From I1 to I0

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31-07-2013

Main Sources: [1], [2]

Main Result: "There exists $j : L_{\lambda^+ + \omega + 1}(V_{\lambda+1}) \prec L_{\lambda^+ + \omega + 1}(V_{\lambda+1})$" strongly implies "There exists $k : L_{\lambda^+}(V_{\lambda+1}) \prec L_{\lambda^+}(V_{\lambda+1})$".

The proof that I0 strongly implies I1 is more or less the same of Σ^1_{n+2} strongly implies Σ^1_n. In fact it is possible to prove even more:

Theorem 0.1 (Laver, [1])**.** *The smallest λ such that there exists $j : L_1(V_{\lambda+1}) \prec L_1(V_{\lambda+1})$ is bigger than the smallest λ such that there exists $j : V_{\lambda+1} \prec V_{\lambda+1}$.*

Looking back at the proof in Chapter Two, the first step to prove is reflection:

Lemma 0.2 (Laver, [1])**.** *Let $j : L_1(V_{\lambda+1}) \prec L_1(V_{\lambda+1})$. For every $A, B \subseteq V_\lambda$, $\beta < \mathrm{crt}(j)$ there exists $k : V_{\lambda+1} \prec V_{\lambda+1}$, with $\beta < \mathrm{crt}(k) < \mathrm{crt}(j)$, such that $k(B) = j(B)$ and there exists $A' \subseteq V_\lambda$ such that $k(A') = A$.*

Proof. The only detail we should care of is the fact that "$j : V_{\lambda+1} \prec V_{\lambda+1}$" must be definable in $L_1(V_{\lambda+1})$. But this is true, because the satisfaction relation in $V_{\lambda+1}$ is definable in $L_1(V_{\lambda+1})$, so the proof is, *mutatis mutandis*, the same as Lemma 0.25 in Chapter Two. □

Proof of Theorem 0.1. The proof follows the same method of Theorem 0.35 in Chapter Two.

Using Lemma 0.2, we build a sequence $\langle k_0, \dots, k_n, \dots \rangle$ such that k_i is Σ^1_n for every n, i.e., $k_i : V_{\lambda+1} \prec V_{\lambda+1}$, and $\mathrm{crt}(k_0) < \mathrm{crt}(k_1) < \dots < \mathrm{crt}(j)$, such that there exist l_i, $k_0(l_0) = j \upharpoonright V_\lambda$ and $k_{i+1}(l_{i+1}) = l_i$. Then $K = k_0 \circ k_1 \circ \dots$ is a Σ^1_ω elementary embedding. Call $\alpha = \sup_{i \in \omega} \mathrm{crt}(k_i)$, then $\alpha < \mathrm{crt}(j)$ and $K : V_{\alpha+1} \prec V_{\lambda+1}$. Then there exists j_α such that $K(j_\alpha) = j \upharpoonright V_\lambda$, so $j_\alpha : V_{\alpha+1} \prec V_{\alpha+1}$. □

Note that in this case there is a fundamental difference between being Σ^1_n for every n and being an elementary embedding from $V_{\lambda+1}$ to itself. Suppose that we have $j_i : V_{\lambda+1} \prec V_{\lambda+1}$. Then the inverse limit J is *not* an elementary embedding from $V_{\lambda'+1}$ to $V_{\lambda+1}$, because the domain of J is just not $V_{\lambda'+1}$. Consider again the definition of the domain of J: $H = \{x \in L_{\lambda^+} : \exists n \in \omega \forall m \geq n\ k_m(x) = x\}$. Following this definition, for example, $\lambda' \notin H$, because it is moved by all j_i, while on the other hand $\lambda \in H$, because it is never moved. What we can prove with the methods provided, is that the unique extension of the inverse limits of the $j_i \upharpoonright V_\lambda$ to $V_{\lambda'}$ is an elementary embedding.

Now that we know there is a gap between I0 and I1, and that we localized that gap in $L_1(V_{\lambda+1})$, we can ask whether this is the only gap, i.e., if there are other gaps between I0 and $L_1(V_{\lambda+1}$. The answer is positive, and the method is still a generalization of the one in Chapter Two, but first we need to learn more aboute the sets like $L_\alpha(V_{\lambda+1})$.

Definition 0.3.

$$\mathrm{OD}_a = \{x : x \textit{ is definable with parameters from } \mathrm{Ord} \cup a\}.$$

Lemma 0.4. $L(V_{\lambda+1}) = \bigcup_{a \in V_{\lambda+1}} \mathrm{OD}_a$.

Proof. This is immediate after Lemma 0.37 in Chapter Three □

Lemma 0.5. *Define $H^{L_\alpha(V_{\lambda+1})}(X)$ as the closure of X under partial Skolem functions. Then*

$$H^{L_\alpha(V_{\lambda+1})}(X) = \{z \in L_\alpha(V_{\lambda+1}) : z \textit{ is definable in } L_\alpha(V_{\lambda+1}) \textit{ with parameters in } X \cup V_{\lambda+1}\}.$$

Proof. Since the partial Skolem functions are definable, the lemma is true. □

Definition 0.6. *We say that α is good if every element of $L_\alpha(V_{\lambda+1})$ is definable in $L_\alpha(V_{\lambda+1})$ from parameters in $V_{\lambda+1}$.*

Some remarks to give an idea about which ordinals are good.

Remark 0.7. *If α is good, then $\alpha + 1$ is good.*

Proof. Let $x \in L_{\alpha+1}(V_{\lambda+1})$. Then there exists a formula φ and some $a_1, \dots, a_n \in L_\alpha(V_{\lambda+1})$ such that

$$x = \{y \in L_\alpha(V_{\lambda+1}) : L_\alpha(V_{\lambda+1}) \vDash \varphi(a_1, \dots, a_n)\}.$$

Therefore x is definable in $L_{\alpha+1}(V_{\lambda+1})$ with parameters $L_\alpha(V_{\lambda+1}), a_1, \dots, a_n$. First note that since α is good, the a_i's are definable in $L_\alpha(V_{\lambda+1})$ with parameters in $V_{\lambda+1}$, then α is definable in $L_{\alpha+1}(V_{\lambda+1})$ (it is the largest ordinal minus an initial $\lambda+1$ segment) with λ as parameter, and therefore also $L_\alpha(V_{\lambda+1})$ is definable in $L_{\alpha+1}(V_{\lambda+1})$ with λ as parameter. This proves the remark. $\square$

Remark 0.8. *If $\beta \leq \lambda$, then β is good.*

Proof. We prove it by induction. The successor step is proved in the previous remark. Suppose $\beta < \lambda$ is limit and every ordinal less then β is good. For every $x \in L_\beta(V_{\lambda+1})$ there exists $\alpha < \beta$ such that $x \in L_\alpha(V_{\lambda+1})$. As α is good, x is definable in $L_\alpha(V_{\lambda+1})$ with parameters from $V_{\lambda+1}$, and therefore is definable in $L_\beta(V_{\lambda+1})$ with parameters from $V_{\lambda+1}$ and $L_\alpha(V_{\lambda+1})$ (i.e., α and λ). Since α is in $V_{\lambda+1}$, the remark is proven. $\square$

Remark 0.9. *If $\beta \leq \lambda^+$, then β is good.*

Proof. The proof is the same as the previous remark, with the added consideration that every ordinal less than λ^+ can be coded by a well-ordering of λ, that is in turn in $V_{\lambda+1}$. Therefore any $\alpha < \lambda^+$ is definable with parameters from $V_{\lambda+1}$. $\square$

Remark 0.10. *There exists an ordinal $< \Theta$ that is not good.*

Proof. Clearly Θ is not good: define $\pi(\langle \ulcorner \varphi \urcorner, a_1, \dots, a_n \rangle)$ as the set definable in $L_\Theta(V_{\lambda+1})$ with the formula φ using parameters $a_1, \dots, a_n \in V_{\lambda+1}$. If Θ were good, that π would be a surjection from $V_{\lambda+1}$ to $L_\Theta(V_{\lambda+1})$, and this is impossible by definition of Θ.

Consider $H^{L_\Theta(V_{\lambda+1})}(V_{\lambda+1})$. Its collapse is a set of the form $L_\gamma(V_{\lambda+1}) \prec L_\Theta(V_{\lambda+1})$. As $L_\Theta(V_{\lambda+1})$ satisfies "there exists x not definable with parameters from $V_{\lambda+1}$", $L_\gamma(V_{\lambda+1})$ satisfies the same, therefore γ is not good. $\square$

Note that if α is good, we can say that every element of $L_\alpha(V_{\lambda+1})$ is definable in $L_\alpha(V_{\lambda+1})$ from *one* parameter in $V_{\lambda+1}$, because a finite sequence in $V_{\lambda+1}$ can be coded as an element of $V_{\lambda+1}$. Moreover, $\alpha < \Theta$, because α good means that $L_\alpha(V_{\lambda+1}) = H^{L_\alpha(V_{\lambda+1})}(V_{\lambda+1}) \subseteq H^{L(V_{\lambda+1})}(V_{\lambda+1})$, and in $L(V_{\lambda+1})$ there is a surjection from $V_{\lambda+1}$ to $H^{L(V_{\lambda+1})}(V_{\lambda+1})$ itself. Finally, note that if $j : L_\alpha(V_{\lambda+1}) \prec L_\alpha(V_{\lambda+1})$, then j is univocally induced by $j \upharpoonright V_\lambda$. This recalls the situation in Chapter Three, where it was proved that if $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ is weakly proper, then it is univocally induced by $j \upharpoonright V_\lambda$. If it is not weakly proper, at least $j \upharpoonright L_\Theta(V_{\lambda+1})$ is univocally induced by $j \upharpoonright V_\lambda$.

Lemma 0.11. *Good ordinals are unbounded in Θ.*

Proof. Note that if there exists a surjection $h : V_{\lambda+1} \twoheadrightarrow L_\beta(V_{\lambda+1})$ and h is definable, then β is good.

Let $\beta_0 < \Theta$. Then there exists a surjection $h : V_{\lambda+1} \twoheadrightarrow L_{\beta_0}(V_{\lambda+1})$. Now, $H^{L(V_{\lambda+1})}(L_{\beta_0}(V_{\lambda+1})) \equiv L(V_{\lambda+1})$, so by condensation there exists $\gamma < \Theta$ such that $H^{L(V_{\lambda+1})}(L_{\beta_0}(V_{\lambda+1})) = L_\gamma(V_{\lambda+1})$. Since $L_\gamma(V_{\lambda+1}) \prec L(V_{\lambda+1})$, by elementarity there exists a surjection $h : V_{\lambda+1} \twoheadrightarrow L_{\beta_0}(V_{\lambda+1})$, with $h \in L_\gamma(V_{\lambda+1})$.

This proves that

$$\beta_1 = \min\{\gamma : \exists h : V_{\lambda+1} \twoheadrightarrow L_{\beta_0}(V_{\lambda+1}),\ h \in L_\gamma(V_{\lambda+1})\} < \Theta.$$

Define

$$A_1 = \{a \in V_{\lambda+1} : \exists h : V_{\lambda+1} \twoheadrightarrow L_{\beta_0}(V_{\lambda+1}),\ h \in L_{\beta_1}(V_{\lambda+1}) \cap \mathrm{OD}_a\},$$

and continue by induction defining

$$\begin{aligned} \beta_{n+1} &= \min\{\gamma : \exists h : V_{\lambda+1} \twoheadrightarrow L_{\beta_n}(V_{\lambda+1}),\ h \in L_\gamma(V_{\lambda+1})\}. \\ A_{n+1} &= \{a \in V_{\lambda+1} : \exists h : V_{\lambda+1} \twoheadrightarrow L_{\beta_n}(V_{\lambda+1}),\ h \in L_{\beta_{n+1}}(V_{\lambda+1}) \cap \mathrm{OD}_a\}, \end{aligned}$$

By DC we choose $\langle a_1, a_2, \dots \rangle$ such that $a_n \in A_n$. Finally, we define h_n as the smallest surjection from $V_{\lambda+1}$ into $L_{\beta_n}(V_{\lambda+1})$ that is in OD_{a_n}, with OD_{a_n} well-ordered in the standard way.

Let $\beta_\omega = \bigcup_{n\in\omega} \beta_n$, Since Θ is regular, $\beta_\omega < \Theta$. Now let $\tilde{h} : \omega \times V_{\lambda+1} \twoheadrightarrow L_{\beta_\omega}(V_{\lambda+1})$ be defined as $\tilde{h}(n, x) = h_n(x)$. Then $\tilde{h}$ is definable, so β_ω is good. □

To implement the method of Chapter Two to this setting, we need to be able to define $j(k)$, when $j, k : L_\beta(V_{\lambda+1}) \prec L_\beta(V_{\lambda+1})$. This is less easy then expected: the most natural definition would be $j(k) = \bigcup_{\gamma<\beta} j(k \upharpoonright L_\gamma(V_{\lambda+1}))$, but this creates problems. First of all, such definition doesn't take in consideration the case when β is a successor ordinal, and moreover there is the possibility that $k \upharpoonright L_\gamma(V_{\lambda+1})$ is not an element of $L_\beta(V_{\lambda+1})$, and so it is not in the domain of j.

Definition 0.12. *Let $j, k : L_\beta(V_{\lambda+1}) \prec L_\beta(V_{\lambda+1})$ and suppose that $k \upharpoonright L_\gamma(V_{\lambda+1}) \in L_\beta(V_{\lambda+1})$ for cofinallyy $\gamma < \beta$.*

- *Suppose $\beta = \gamma + 1$. Then $j(\gamma) = k(\gamma) = \gamma)$ and*

  $$j(k \upharpoonright L_\gamma(V_{\lambda+1})) : L_\gamma(V_{\lambda+1}) \prec L_\gamma(V_{\lambda+1}).$$

  For every $x \in L_{\gamma+1}(V_{\lambda+1})$, there exist φ formula and $y_1, \dots, y_n \in L_\gamma(V_{\lambda+1})$ such that

  $$x = \{y \in L_\gamma(V_{\lambda+1}) : L_\gamma(V_{\lambda+1}) \vDash \varphi(y, y_1, \dots, y_n)\}.$$

Then define

$$j(k)(x) = \{y \in L_\gamma(V_{\lambda+1}) : \\ L_\gamma(V_{\lambda+1}) \vDash \varphi(y, j(k \restriction L_\gamma(V_{\lambda+1}))(y_1), \dots, j(k \restriction L_\gamma(V_{\lambda+1}))(y_n)\}.$$

- *Suppose β is limit. Then define $j(k) = \bigcup_{\gamma<\beta} j(k \restriction L_\gamma(V_{\lambda+1}))$.*

Lemma 0.13. *When it is defined, $j(k) : L_\beta(V_{\lambda+1}) \prec L_\beta(V_{\lambda+1})$.*

Proof.

- Suppose $\beta = \gamma+1$. Notice that k uniquely extends $k \restriction L_\gamma(V_{\lambda+1})$, so we can consider k as a class in $L_\beta(V_{\lambda+1})$ defined from the parameter $L_\gamma(V_{\lambda+1})$. Then $L_\beta(V_{\lambda+1}) \vDash \forall x\ \varphi(x) \leftrightarrow \varphi(k(x))$, so

$$L_\beta(V_{\lambda+1}) \vdash \forall\psi\forall y_1, \dots, y_n \in L_\gamma(V_{\lambda+1}) \\ (\forall x \quad \text{“}x \text{ is defined with } \psi, y_1, \dots, y_n\text{”}) \to \varphi(x) \leftrightarrow \varphi(k(x)).$$

Then

$$L_\beta(V_{\lambda+1}) \vDash \forall\psi\forall y_1, \dots, y_n \in L_\gamma(V_{\lambda+1}) \\ (\forall x \quad \text{“}x \text{ is defined with } \psi, y_1, \dots, y_n\text{”} \wedge \\ \forall z \quad \text{“}z \text{ is defined with } \psi, k(y_1), \dots, k(y_n)\text{”}) \to \varphi(x) \leftrightarrow \varphi(z),$$

therefore

$$L_\beta(V_{\lambda+1}) \vDash \forall\psi\forall y_1, \dots, y_n \in L_\gamma(V_{\lambda+1}) \\ (\forall x \quad \text{“}x \text{ is defined with } \psi, y_1, \dots, y_n\text{”} \wedge \\ \forall z \quad \text{“}z \text{ is defined with } \psi, j(k)(y_1), \dots, j(k)(y_n)\text{”}) \to \varphi(x) \leftrightarrow \varphi(z),$$

but z is in fact $j(k)(x)$, so the lemma is proved.

- Suppose β is limit. Let $A_k = \{\gamma < \beta : K \restriction L_\gamma(V_{\lambda+1}) \in L_\beta(V_{\lambda+1})\}$. Then

$$\forall\gamma \in A_k\ L_\beta(V_{\lambda+1}) \vDash \forall x \in L_\gamma(V_{\lambda+1})\ \varphi(x) \leftrightarrow \varphi(k \restriction L_\gamma(V_{\lambda+1})(x)),$$

therefore

$$\forall\gamma \in A_k\ L_\beta(V_{\lambda+1}) \vDash \forall\gamma < \beta\ \forall x \in L_\gamma(V_{\lambda+1})\ \varphi(x) \leftrightarrow \varphi(j(k \restriction L_\gamma(V_{\lambda+1}))(x))$$

that, again, is $j(k)(x)$. Since A_k is cofinal in β for every $x \in L_\beta(V_{\lambda+1})$ there exists a $\gamma \in A_k$ such that $x \in L_\gamma(V_{\lambda+1})$, so the lemma is proved.

$\square$

Note that if $\beta \leq \Theta$, the conditions for the definition of $j(k)$ are not really bounding. If, for example, β is good, then the conditions are satisfied: consider that

$$k \upharpoonright L_\gamma(V_{\lambda+1}) = \{(x,y) : x \in L_\gamma(V_{\lambda+1}), \forall\psi\forall y \in V_{\lambda+1} \\ \text{“}x\text{ is defined by }\psi, y\text{''} \to \text{“}y\text{ is defined by }\psi, k(y)\text{''}\},$$

and as $L_\gamma(V_{\lambda+1}) \in L_\beta(V_{\lambda+1})$, $k \upharpoonright L_\gamma(V_{\lambda+1})$ is definable with elements from $V_{\lambda+1}$, so $k \upharpoonright L_\gamma(V_{\lambda+1}) \in L_\beta(V_{\lambda+1})$.

This is also true for β that is limit of good ordinals, such as Θ.

Observe that if $j : L_\Theta(V_{\lambda+1}) \prec L_\Theta(V_{\lambda+1})$, then for every $\alpha < \Theta$, $j \upharpoonright L_\alpha(V_{\lambda+1}) \in L(V_{\lambda+1})$. In fact if β is good, $j \upharpoonright L_\beta(V_{\lambda+1}) \in L_{j(\beta)+1}(V_{\lambda+1})$, because it is definable in $L_{j(\beta)}(V_{\lambda+1})$ with parameters $\beta, j \upharpoonright V_\lambda$. If β is not good, then there exists a $\beta < \alpha < \Theta$ such that α is good, and so $j \upharpoonright L_\alpha(V_{\lambda+1}) \in L(V_{\lambda+1})$, but then also $j \upharpoonright L_\beta(V_{\lambda+1}) \in L(V_{\lambda+1})$.

This is in sharp contrast with the fact, noted in Chapter Three, that if $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ then $j \upharpoonright L_\Theta(V_{\lambda+1}) \notin L(V_{\lambda+1})$. This is one of the many indicators of the special role that Θ has in the analysis of I0.

Note also that if $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$, then for every $\alpha < \Theta$ there exists $\alpha < \beta < \Theta$ and $k : L_\beta(V_{\lambda+1}) \prec L_\beta(V_{\lambda+1})$ such that $k \upharpoonright L_\alpha(V_{\lambda+1}) = j \upharpoonright L_\alpha(V_{\lambda+1})$, but $k \upharpoonright L_\beta(V_{\lambda+1}) \neq j \upharpoonright L_\beta(V_{\lambda+1})$, because otherwise it would be possible to define $j \upharpoonright L_\Theta(V_{\lambda+1})$ in $L(V_{\lambda+1})$. Of course, by the considerations made after Definition 0.6, it is not possible to extend k to $L(V_{\lambda+1})$.

Let's go back to the method of inverse limits now. We can summarize it in five steps:

- fix an elementary embedding j, with the property you want to be stronger, in this case $j : L_\beta(V_{\lambda+1}) \prec L_\beta(V_{\lambda+1})$, and also with the property that you want to be weaker, in this case being an elementary embedding from $L_\alpha(V_{\lambda+1})$ to itself, with $\alpha < \beta$;

- with some kind of reflection, find another elementary embedding k, even weaker, such that there exists j_1 with $k(j_1) = j$;

- by induction, find k_i such that $k_i(j_{i+1}) = j_i$;

- prove that the inverse limit of the k_i's, K, is an elementary embedding enough strong to “recognize” the weaker property;

- then pick j' such that $K(j') = j$ as a witness.

Unfortunately, this method as it is does not work for our case. The problem is in the fourth point: it is not always the case that K is an elementary embedding. Pick for example $k_0, k_1, \dots$ such that $k_i : L_{\lambda^+}(V_{\lambda+1}) \prec L_{\lambda^+}(V_{\lambda+1})$ such that $\lambda' = \sup_{i<\omega} \operatorname{crt}(k_i) < \lambda$, and let K be their inverse limit. The domain of K is

$$H = \{x \in L_{\lambda^+}(V_{\lambda+1}) : \exists n \in \omega \forall m \geq n\ k_m(x) = x\}.$$

Note that H is not a V_α anymore, because being a member of H does not depend exclusively from the rank. But $V_{\lambda'} \subseteq H$. Now, since λ^+ is regular, for every $i \in \omega$ the set of fixed points of k_i is an ω-club $C_i \subset \lambda^+$. Then $C = \bigcap_{i\in\omega} C_i$ is an ω-club and for every $\alpha \in C$, $K(\alpha) = \alpha$. Suppose now that $K : H \prec L_{\lambda^+}(V_{\lambda+1})$. Then for every $\alpha \in C$, since $L_{\lambda^+}(V_{\lambda+1}) \vDash$ "α is a well-order in V_λ", we have that $H \vDash$ "α is a well-order in $V_{\lambda'}$", but this is impossible, because λ' has less many well-orders respect to λ.

Anyway, not everything is lost. The second point still works in this setting.

Lemma 0.14 (Woodin [3])**.** *If there exists $j : L_\Theta(V_{\lambda+1}) \prec L_\Theta(V_{\lambda+1})$, then for every $\beta_0 < \beta_1 < \dots < \beta_n < \Theta$ and for every $x \in L_{\beta_n}(V_{\lambda+1})$ there exists an elementary embedding $k : L_{\beta_n}(V_{\lambda+1}) \prec L_{\beta_n}(V_{\lambda+1})$ such that for every i $k(\beta_i) = \beta_i$ and there exists $y \in L_{\beta_n}(V_{\lambda+1})$ such that $k(y) = x$.*

Proof. Let $j : L_\Theta(V_{\lambda+1}) \prec L_\Theta(V_{\lambda+1})$ and suppose the lemma is false for n. Then there are least $\beta_0 < \dots < \beta_n$ such that the lemma is false, i.e.,

$$\exists x \in L_{\beta_n}(V_{\lambda+1})\ \forall k : L_{\beta_n}(V_{\lambda+1}) \prec L_{\beta_n}(V_{\lambda+1})$$
$$(\exists i k\ (\beta_i) \neq \beta_i) \vee (\forall y \in L_{\beta_n}(V_{\lambda+1})\ k(y) \neq x).$$

If $k : L_{\beta_n}(V_{\lambda+1}) \prec L_{\beta_n}(V_{\lambda+1})$ then $k \in L_\Theta(V_{\lambda+1})$, so the definition of the β_i's is a formula in $L_\Theta(V_{\lambda+1})$ with λ as a parameter, so $j(\beta_i) = \beta_i$ for all i and

$$j \restriction L_{\beta_n}(V_{\lambda+1}) : L_{\beta_n}(V_{\lambda+1}) \prec L_{\beta_n}(V_{\lambda+1}).$$

Let $x \in L_{\beta_n}(V_{\lambda+1})$ be the witness of the definition of the β_i's. Then by elementarity

$$L_\Theta(V_{\lambda+1}) \vDash \forall k : L_{\beta_n}(V_{\lambda+1}) \prec L_{\beta_n}(V_{\lambda+1})$$
$$(\exists i k\ (\beta_i) \neq \beta_i) \vee (\forall y \in L_{\beta_n}(V_{\lambda+1})\ k(y) \neq j(x)),$$

but this is false because $j \restriction L_{\beta_n}(V_{\lambda+1})$ does not satisfy it. $\square$

Lemma 0.15 (Laver, [2]). *Let δ good, $\beta > \delta$, $j : L_\beta(V_{\lambda+1}) \prec L_\beta(V_{\lambda+1})$, $x, y \in V_{\lambda+1}$, $\mu < \mathrm{crt}(j)$. Then there exists a $\delta' < \delta$ good, an elementary embedding $k : V_{\lambda+1} \prec V_{\lambda+1}$ extendible to a $\hat{k} : L_{\delta'}(V_{\lambda+1}) \prec L_\delta(V_{\lambda+1})$ and a $y' \in V_{\lambda+1}$ such that $k(x) = j(x)$, $k(y') = y$ and $\mu < \mathrm{crt}(k) < \mathrm{crt}(j)$.*

Proof. Note that "δ is good" is definable in $L_\beta(V_{\lambda+1})$, so surely

$$L_\beta(V_{\lambda+1}) \vDash \delta \text{ is good, } \delta \leq j(\delta),\ j \upharpoonright V_{\lambda+1} : V_{\lambda+1} \prec V_{\lambda+1} \text{ and it extends to } j : L_\delta(V_{\lambda+1}) \prec L_{j(\delta)}(V_{\lambda+1}),\ j(j(x)) = j(j(x)),\ j(y) = j(y),\ \mu < \mathrm{crt}(j) < j(\mathrm{crt}(j)).$$

With a smart quantification we get

$$L_\beta(V_{\lambda+1}) \vDash \exists \delta'\ \delta' \text{ is good, } \delta' \leq j(\delta), \exists k\ k : V_{\lambda+1} \prec V_{\lambda+1} \text{ and it extends to } \hat{k} : L_{\delta'}(V_{\lambda+1}) \prec L_{j(\delta)}(V_{\lambda+1}),\ k(j(x)) = j(j(x)),\ \exists y' \in V_{\lambda+1}\ k(y') = j(y),\ \mu < \mathrm{crt}(k) < j(\mathrm{crt}(j)).$$

Then by elementarity

$$L_\beta(V_{\lambda+1}) \vDash \exists \delta'\ \delta' \text{ is good, } \delta' \leq \delta, \exists k\ k : V_{\lambda+1} \prec V_{\lambda+1} \text{ and it extends to } \hat{k} : L_{\delta'}(V_{\lambda+1}) \prec L_\delta(V_{\lambda+1}),\ k(x) = j(x),\ \exists y' \in V_{\lambda+1}\ k(y') = y,\ \mu < \mathrm{crt}(k) < \mathrm{crt}(j).$$

□

Note that if δ is definable in $L_\beta(V_{\lambda+1})$, then $\delta' = \delta$. The goodness, here, is a bonus, not a necessary condition. One can see immediately that the Lemma works even without goodness (both in the hypotheses and thesis).

So, the only obstacle seems the fact that the inverse limit can be not an elementary embedding. The trick used to get round this difficulty will be defining the inverse limit respect to a pwo.

Definition 0.16. *Let $\ll$ be a pwo and a in its domain. Then $\|a\| = \mathrm{ot} \ll\upharpoonright \{b : b \ll a\}$ is the* norm *of a.*

Definition 0.17. *A $\ll$ pwo of $V_{\lambda+1}$ is J-projective if it is definable in $V_{\lambda+1}$ with parameters from $J''V_{\lambda'+1}$. A $\ll$ pwo in $V_{\lambda+1}$ is λ-projective iff it is definable in $V_{\lambda+1}$ with parameters.*

From a J-projective pwo $\ll$, we construct now a sequence of pwos that will be the backbone of the construction of the inverse limit.

Let $J = j_0 \circ j_1 \circ \dots$, $\operatorname{crt}(j_0) < \operatorname{crt}(j_1) < \dots < \lambda$, $\lim_{n\in\omega} \operatorname{crt}(j_n) = \lambda' < \lambda$, with $j_0, j_1, \dots : L_\beta(V_{\lambda+1}) \prec L_\beta(V_{\lambda+1})$. Call $J_n = j_n \circ j_{n+1} \circ \dots$.

Let $\ll$ be a J-projective pwo, and fix φ and $E \in V_{\lambda'+1}$ such that

$$\ll = \{(x,y) \in V_{\lambda+1} : V_{\lambda+1} \vDash \varphi(x,y,J(E))\}.$$

Suppose that the ordertype of $\ll$ is less then β.

Define

- $\ll_n = \{(x,y) \in V_{\lambda+1} : V_{\lambda+1} \vDash \varphi(x,y,J_n(E))\}$ and let $\|\cdot\|_n$ be its associated norm;
- $\ll_{\lambda'} = \{(x,y) \in V_{\lambda'+1} : V_{\lambda'+1} \vDash \varphi(x,y,E)\}$ and let $\|\cdot\|_{\lambda'}$ be its associated norm.

Lemma 0.18. *1. $j_n(\ll_{n+1}) = \ll_n$ and $\ll_n$ is a pwo of $V_{\lambda+1}$;*

2. $\forall a,b \in V_{\lambda'+1}\ \forall n \in \omega\ a \ll_{\lambda'} b$ iff $J_n(a) \ll_n J_n(b)$;

3. $\ll_{\lambda'}$ is a pwo of $V_{\lambda'+1}$;

4. $\forall a \in V_{\lambda+1}\ \forall n \in \omega\ \|a\|_{\ll_{n+1}} = \gamma$ iff $\|j_n(a)\|_n = j_n(\gamma)$;

5. $\|J(F)\|_\ll = \alpha_0 \to \forall n \in \omega\ \|J_n(F)\|_n = \alpha_n$, where $j_n(\alpha_{n+1}) = \alpha_n$.

Proof. Note that being a pwo is a Δ^1_1 property, so if $\ll$ is a pwo and j is elementary, one cannot take for granted that $j(\ll)$ is a pwo.

1. The pwo $\ll_{n+1}$ is a member of $L_\beta(V_{\lambda+1})$, so it is in the domain of j. By definition

$$j_n(\ll_{n+1}) = j_n(\{(x,y) \in V_{\lambda+1} : V_{\lambda+1} \vDash \varphi(x,y,J_{n+1}(E))\}) = \\ = \{(x,y) \in V_{\lambda+1} : V_{\lambda+1} \vDash \varphi(x,y,j_n(J_{n+1}(E)))\}.$$

   But $j_n \circ J_{n+1} = J_n$, so the first part is proved. To prove that $\ll_n$ is a pwo, we just need to prove that is well-founded, since the rest comes by elementarity. Suppose $A \subseteq \ll_{n+1}$ is ill-founded. By DC, we can find a descending chain C. Naturally C can be coded as an element of $V_{\lambda+1}$, therefore $j_n(C)$ is a descending chain in $\ll_n$, and $j_0 \circ \dots j_n(C)$ is a descending chain in $\ll$, contradiction.

2. J_n is an elementary embedding between $V_{\lambda'+1}$ and $V_{\lambda+1}$, so for all $a,b \in V_{\lambda'+1}$ $V_{\lambda'+1} \vDash \varphi(a,b,E)$ iff $V_{\lambda+1} \vDash \varphi(J_n(a), J_n(B), J_n(E)\}$.

3. Suppose that $A \subseteq \ll_{\lambda'}$ is ill-founded, and fix a descending chain C. As before, C is coded as an element of $V_{\lambda'+1}$, so by (2) $J(C)$ is a descending chain in $\ll$. Contradiction.

4. By elementarity
$$\mathrm{ot}(\{b \in V_{\lambda+1} : b \ll_{n+1} a\}) = \gamma \text{ iff } \mathrm{ot}(\{b \in V_{\lambda+1} : b \ll_n j_n(a)\}) = j_n(\gamma).$$

5. By the results above:
$$\|J_n(F)\|_{\ll_n} = \|j_n(J_{n+1})(F)\|_{\ll_n} = j_n(\|J_{n+1}(F)\|_{\ll_{n+1}}).$$

□

Let $J : V_{\lambda'+1} \prec V_{\lambda+1}$, and fix $\ll_{\lambda'}$ and $\ll$, pwos respectively on $V_{\lambda'+1}$ and $V_{\lambda+1}$, such that $\forall a, b \in V_{\lambda'+1}$ $a \ll_{\lambda'} b$ iff $J(a) \ll J(b)$. Suppose $a \in V_{\lambda'+1}$ is such that $\|a\|_{\lambda'} = \alpha'$ and $\|J(a)\| = \alpha$. We define $\hat{J} : L_{\alpha'}(V_{\lambda'+1}) \to L_\alpha(V_{\lambda+1})$ in the following way:

fix an $x \in L_{\alpha'}(V_{\lambda'+1})$. This will be definable with an ordinal $\gamma' < \alpha'$ and a $b \in V_{\lambda'+1}$. Then define $J(x)$ as the element defined by $\gamma = \|J(E)\|$, where $\gamma' = \|E\|_{\lambda'}$, and $J(a)$. More specifically, if $x = \{y : \varphi(y, \|E\|_{\lambda'}, a)\}$, then $\hat{J}(x) = \{y : \varphi(y, \|J(E)\|, J(a))\}$.

It is easy to see that $\hat{J} \supseteq J$ and $\hat{J} \upharpoonright L_1(V_{\lambda'+1})$ is a Δ_0-elementary embedding, but beyond that it is difficult to see even if it is a well-defined function.

Theorem 0.19 (Laver, [2])**.** *Let $j_i : L_\beta(V_{\lambda+1}) \prec L_\beta(V_{\lambda+1})$, with $\mathrm{crt}(j_0) < \mathrm{crt}(j_1) < \ldots \uparrow \lambda'$. Let J be the inverse limit of the elementary embeddings $j_i \upharpoonright V_{\lambda+1}$. Let $\ll$ a J-projective pwo, with $\ll_{\lambda'}$ the respective pwo in $V_{\lambda'+1}$. Let $a \in V_{\lambda'+1}$ with $\|a\|_{\lambda'} = \alpha'$ and $\|J(a)\| = \alpha$. If there exists a good ordinal $\rho \geq \alpha$ such that $\rho + \omega \leq \beta$, then $\hat{J} : L_{\alpha'}(V_{\lambda'+1}) \prec L_\alpha(V_{\lambda+1})$, with $\hat{J}$ induced by J and $\ll$.*

Proof. Let us fix the notations: $\ll = \{(x, y) : V_{\lambda+1} \vDash \varphi(x, y, J(E))\}$, $\ll_{\lambda'} = \{(x, y) : V_{\lambda'+1} \vDash \varphi(x, y, E)\}$, $E_n = J_n(E)$ and $a_n = J_n(a)$.

The proof is by induction on α.

Suppose $\alpha_n = \|a_n\|_{\ll_n}$. Since by Lemma 0.18 $j_i(\alpha_{i+1}) = \alpha_i$, we have $\alpha_i \geq \alpha_{i+1}$, so $\exists n\ \forall m \geq n\ \alpha_n = \alpha_m$. If $\alpha_n < \alpha$, then by inductive hypothesis $\hat{J}_n : L_{\alpha'}(V_{\lambda'+1}) \prec L_{\alpha_n}(V_{\lambda+1})$. It is easy to see that $j_0 \circ \cdots \circ j_{n+1} \circ \hat{J}_n = \hat{J}$, and so the theorem is proved.

Therefore we can suppose $j_i(\alpha) = \alpha$ for every $i \in \omega$.

For every $\gamma' < \alpha'$, there exists $G \in V_{\lambda+1}$ such that $\|G\|_{\lambda'} = \gamma'$. Then by induction for every $\gamma' < \alpha'$ $\hat{J}_{\gamma'} : L_{\gamma'}(V_{\lambda'+1}) \prec L_\gamma(V_{\lambda+1})$, all the $\hat{J}_{\gamma'}$ agree and so is it possible to take the union. If α' is a limit ordinal, then $\hat{J} = \bigcup_{\gamma'<\alpha'} \hat{J}_{\gamma'}$, so $\hat{J}$ is Δ_0. If α' is a successor, then $\hat{J}$ is naturally defined from $\hat{J}_{\gamma'}$ with $\gamma' + 1 = \alpha'$, and $\hat{J}$ is Δ_0.

Now we prove by induction on n the following claim:

(*) Let $j_i : L_\beta(V_{\lambda+1}) \prec L_\beta(V_{\lambda+1})$, with $\mathrm{crt}(j_0) < \mathrm{crt}(j_1) < \ldots \uparrow \lambda'$. Let J be the inverse limit of the elementary embeddings $j_i \upharpoonright V_{\lambda+1}$. Let $\ll$ a J-projective pwo, with $\ll_{\lambda'}$ the respective pwo in $V_{\lambda'+1}$. Let $a \in V_{\lambda'+1}$ with $\|a\|_{\lambda'} = \alpha'$ and $\|J(a)\| = \alpha$. Suppose that $j_i(\alpha) = \alpha$ for every $i \in \omega$ and that $\hat{J} : L_{\alpha'}(V_{\lambda'+1}) \prec_0 L_\alpha(V_{\lambda+1})$. If there exists ρ good such that $\rho + n \leq \beta$, then $\hat{J} : L_{\alpha'}(V_{\lambda'+1}) \prec_n L_\alpha(V_{\lambda+1})$, with $\hat{J}$ induced by J and $\ll$.

Let ρ be the minimum good ordinal such that $\alpha \leq \rho + n + 1 \leq \beta$. Then $j(\rho) = \rho$. Suppose that $\hat{J}$ is a Σ_n-elementary embedding. Then we just need to prove that if $L_\alpha(V_{\lambda+1}) \vDash \exists x\ \varphi'(x, \hat{J}(y))$ then $L_{\alpha'}(V_{\lambda'+1}) \vDash \exists x\ \varphi'(x, y)$, with $y \in L_{\alpha'}(V_{\lambda'+1})$ and φ' a Π_n formula. For any element of $L_{\alpha'}(V_{\lambda'+1})$, there exist $C_0, C_1 \in V_{\lambda'+1}$ and φ formula such that

$$y = \{x : L_{\|C_0\|_{\lambda'}}(V_{\lambda'+1}) \vDash \varphi(x, C_1)\},$$

with $\|C_0\|_{\lambda'} < \alpha'$. So let x^* be a witness for φ', i.e., $L_\alpha(V_{\lambda+1}) \vDash \varphi'(x^*, \hat{J}(y))$. Then, as before, there exist $D^*_0, D^*_1 \in V_{\lambda+1}$ and a formula ψ such that

$$x^* = \{x : L_{\|D^*_0\|}(V_{\lambda+1}) \vDash \psi(x, D^*_1)\}.$$

Using Lemma 0.15 we define k_i, $D^*_{r,i}$ with $r = 0, 1$ such that:

- $k_i : L_{\rho+n}(V_{\lambda+1}) \prec L_{\rho+n}(V_{\lambda+1})$;
- $\mathrm{crt}(k_0) < \mathrm{crt}(j_0) < \mathrm{crt}(k_1) < \mathrm{crt}(j_1) < \ldots$
- $k_i(J_{i+1}(C_r)) = j_i(J_{i+1}(C_r))$, $k_i(J_{i+1}(E)) = j_i(J_{i+1}(E))$;
- $k_i(D^*_{r,i+1}) = D^*_{r,i}$, $D^*_{r,0} = D^*_r$.

The definition is, as always, by induction. Since we are supposing $\rho+n+1 \geq \beta$, we have that $\rho + n < \beta$ and it is definable in $L_\beta(V_{\lambda+1})$, so by Lemma 0.15 there exist $k_i : L_{\rho+n}(V_{\lambda+1}) \prec L_{\rho+n}(V_{\lambda+1})$ and $D^*_{r,i+1} \in V_{\lambda+1}$ such that $k_i(J_{i+1}(C_0)) = j_i(J_{i+1}(C_0))$, $k_i(J_{i+1}(C_1)) = j_i(J_{i+1}(C_1))$, $k_i(J_{i+1}(E)) = j_i(J_{i+1}(E))$, $k_i(J_{i+1}(a)) = j_i(J_{i+1}(a))$, $k_i(D^*_{r,i+1}) = D^*_{r,i}$ and $\mathrm{crt}(j_{i-1}) < \mathrm{crt}(k_i) < \mathrm{crt}(j_i)$. Now let K be the inverse limit of $k_i \upharpoonright V_{\lambda+1}$. Then $\sup \mathrm{crt}\, k_i = \lambda'$, $K(C_r) = J(C_r)$, $K(E) = J(E)$, $K(a) = J(a)$ (and therefore $\ll$ is also K-projective and the respective $\ll_{\lambda'}$ does not change), and there exist D_r such that $K(D_r) = D_r^*$. By induction $\hat{K} : L_{\alpha'}(V_{\lambda+1}) \prec_n L_\alpha(V_{\lambda+1})$. If we define

$$x = \{x : L_{\|D_0\|_{\lambda'}}(V_{\lambda'+1}) \vDash \psi(x, D_1)\}$$

then by definition $\hat{K}(x) = x^*$. Since $K(C_r) = J(C_r)$, it is also true that $\hat{K}(y) = \hat{J}(y)$, so $L_\alpha(V_{\lambda+1}) \vDash \varphi(\hat{K}(x), \hat{K}(y))$. By elementarity we have $L_{\alpha'}(V_{\lambda'+1}) \vDash \varphi(x, y)$ so $L_{\alpha'}(V_{\lambda'+1}) \vDash \exists x\ \varphi(x, y)$. □

The previous Theorem gives a very precise idea on how much an inverse limit can be extended, with the only limitation being the fact that α must be "measured" by some J-projective pwo. How large can α be?

Theorem 0.20 (Woodin [3])**.** *Under I0, the supremum of the lengths of λ-projective pwos of $V_{\lambda+1}$ is $\geq \lambda^{+\lambda}$.*

But thanks to Theorem 0.15, any λ-projective $\ll$ can be J-projective, and this leads to several results.

Theorem 0.21 (Laver, [2])**.** *"There exists $j : L_{\lambda^+ + \omega + 1}(V_{\lambda+1}) \prec L_{\lambda^+ + \omega + 1}(V_{\lambda+1})$" strongly implies "There exists $k : L_{\lambda^+}(V_{\lambda+1}) \prec L_{\lambda^+}(V_{\lambda+1})$".*

Proof. Let $\ll$ be any λ-projective pwo longer than λ^+. Let $E^* \in V_{\lambda+1}$ be the parameter used in its definition, and let $a^* \in V_{\lambda+1}$ such that $\|a^*\| = \lambda^+$. We construct by Lemma 0.15 $k_0, k_1, \cdots : L_{\lambda^+ + \omega}(V_{\lambda+1}) \prec L_{\lambda^+ + \omega}(V_{\lambda+1})$ such that the inverse limit K of their restrictions to $V_{\lambda+1}$ is such that there exist E, a, j' with $K(E) = E^*$, $K(a) = a^*$ and $K(j') = j \upharpoonright V_\lambda$. As $K(E) = E^*$, $\ll$ is K-projective. Let $\|a\|_{\lambda'} = \alpha'$. Since $\|K(a)\| = \|a^*\| = \lambda^+$, by Theorem 0.19 K induces a $\hat{K} : L_{\alpha'}(V_{\lambda'+1}) \prec L_{\lambda^+}(V_{\lambda+1})$. But

$$\begin{aligned} L_{\lambda^+}(V_{\lambda+1}) \vDash \forall \alpha \in \mathrm{Ord}\ \exists \text{ well-ordering of } V_\lambda \text{ of order-type } \alpha \wedge \\ \wedge \text{ every well-ordering of } V_\lambda \text{ is codified by some } \alpha \in \mathrm{Ord} \quad (1) \end{aligned}$$

therefore $L_{\alpha'}(V_{\lambda'+1})$ satisfies the same thing and $\alpha' = (\lambda')^+$, so $\hat{K} : L_{(\lambda')^+}(V_{\lambda'+1}) \prec L_{\lambda^+}(V_{\lambda+1})$. Since $j(\lambda^+) = \lambda^+$, $j \upharpoonright L_{\lambda^+}(V_{\lambda+1})$ is an elementary embedding in $L_{\lambda^+}(V_{\lambda+1})$, i.e.,

$$L_{\lambda^+}(V_{\lambda+1}) \vDash \forall x_1, \dots, x_n \in V_{\lambda+1}\ \varphi(x_1, \dots, x_n) \leftrightarrow \varphi(j(x_1), \dots, j(x_n)),$$

and by elementarity

$$L_{(\lambda')^+}(V_{\lambda'+1}) \vDash \forall x_1, \dots, x_n \in V_{\lambda'+1}\ \varphi(x_1, \dots, x_n) \leftrightarrow \varphi(j'(x_1), \dots, j'(x_n)).$$

Since $(\lambda')^+$ is good, this naturally translates as

$$L_{(\lambda')^+}(V_{\lambda'+1}) \vDash \forall x_1, \dots, x_n\ \varphi(x_1, \dots, x_n) \leftrightarrow \varphi(j'(x_1), \dots, j'(x_n)),$$

i.e., $j' : L_{(\lambda')^+}(V_{\lambda'+1}) \prec L_{(\lambda')^+}(V_{\lambda'+1})$. □

It is worth to note that λ^+ has a special role in the proof, but not a unique one. It is possible to generalize the proof to any ordinal that has the same properties of λ^+ that are used: first, the ordinal must be good; second, it must be definable enough. The key of the proof is display 1: there exists a sentence σ such that if $L_\alpha(V_{\lambda+1}) \vDash \sigma$ then $\alpha = \lambda^+$. We call all such ordinals *uniformly definable*. Then:

Corollary 0.22. *Let α be a good uniformly definable ordinal. Then "There exists $j : L_{\alpha+\omega+1}(V_{\lambda+1}) \prec L_{\alpha+\omega+1}(V_{\lambda+1})$" strongly implies "There exists $k : L_\alpha(V_{\lambda+1}) \prec L_\alpha(V_{\lambda+1})$".*

The smallest application of this is with $\alpha = n$, so "There $j : L_{\omega+1}(V_{\lambda+1}) \prec L_{\omega+1}(V_{\lambda+1})$" strongly implies for every $n \in \omega$ "There exists $k : L_n(V_{\lambda+1}) \prec L_n(V_{\lambda+1})$". It is easy to find many other examples of this kind.

Since by Lemma 0.14 also elementary embeddings from $L_\Theta(V_{\lambda+1})$ to itself can be reflected, "There exists $j : L_\Theta(V_{\lambda+1}) \prec L_\Theta(V_{\lambda+1})$" strongly implies for any α good, uniformly definable and λ-projective "There exists $k : L_\alpha(V_{\lambda+1}) \prec L_\alpha(V_{\lambda+1})$"

For further results on strong implications of hypothesis between I1 and I0, see [2].

References

[1] R. Laver, *Implications Between Strong Large Cardinal Axioms.* Annals of Pure and Applied Logic **90** (1997) 79–90.

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