weakly proper elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ is proper.

We've seen that properness is quite important for establishing Determinacy results.
But is it really a property?

Theorem

Suppose $\alpha < \Upsilon$. If

- $\alpha = 0$, or
- α is a successor ordinal,

then every weakly proper elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ is proper.

We've seen that properness is quite important for establishing Determinacy results.

But is it really a property?

Theorem

Suppose $\alpha < \Upsilon$. If

- $\alpha = 0$, or
- α is a successor ordinal, or
- α is a limit ordinal with cofinality $> \omega$

then every weakly proper elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ is proper.

We can think of two possible scenarios

We can think of two possible scenarios:

Definition

- α is partially non-proper if there exist $j, k : L(E^0_\alpha) \prec L(E^0_\alpha)$ such that j is proper and k is not proper;

We can think of two possible scenarios:

Definition

- α is partially non-proper if there exist $j, k : L(E^0_\alpha) \prec L(E^0_\alpha)$ such that j is proper and k is not proper;
- α is totally non-proper if every elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ is not proper.

We can think of two possible scenarios:

Definition

- α is partially non-proper if there exist $j, k : L(E^0_\alpha) \prec L(E^0_\alpha)$ such that j is proper and k is not proper;
- α is totally non-proper if every elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ is not proper.

We will prove that both exist.

We can think of two possible scenarios:

Definition

- α is partially non-proper if there exist $j, k : L(E^0_\alpha) \prec L(E^0_\alpha)$ such that j is proper and k is not proper;
- α is totally non-proper if every elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ is not proper.

We will prove that both exist. The key Lemma is the following:

Lemma

Suppose $\alpha < \Upsilon$ and $\Theta^{E^0_\alpha}$ is regular in $L(E^0_\alpha)$. If $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ is proper then the set of fixed points of j is cofinal in $\Theta^{E^0_\alpha}$.

Informal definition of $X^\sharp$:
Suppose that there exists a class I of indiscernibles of $(L(X), \in, \{a : a \in X\}, X)$ such that every cardinal $> |X|$ is in I.
Then $X^\sharp$ is the theory of the indiscernibles in the language $\{\in\} \cup \{a : a \in X\} \cup \{X\}$, i.e.

$$\begin{aligned} X^\sharp = \{\varphi(a_1, \dots, a_n, X, i_1, \dots, i_n) : a_1, \dots, a_n \in X, \\ L(X) \vDash \varphi(a_1, \dots, a_n, X, i_1, \dots, i_n) \text{ for some (any)} \\ \text{indiscernibles } i_1 < \dots < i_n \in I\} \end{aligned}$$

Informal definition of $X^\sharp$:

Suppose that there exists a class I of indiscernibles of $(L(X), \in, \{a : a \in X\}, X)$ such that every cardinal $> |X|$ is in I. Then $X^\sharp$ is the theory of the indiscernibles in the language $\{\in\} \cup \{a : a \in X\} \cup \{X\}$, i.e.

$$X^\sharp = \{\varphi(a_1, \dots, a_n, X, i_1, \dots, i_n) : a_1, \dots, a_n \in X, \\ L(X) \vDash \varphi(a_1, \dots, a_n, X, i_1, \dots, i_n) \text{ for some (any)} \\ \text{indiscernibles } i_1 < \dots < i_n \in I\}$$

$X^\sharp$ contains the "truth" of $L(X)$, so it cannot be in $L(X)$.

In our case, for every α, $(E^0_\alpha)^\sharp \notin L(E^0_\alpha)$.

In our case, for every α, $(E^0_\alpha)^\sharp \notin L(E^0_\alpha)$.
If α is a limit and $E^0_\alpha = \bigcup E^0_\beta$, then we can slice $(E^0_\alpha)^\sharp$ in smaller pieces, digestible by $L(E^0_\alpha)$

In our case, for every α, $(E^0_\alpha)^\sharp \notin L(E^0_\alpha)$.
If α is a limit and $E^0_\alpha = \bigcup E^0_\beta$, then we can slice $(E^0_\alpha)^\sharp$ in smaller pieces, digestible by $L(E^0_\alpha)$:

$$(E^0_\alpha)^\sharp_{\beta,n} = (E^0_\alpha)^\sharp \cap (\{\in\} \cup \{a : a \in E^0_\beta\} \cup \{X\} \cup \{i_1, \dots, i_n\})$$

In our case, for every α, $(E^0_\alpha)^\sharp \notin L(E^0_\alpha)$.
If α is a limit and $E^0_\alpha = \bigcup E^0_\beta$, then we can slice $(E^0_\alpha)^\sharp$ in smaller pieces, digestible by $L(E^0_\alpha)$:

$$(E^0_\alpha)^\sharp_{\beta,n} = (E^0_\alpha)^\sharp \cap (\{\in\} \cup \{a : a \in E^0_\beta\} \cup \{X\} \cup \{i_1, \dots, i_n\})$$

For every $\beta < \alpha$, $n \in \omega$ $(E^0_\alpha)^\sharp_{\beta,n} \in E^0_\alpha$, but $L(E^0_\alpha)$ doesn't know that they are sharp fragments.

In our case, for every α, $(E^0_\alpha)^\sharp \notin L(E^0_\alpha)$.
If α is a limit and $E^0_\alpha = \bigcup E^0_\beta$, then we can slice $(E^0_\alpha)^\sharp$ in smaller pieces, digestible by $L(E^0_\alpha)$:

$$(E^0_\alpha)^\sharp_{\beta,n} = (E^0_\alpha)^\sharp \cap (\{\in\} \cup \{a : a \in E^0_\beta\} \cup \{X\} \cup \{i_1, \dots, i_n\})$$

For every $\beta < \alpha$, $n \in \omega$ $(E^0_\alpha)^\sharp_{\beta,n} \in E^0_\alpha$, but $L(E^0_\alpha)$ doesn't know that they are sharp fragments.
So if $k : E^0_\beta \prec E^0_\alpha$, k(sharp fragment) can be anything.

Definition

We say that $k : E^0_\beta \prec E^0_\alpha$ is *sharp-friendly* if it maps sharp fragments to sharp fragments.

Definition

We say that $k : E^0_\beta \prec E^0_\alpha$ is *sharp-friendly* if it maps sharp fragments to sharp fragments.

Lemma

If $k : E^0_\beta \prec E^0_\alpha$ is sharp-friendly, then it's possible to extend it to $\hat{k} : L(E^0_\beta) \prec L(E^0_\alpha)$.

In this subsection we work in $I = \{\beta < \Upsilon : \forall\gamma < \beta\ L(E^0_\gamma) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}\}$.

In this subsection we work in
$I = \{\beta < \Upsilon : \forall\gamma < \beta\ L(E^0_\gamma) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}\}$.
Beyond $(E^0_\beta)^\sharp_{\gamma,n}$, we can define also $(E^0_\beta)^\sharp_\gamma$, that it's a theory in the language with constants from E^0_γ.

In this subsection we work in
$I = \{\beta < \Upsilon : \forall\gamma < \beta \ L(E^0_\gamma) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}\}$.
Beyond $(E^0_\beta)^\sharp_{\gamma,n}$, we can define also $(E^0_\beta)^\sharp_\gamma$, that it's a theory in the language with constants from E^0_γ.
But $(E^0_\gamma)^\sharp$ is also a theory in that language. What if they are equal, i.e. what if the sharp reflects on γ?

In this subsection we work in
$I=\{\beta<\Upsilon:\forall\gamma<\beta\ L(E^0_\gamma)\vDash V=\mathrm{HOD}_{V_{\lambda+1}}\}$.
Beyond $(E^0_\beta)^\sharp_{\gamma,n}$, we can define also $(E^0_\beta)^\sharp_\gamma$, that it's a theory in the language with constants from E^0_γ.
But $(E^0_\gamma)^\sharp$ is also a theory in that language. What if they are equal, i.e. what if the sharp reflects on γ?
This is something less than asking that $L(E^0_\gamma)\prec L(E^0_\beta)$, but something more than $E^0_\gamma\prec E^0_\beta$.

In this subsection we work in
$I = \{\beta < \Upsilon : \forall\gamma < \beta\ L(E^0_\gamma) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}\}$.
Beyond $(E^0_\beta)^\sharp_{\gamma,n}$, we can define also $(E^0_\beta)^\sharp_\gamma$, that it's a theory in the language with constants from E^0_γ.
But $(E^0_\gamma)^\sharp$ is also a theory in that language. What if they are equal, i.e. what if the sharp reflects on γ?
This is something less than asking that $L(E^0_\gamma) \prec L(E^0_\beta)$, but something more than $E^0_\gamma \prec E^0_\beta$.
In fact, it's equivalent to the sharp-friendliness of the identity from E^0_γ to E^0_β.

Let $\beta \in I$.

Let $\beta \in I$. Define I_β as the set of all γ's such that the sharp in β reflects on γ.

Let $\beta \in I$. Define I_β as the set of all γ's such that the sharp in β reflects on γ.

Lemma

For every $\beta \in I$:

- If $I_\beta \neq \emptyset$ then β is a limit and $\beta = \Theta^{E^0_\beta} = \sup_{\gamma<\beta} \Theta^{E^0_\gamma}$;

Let $\beta \in I$. Define I_β as the set of all γ's such that the sharp in β reflects on γ.

Lemma

For every $\beta \in I$:

- If $I_\beta \neq \emptyset$ then β is a limit and $\beta = \Theta^{E^0_\beta} = \sup_{\gamma<\beta} \Theta^{E^0_\gamma}$;
- if $\gamma \in I_\beta$, then $I_\beta \cap \gamma = I_\gamma$;

Let $\beta \in I$. Define I_β as the set of all γ's such that the sharp in β reflects on γ.

Lemma

For every $\beta \in I$:

- If $I_\beta \neq \emptyset$ then β is a limit and $\beta = \Theta^{E^0_\beta} = \sup_{\gamma<\beta} \Theta^{E^0_\gamma}$;
- if $\gamma \in I_\beta$, then $I_\beta \cap \gamma = I_\gamma$;
- I_β is closed.

The following Lemma is a key point:

The following Lemma is a key point:

Lemma

For every $\gamma < \beta \in I$,

- for every $j : L(E^0_\beta) \prec L(E^0_\beta)$, $j \upharpoonright E^0_\beta$ is sharp-friendly;

The following Lemma is a key point:

Lemma

For every $\gamma < \beta \in I$,

- for every $j : L(E^0_\beta) \prec L(E^0_\beta)$, $j \upharpoonright E^0_\beta$ is sharp-friendly;
- for every $j : L(E^0_\gamma) \prec L(E^0_\beta)$, $j \upharpoonright E^0_\gamma$ is sharp-friendly.

The following Lemma is a key point:

Lemma

For every $\gamma < \beta \in I$,

- for every $j : L(E^0_\beta) \prec L(E^0_\beta)$, $j \upharpoonright E^0_\beta$ is sharp-friendly;
- for every $j : L(E^0_\gamma) \prec L(E^0_\beta)$, $j \upharpoonright E^0_\gamma$ is sharp-friendly.

So every $j : L(E^0_\beta) \prec L(E^0_\beta)$ maps in a good way the initial segments of I_β,

The following Lemma is a key point:

Lemma

For every $\gamma < \beta \in I$,

- for every $j : L(E^0_\beta) \prec L(E^0_\beta)$, $j \restriction E^0_\beta$ is sharp-friendly;
- for every $j : L(E^0_\gamma) \prec L(E^0_\beta)$, $j \restriction E^0_\gamma$ is sharp-friendly.

So every $j : L(E^0_\beta) \prec L(E^0_\beta)$ maps in a good way the initial segments of I_β,
i.e. for every $\gamma \in I_\beta$ $j(\gamma) \in I_\beta$.

The following Lemma is a key point:

Lemma

For every $\gamma < \beta \in I$,

- for every $j : L(E^0_\beta) \prec L(E^0_\beta)$, $j \upharpoonright E^0_\beta$ is sharp-friendly;
- for every $j : L(E^0_\gamma) \prec L(E^0_\beta)$, $j \upharpoonright E^0_\gamma$ is sharp-friendly.

So every $j : L(E^0_\beta) \prec L(E^0_\beta)$ maps in a good way the initial segments of I_β,
i.e. for every $\gamma \in I_\beta$ $j(\gamma) \in I_\beta$. (Note that $j(I_\gamma) = I_{j(\gamma)}$).

Theorem

Let $\beta \in I$ such that $\mathrm{ot}(I_\beta) = \lambda$.

Theorem

Let $\beta \in I$ such that $\mathrm{ot}(I_\beta) = \lambda$. Then β is totally non-proper.

Theorem

Let $\beta \in I$ such that $\text{ot}(I_\beta) = \lambda$. Then β is totally non-proper.

Proof.

Since I_β is closed, $\sup I_\beta = \beta = \Theta^{E^0_\beta}$.

Theorem

Let $\beta \in I$ such that $\mathrm{ot}(I_\beta) = \lambda$. Then β is totally non-proper.

Proof.

Since I_β is closed, $\sup I_\beta = \beta = \Theta^{E^0_\beta}$. Define γ_n as the κ_n-th element of I_β.

Theorem

Let $\beta \in I$ such that $\mathrm{ot}(I_\beta) = \lambda$. Then β is totally non-proper.

Proof.

Since I_β is closed, $\sup I_\beta = \beta = \Theta^{E^0_\beta}$. Define γ_n as the κ_n-th element of I_β. So $j(\gamma_n) = \gamma_{n+1}$ (we can see γ_n as the κ_n-th element of $I_{\gamma_{n+2}}$), and j cannot be proper. □

By the Lemma above, we only have to find an α such that we know that there exists a proper elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$, By the Lemma above, we only have to find an α such that we know that there exists a proper elementary embedding $j : L(E^0_\alpha) \prec L(E^0_\alpha)$, and a sharp-friendly elementary embedding $k : E^0_\alpha \prec E^0_\alpha$ whose extension is not proper.

Define the game G_α in $L((E^0_\alpha)^\sharp)$:

$$\begin{array}{lccccc} I & \langle k_0,\beta_0\rangle & & \langle k_1,\beta_1\rangle & & \langle k_2,\beta_2\rangle \quad \dots \\ II & & \eta_0 & & \eta_1 & \end{array}$$

with the following rules:

- $k_0=\emptyset$;
- $k_{i+1}: E^0_{\beta_i}\prec E^0_{\beta_{i+1}}$;
- for every $\gamma<\beta_i$, $k_{i+1}((E^0_\alpha)^\sharp_{\gamma,n})=(E^0_\alpha)^\sharp_{k_{i+1}(\gamma),n}$;
- $\beta_i,\eta_i<\alpha$;
- $\beta_{i+1}>\eta_i$;
- $k_i\subseteq k_{i+1}$ and $k_{i+1}(\beta_i)=\beta_{i+1}$;
- II wins if and only if I at a certain point can't play anymore.

So we have to find an α such that $\alpha = \Theta^{E^0_\alpha}$, $\mathrm{cof}(\alpha) = \omega$ and G_α is determined for I.

So we have to find an α such that $\alpha = \Theta^{E^0_\alpha}$, $\mathrm{cof}(\alpha) = \omega$ and G_α is determined for I. Let $\xi < \Upsilon$ and define the *closed* initial segment

$$H_\xi = \{\gamma \leq \xi : E^0_\gamma \subseteq (\mathrm{HOD}_{V_{\lambda+1}})^{L(E^0_\xi)}\}.$$

So we have to find an α such that $\alpha = \Theta^{E^0_\alpha}$, $\mathrm{cof}(\alpha) = \omega$ and G_α is determined for I. Let $\xi < \Upsilon$ and define the *closed* initial segment

$$H_\xi = \{\gamma \leq \xi : E^0_\gamma \subseteq (\mathrm{HOD}_{V_{\lambda+1}})^{L(E^0_\xi)}\}.$$

Lemma

Let $\xi < \Upsilon$. Let $\eta = \sup H_\xi$.

So we have to find an α such that $\alpha = \Theta^{E^0_\alpha}$, $\mathrm{cof}(\alpha) = \omega$ and G_α is determined for I. Let $\xi < \Upsilon$ and define the *closed* initial segment

$$H_\xi = \{\gamma \leq \xi : E^0_\gamma \subseteq (\mathrm{HOD}_{V_{\lambda+1}})^{L(E^0_\xi)}\}.$$

Lemma

Let $\xi < \Upsilon$. Let $\eta = \sup H_\xi$. If $\eta < \xi$, then η is a limit ordinal

So we have to find an α such that $\alpha = \Theta^{E^0_\alpha}$, $\mathrm{cof}(\alpha) = \omega$ and G_α is determined for I. Let $\xi < \Upsilon$ and define the *closed* initial segment

$$H_\xi = \{\gamma \leq \xi : E^0_\gamma \subseteq (\mathrm{HOD}_{V_{\lambda+1}})^{L(E^0_\xi)}\}.$$

Lemma

Let $\xi < \Upsilon$. Let $\eta = \sup H_\xi$. If $\eta < \xi$, then η is a limit ordinal and

- $\eta = \Theta^{E^0_\eta} = \Theta^{(\mathrm{HOD}_{V_{\lambda+1}})^{E^0_\xi}}$;

So we have to find an α such that $\alpha = \Theta^{E^0_\alpha}$, $\mathrm{cof}(\alpha) = \omega$ and G_α is determined for I. Let $\xi < \Upsilon$ and define the *closed* initial segment

$$H_\xi = \{\gamma \leq \xi : E^0_\gamma \subseteq (\mathrm{HOD}_{V_{\lambda+1}})^{L(E^0_\xi)}\}.$$

Lemma

Let $\xi < \Upsilon$. Let $\eta = \sup H_\xi$. If $\eta < \xi$, then η is a limit ordinal and

- $\eta = \Theta^{E^0_\eta} = \Theta^{(\mathrm{HOD}_{V_{\lambda+1}})^{E^0_\xi}}$;
- $L((E^0_\eta)^\sharp) \cap V_{\lambda+2} = E^0_\eta$.

Notation

From now on, we call α the minimum ordinal such that $L((E^0_\alpha)^\sharp) \cap V_{\lambda+2} = E^0_\alpha$.

Notation

From now on, we call α the minimum ordinal such that $L((E^0_\alpha)^\sharp) \cap V_{\lambda+2} = E^0_\alpha$.

Then both $L(E^0_\alpha)$ and $L((E^0_\alpha)^\sharp)$ have good qualities, and α is "large" in $L((E^0_\alpha)^\sharp)$.

Notation

From now on, we call α the minimum ordinal such that $L((E^0_\alpha)^\sharp) \cap V_{\lambda+2} = E^0_\alpha$.

Then both $L(E^0_\alpha)$ and $L((E^0_\alpha)^\sharp)$ have good qualities, and α is “large” in $L((E^0_\alpha)^\sharp)$.

Lemma

- $\alpha = \Theta^{E^0_\alpha} = \Theta^{(E^0_\alpha)^\sharp}$;

Notation

From now on, we call α the minimum ordinal such that $L((E^0_\alpha)^\sharp)\cap V_{\lambda+2}=E^0_\alpha$.

Then both $L(E^0_\alpha)$ and $L((E^0_\alpha)^\sharp)$ have good qualities, and α is "large" in $L((E^0_\alpha)^\sharp)$.

Lemma

- $\alpha=\Theta^{E^0_\alpha}=\Theta^{(E^0_\alpha)^\sharp}$;
- $L(E^0_\alpha), L((E^0_\alpha)^\sharp)\vDash V=\mathrm{HOD}_{V_{\lambda+1}}$;

Notation

From now on, we call α the minimum ordinal such that $L((E^0_\alpha)^\sharp) \cap V_{\lambda+2} = E^0_\alpha$.

Then both $L(E^0_\alpha)$ and $L((E^0_\alpha)^\sharp)$ have good qualities, and α is "large" in $L((E^0_\alpha)^\sharp)$.

Lemma

- $\alpha = \Theta^{E^0_\alpha} = \Theta^{(E^0_\alpha)^\sharp}$;
- $L(E^0_\alpha), L((E^0_\alpha)^\sharp) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$;
- α is regular in $L((E^0_\alpha)^\sharp)$.

Notation

From now on, we call α the minimum ordinal such that $L((E^0_\alpha)^\sharp) \cap V_{\lambda+2} = E^0_\alpha$.

Then both $L(E^0_\alpha)$ and $L((E^0_\alpha)^\sharp)$ have good qualities, and α is "large" in $L((E^0_\alpha)^\sharp)$.

Lemma

- $\alpha = \Theta^{E^0_\alpha} = \Theta^{(E^0_\alpha)^\sharp}$;
- $L(E^0_\alpha), L((E^0_\alpha)^\sharp) \vDash V = \text{HOD}_{V_{\lambda+1}}$;
- α is regular in $L((E^0_\alpha)^\sharp)$.

Note that, since there exists $j : L(E^0_{\alpha+2}) \prec L(E^0_{\alpha+2})$, $j \upharpoonright L((E^0_\alpha)^\sharp)$ is an elementary embedding, so in $L((E^0_\alpha)^\sharp)$ the first and second degree analogies hold.

Theorem

In $L((E^0_\alpha)^\sharp)$ II cannot have a winning strategy for the game G_α.

Theorem

In $L((E^0_\alpha)^\sharp)$ II cannot have a winning strategy for the game G_α.

Proof.

We fix $j : L((E^0_\alpha)^\sharp) \prec L((E^0_\alpha)^\sharp)$.

Theorem

In $L((E^0_\alpha)^\sharp)$ II cannot have a winning strategy for the game G_α.

Proof.

We fix $j : L((E^0_\alpha)^\sharp) \prec L((E^0_\alpha)^\sharp)$.

Claim. For every $\beta_n < \alpha$, there is a surjection in $L((E^0_\alpha)^\sharp)$ from $V_{\lambda+1}$ to the set of all the k_n such that $\langle k_n, \beta_n \rangle$ is a legal move for I.

Theorem

In $L((E^0_\alpha)^\sharp)$ II cannot have a winning strategy for the game G_α.

Proof.

We fix $j : L((E^0_\alpha)^\sharp) \prec L((E^0_\alpha)^\sharp)$.

Claim. For every $\beta_n < \alpha$, there is a surjection in $L((E^0_\alpha)^\sharp)$ from $V_{\lambda+1}$ to the set of all the k_n such that $\langle k_n, \beta_n \rangle$ is a legal move for I.

This is because for every $\beta < \alpha$ every element of E^0_β is definable with parameters from $\Theta^{E^0_\beta} \cup V_{\lambda+1}$.

Theorem

In $L((E^0_\alpha)^\sharp)$ II cannot have a winning strategy for the game G_α.

Proof.

We fix $j : L((E^0_\alpha)^\sharp) \prec L((E^0_\alpha)^\sharp)$.

Claim. For every $\beta_n < \alpha$, there is a surjection in $L((E^0_\alpha)^\sharp)$ from $V_{\lambda+1}$ to the set of all the k_n such that $\langle k_n, \beta_n \rangle$ is a legal move for I.

This is because for every $\beta < \alpha$ every element of E^0_β is definable with parameters from $\Theta^{E^0_\beta} \cup V_{\lambda+1}$. So every elementary embedding $k : E^0_{\beta_{n-1}} \prec E^0_{\beta_n}$ is defined by its behaviour on $\Theta^{E^0_{\beta_{n-1}}}$, and we have few of this behaviours because of the Coding Lemma. □

Proof.

If II had a winning strategy τ, it would be definable.

Proof.

If II had a winning strategy τ, it would be definable.
So we can define the set C of the ordinals closed under τ, i.e. of the ordinals η such that if $\beta_n < \eta$, then for every k_n $\tau(\langle k_n, \beta_n \rangle) < \eta$.

Proof.

If II had a winning strategy τ, it would be definable.
So we can define the set C of the ordinals closed under τ, i.e. of the ordinals η such that if $\beta_n < \eta$, then for every k_n $\tau(\langle k_n, \beta_n \rangle) < \eta$. By the first claim C is a club.

Proof.

If II had a winning strategy τ, it would be definable.
So we can define the set C of the ordinals closed under τ, i.e. of the ordinals η such that if $\beta_n < \eta$, then for every k_n $\tau(\langle k_n, \beta_n \rangle) < \eta$. By the first claim C is a club.
Since C is definable and α is regular, C has ordertype α.

Proof.

If II had a winning strategy τ, it would be definable.
So we can define the set C of the ordinals closed under τ, i.e. of the ordinals η such that if $\beta_n < \eta$, then for every k_n $\tau(\langle k_n, \beta_n \rangle) < \eta$. By the first claim C is a club.
Since C is definable and α is regular, C has ordertype α.
But then if I plays the κ_n-th element of C as β_n and $j \upharpoonright E^0_{\beta_{n-1}}$ as k_n, I wins, and that's a contradiction.

Proof.

If II had a winning strategy τ, it would be definable.
So we can define the set C of the ordinals closed under τ, i.e. of the ordinals η such that if $\beta_n < \eta$, then for every k_n $\tau(\langle k_n, \beta_n \rangle) < \eta$. By the first claim C is a club.
Since C is definable and α is regular, C has ordertype α.
But then if I plays the κ_n-th element of C as β_n and $j \upharpoonright E^0_{\beta_{n-1}}$ as k_n, I wins, and that's a contradiction. □

Since $\mathrm{cof}(\alpha) = \omega$ and for every $j : L(E^0_{\alpha+2}) \prec L(E^0_{\alpha+2})$ $j \upharpoonright L(E^0_\alpha)$ is proper, then α is a partially non-proper ordinal.

What is the correlation between α and β?

What is the correlation between α and β?

Lemma

The ordertype of I_α is α, so there exists an $\alpha_0 < \alpha$ such that $\text{ot}(I_{\alpha_0}) = \lambda$.

What is the correlation between α and β?

Lemma

The ordertype of I_α is α, so there exists an $\alpha_0 < \alpha$ such that $\text{ot}(I_{\alpha_0}) = \lambda$.

Proof.

This is because in $(E^0_\alpha)^\sharp$ there are few partial Skolem functions.

What is the correlation between α and β?

Lemma

The ordertype of I_α is α, so there exists an $\alpha_0 < \alpha$ such that $\mathrm{ot}(I_{\alpha_0}) = \lambda$.

Proof.

This is because in $(E^0_\alpha)^\sharp$ there are few partial Skolem functions. Let $\gamma < \alpha$. Then $H = H^{(E^0_\alpha)^\sharp}((E^0_\alpha)^\sharp \cap E^0_\gamma)$ is small, so the least η such that $H \subseteq E^0_\eta$ is less than α.

What is the correlation between α and β?

Lemma

The ordertype of I_α is α, so there exists an $\alpha_0 < \alpha$ such that $\mathrm{ot}(I_{\alpha_0}) = \lambda$.

Proof.

This is because in $(E^0_\alpha)^\sharp$ there are few partial Skolem functions. Let $\gamma < \alpha$. Then $H = H^{(E^0_\alpha)^\sharp}((E^0_\alpha)^\sharp \cap E^0_\gamma)$ is small, so the least η such that $H \subseteq E^0_\eta$ is less than α.
So we can build a club of γ's such that
$(E^0_\alpha)^\sharp \cap \bigcup_{\eta<\gamma} E^0_\eta \prec (E^0_\alpha)^\sharp$.

What is the correlation between α and β?

Lemma

The ordertype of I_α is α, so there exists an $\alpha_0 < \alpha$ such that $\mathrm{ot}(I_{\alpha_0}) = \lambda$.

Proof.

This is because in $(E^0_\alpha)^\sharp$ there are few partial Skolem functions. Let $\gamma < \alpha$. Then $H = H^{(E^0_\alpha)^\sharp}((E^0_\alpha)^\sharp \cap E^0_\gamma)$ is small, so the least η such that $H \subseteq E^0_\eta$ is less than α.
So we can build a club of γ's such that $(E^0_\alpha)^\sharp \cap \bigcup_{\eta<\gamma} E^0_\eta \prec (E^0_\alpha)^\sharp$. Since "being a sharp" is a local property, this means that $(E^0_\alpha)^\sharp$ reflects in γ, i.e. $\gamma \in I_\alpha$.

What is the correlation between α and β?

Lemma

The ordertype of I_α is α, so there exists an $\alpha_0 < \alpha$ such that $\mathrm{ot}(I_{\alpha_0}) = \lambda$.

Proof.

This is because in $(E^0_\alpha)^\sharp$ there are few partial Skolem functions. Let $\gamma < \alpha$. Then $H = H^{(E^0_\alpha)^\sharp}((E^0_\alpha)^\sharp \cap E^0_\gamma)$ is small, so the least η such that $H \subseteq E^0_\eta$ is less than α.

So we can build a club of γ's such that $(E^0_\alpha)^\sharp \cap \bigcup_{\eta<\gamma} E^0_\eta \prec (E^0_\alpha)^\sharp$. Since "being a sharp" is a local property, this means that $(E^0_\alpha)^\sharp$ reflects in γ, i.e. $\gamma \in I_\alpha$.

This proves that I_α is a club in α. Since $I_\alpha \in L((E^0_\alpha)^\sharp)$ and α is regular in $L((E^0_\alpha)^\sharp)$, we're done. □

Are there other differences?

Are there other differences?

Lemma

Let α and β as above.

Are there other differences?

Lemma

Let α and β as above.

- Let $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ weakly proper.

Are there other differences?

Lemma

Let α and β as above.

- Let $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ weakly proper. Then there exist at least 2^λ different weakly proper non-proper (proper) elementary embeddings $k : L(E^0_\alpha) \prec L(E^0_\alpha)$ such that $k \upharpoonright V_\lambda = j \upharpoonright V_\lambda$.

Are there other differences?

Lemma

Let α and β as above.

- Let $j : L(E^0_\alpha) \prec L(E^0_\alpha)$ weakly proper. Then there exist at least 2^λ different weakly proper non-proper (proper) elementary embeddings $k : L(E^0_\alpha) \prec L(E^0_\alpha)$ such that $k \upharpoonright V_\lambda = j \upharpoonright V_\lambda$.
- For every $j, k : L(E^0_\beta) \prec L(E^0_\beta)$ weakly proper if $j \upharpoonright V_\lambda = k \upharpoonright V_\lambda$, then $j = k$.

Proof.

- Remember the game G_α.

Proof.

- Remember the game G_α. In $L((E^0_\alpha)^\sharp)$ I has a winning quasistrategy, so the set of all the winning successor moves is in $L((E^0_\alpha)^\sharp)$ and it's cofinal in α.

Proof.

- Remember the game G_α. In $L((E^0_\alpha)^\sharp)$ I has a winning quasistrategy, so the set of all the winning successor moves is in $L((E^0_\alpha)^\sharp)$ and it's cofinal in α. This means that the possible different winning plays for I are at least $|^\alpha 2| > 2^\lambda$.

Proof.

- Remember the game G_α. In $L((E^0_\alpha)^\sharp)$ I has a winning quasistrategy, so the set of all the winning successor moves is in $L((E^0_\alpha)^\sharp)$ and it's cofinal in α. This means that the possible different winning plays for I are at least $|{}^\alpha 2| > 2^\lambda$.
- It's possible to prove that every element of E^0_β is definable with parameters from I_β and $V_{\lambda+1}$.

Proof.

- Remember the game G_α. In $L((E^0_\alpha)^\sharp)$ I has a winning quasistrategy, so the set of all the winning successor moves is in $L((E^0_\alpha)^\sharp)$ and it's cofinal in α. This means that the possible different winning plays for I are at least $|{}^\alpha 2| > 2^\lambda$.
- It's possible to prove that every element of E^0_β is definable with parameters from I_β and $V_{\lambda+1}$. So the behaviours of j and k depend only on their behaviours on I_β and V_λ, that in turn depend on their behaviour on λ and V_λ, that are equal.

□

The structure of the previous proofs is the following:

The structure of the previous proofs is the following:

- If E^0_∞ exists, then $I \subsetneq \Upsilon$;

The structure of the previous proofs is the following:

- If E^0_∞ exists, then $I \subsetneq \Upsilon$;
- if $I \subsetneq \Upsilon$, then there exists η such that $L((E^0_\eta)^\sharp) \cap V_{\lambda+2} = E^0_\eta$, and we can define α;

The structure of the previous proofs is the following:

- If E^0_∞ exists, then $I \subsetneq \Upsilon$;
- if $I \subsetneq \Upsilon$, then there exists η such that $L((E^0_\eta)^\sharp) \cap V_{\lambda+2} = E^0_\eta$, and we can define α;
- α is a partially non-proper ordinal, and there exists a totally non-proper ordinal below it

The structure of the previous proofs is the following:

- If E^0_∞ exists, then $I \subsetneq \Upsilon$;
- if $I \subsetneq \Upsilon$, then there exists η such that $L((E^0_\eta)^\sharp) \cap V_{\lambda+2} = E^0_\eta$, and we can define α;
- α is a partially non-proper ordinal, and there exists a totally non-proper ordinal below it

Some of these implications cannot be reversed.

There are plenty of open problems, and most of them seems very difficult:

There are plenty of open problems, and most of them seems very difficult:

- Are there other partially or totally non-proper ordinals?

There are plenty of open problems, and most of them seems very difficult:

- Are there other partially or totally non-proper ordinals?
- Is it possible to have consistency-like results?

There are plenty of open problems, and most of them seems very difficult:

- Are there other partially or totally non-proper ordinals?
- Is it possible to have consistency-like results?
- Are there non-proper elementary embeddings between models like $L(X, V_{\lambda+1})$?

There are plenty of open problems, and most of them seems very difficult:

- Are there other partially or totally non-proper ordinals?
- Is it possible to have consistency-like results?
- Are there non-proper elementary embeddings between models like $L(X, V_{\lambda+1})$?
- Is the existence of E^0_∞ inconsistent?