Introduction to I0: Elementary Embeddings

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Main Sources: [1]

Main Results: Basic notions about $L(V_{\lambda+1})$. Definition of weakly proper elementary embedding. A weakly proper elementary embedding depends only on its behaviour on V_λ.

The next phase in the analysis of the rank-into-rank axioms involves the scanning of the territory between I1 and I0. This will take the next chapter, before that we will fix some notion and spend some efforts for a better understanding for I0.

Definition 0.1. I0 *There exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with critical point less then λ.*

The added assumption for the critical point is necessary to put I0 in the same branch of the other rank-into-rank axioms. If j witness I0, in fact, $j \upharpoonright V_{\lambda+1}$ witness I1, and so λ is the supremum of the critical sequence.

Note that if I0 is true, then $L(V_{\lambda+1}) \nvDash \mathsf{AC}$, because otherwise we could use Kunen's Theroem to prove that there is no elementary embedding.

One of the big peculiarities of I0 is its affinity with AD in $L(\mathbb{R})$ (we'll se these in Chapter Five). In fact, this similarities are grounded on some basic ones between $L(V_{\lambda+1})$ and $L(\mathbb{R})$ themselves.

Lemma 0.2. *There exists a definable surjection $\Phi : Ord \times V_{\lambda+1} \twoheadrightarrow L(V_{\lambda+1})$.*

Proof. This is immediate from the theory of relative constructibility. □

The first application of this Lemma comes in form of partial Skolem functions. Since $L(V_{\lambda+1}) \nvDash \mathsf{AC}$, we possibly cannot have Skolem function. But since by the previous Lemma $L(V_{\lambda+1}) \vDash V = \mathrm{HOD}_{V_{\lambda+1}}$ we can define for every formula $\varphi(x, x_1, \dots, x_n)$, $a \in V_{\lambda+1}$, $a_1, \dots, a_n \in L(V_{\lambda+1})$:

$$h_{\varphi,a}(a_1, \dots, a_n) = y \quad \text{where } y \text{ is the minimum in } (\mathrm{OD}_a)^{L(V_{\lambda+1})} \text{ such that } L(V_{\lambda+1}) \vDash \varphi(y, a_1, \dots, a_n).$$

These are partial Skolem functions, and the Skolem Hull of a set is its closure under all the Skolem functions.

Definition 0.3.

$$\Theta^{L(V_{\lambda+1})}_{V_{\lambda+1}} = \sup\{\gamma : \exists f : V_{\lambda+1} \twoheadrightarrow \gamma, f \in L(V_{\lambda+1})\}$$

In the following, we will call $\Theta^{L(V_{\lambda+1})}_{V_{\lambda+1}} = \Theta$, because it is a lighter notation and there is no possibility of misinterpretation.

The role of Θ in $L(V_{\lambda+1})$ is exactly the same of its correspondent in $L(\mathbb{R})$. In the usual setting, under AC, to measure the largeness of a set we fix a bijection from this set to a cardinal or, equivalently, the ordertype of a well-ordering of the set. Since there is no Axiom of Choice in $L(V_{\lambda+1})$, it is not always possible to define cardinality for sets that are not in $V_{\lambda+1}$ in the usual way, so to quantify the "largeness" of a subset of $V_{\lambda+1}$ we will not use bijections, but surjections, or, equivalently, not well-orders, but prewellorderings (pwo for short).

An order is a pwo if it satisfies antireflexivity, transitivity, and every subset has a least element; in other words, it is a well-order without the antisymmetric property. It is easy to see that the counterimage of a surjective function is a pwo. One can image a pwo as an order whose equivalence classes are well-ordered, or a well-ordered partition. This creates a strong connection between subsets of $V_{\lambda+1}$ and ordinals in Θ:

Lemma 0.4. *1. For every $\alpha < \Theta$, there exists in $L(V_{\lambda+1})$ a pwo in $V_{\lambda+1}$ with ordertype α, that is codeable as a subset of $V_{\lambda+1}$;*

2. for every $Z \subseteq V_{\lambda+1}$, $Z \in L(V_{\lambda+1})$ there exists $\alpha < \Theta$ such that $Z \in L_\alpha(V_{\lambda+1})$.

Proof. 1. Let $\rho : V_{\lambda+1} \twoheadrightarrow \alpha$. Then

$$R_\alpha = \{(a,b) \in V_{\lambda+1} \times V_{\lambda+1} : \rho(a) \leq \rho(b)\}$$

is a pwo in $V_{\lambda+1}$. Moreover, $V_{\lambda+1} \times V_{\lambda+1}$ can be codified as a subset of $V_{\lambda+1}$, so also R_α can.

2. Let γ be such that $Z \in L_\gamma(V_{\lambda+1})$ and consider $H^{L_\gamma(V_{\lambda+1})}(V_{\lambda+1}, Z)$ the Skolem Hull in $L_\gamma(V_{\lambda+1})$ of $V_{\lambda+1}$ and Z. Then, since $H^{L_\gamma(V_{\lambda+1})}(V_{\lambda+1}, Z) \cong L_\gamma(V_{\lambda+1})$, by condensation its collapse $\mathcal{X} = L_\alpha(V_{\lambda+1})$ for some α. But $H^{L_\gamma(V_{\lambda+1})}(V_{\lambda+1}, Z)$ is the closure under the Skolem functions, and since there is a surjection from $V_{\lambda+1}$ to the Skolem functions, this surjection transfers to $H^{L_\gamma(V_{\lambda+1})}(V_{\lambda+1}, Z)$ and to $L_\alpha(V_{\lambda+1})$, so $\alpha < \Theta$. Since Z

and all its elements are in $H^{L_\gamma(V_{\lambda+1})}(V_{\lambda+1}, Z)$, Z is not collapsed and then $Z \in L_\alpha(V_{\lambda+1})$.

$\square$

Definition 0.5.

$$\mathsf{DC}_\lambda : \quad \forall X\ \forall F : (X)^{<\lambda} \to \mathcal{P}(X) \setminus \emptyset\ \exists g : \lambda \to X\ \forall \gamma < \lambda\ g(\gamma) \in F(g \restriction \gamma).$$

Note that this is a generalization of DC, since $\mathsf{DC} = \mathsf{DC}_\omega$: we use directly a function on $< \lambda$-sequences instead of considering a binary relation because binary relations cannot handle the limit stages.

Lemma 0.6. *In $L(V_{\lambda+1})$ the following hold:*

1. *Θ is regular;*

2. *DC_λ.*

Proof. 1. We fix a definable surjection $\Phi : Ord \times V_{\lambda+1} \twoheadrightarrow L(V_{\lambda+1})$. For every $\xi < \Theta$ there is a surjection $h : V_{\lambda+1} \twoheadrightarrow \xi$. First of all, we choose one surjection for each ξ: we define $t : \Theta \setminus \emptyset \to Ord$, where for every $\xi < \Theta$, $t(\xi)$ is the least γ such that there exists $x \in V_{\lambda+1}$ such that $\Phi(\gamma, x)$ is a surjection from $V_{\lambda+1}$ to ξ. Then we define

$$h_\xi(\langle x, y\rangle) = \begin{cases} \Phi(t(\xi), x)(y) & \text{if } \Phi(t(\xi), x) \text{ is a map in } \xi; \\ \emptyset & \text{else.} \end{cases}$$

We have that $h_\xi : V_{\lambda+1} \to \xi$ is well defined, because $\langle x, y\rangle$ is codeable in $V_{\lambda+1}$, and it is indeed a surjection: by definition there exists $x \in V_{\lambda+1}$ such that $\Phi(t(\xi), x)$ is a surjection from $V_{\lambda+1}$ to ξ, so for every $\beta < \xi$ there exists $y \in V_{\lambda+1}$ such that $\Phi(t(\xi), x)(y) = h_\xi(\langle x, y\rangle) = \beta$. Moreover h_ξ is definable in $L(V_{\lambda+1})$.

Now, suppose that Θ is not regular, i.e., there exists $\pi : \alpha \to \Theta$ cofinal in Θ with $\alpha < \Theta$. Then we claim that $H(\langle x, y\rangle) = h_{\pi \circ h_\alpha(x)}(y)$ is a surjection from $V_{\lambda+1}$ to Θ: let $\beta < \Theta$; then there exists $\gamma < \alpha$ such that $\pi(\gamma) > \beta$ and there must exist $x \in V_{\lambda+1}$ such that $h_\alpha(x) = \gamma$; so $\pi(h_\alpha(x)) > \beta$, and there exists $y \in V_{\lambda+1}$ such that $H(\langle x, y\rangle) = h_{\pi \circ h_\alpha(x)}(y) = \beta$. Contradiction.

2. We have to prove that

$$\forall X\ \forall F : (X)^{<\lambda} \to \mathcal{P}(X) \setminus \emptyset\ \exists g : \lambda \to X\ \forall \gamma < \lambda\ g(\gamma) \in F(g \restriction \gamma).$$

The proof is through several steps. First of all, $\mathsf{DC}_\lambda(V_{\lambda+1})$, that is DC_λ only for $X = V_{\lambda+1}$, is quite obvious, because for every F as above by AC there exists a g as above in V, but since g is a λ-sequence of elements in $V_{\lambda+1}$ we have that g is codeable in $V_{\lambda+1}$, so $g \in L(V_{\lambda+1})$.

Then we prove $\mathsf{DC}_\lambda(\alpha \times V_{\lambda+1})$ for every ordinal α. The idea is roughly to divide F in two parts, and to define g using the minimum operator for the ordinal part, and $\mathsf{DC}_\lambda(V_{\lambda+1})$ for the other part. For every $s \in (\alpha \times V_{\lambda+1})^{<\lambda}$, we define $m(s)$ as the minimum γ such that there exists $x \in V_{\lambda+1}$ such that $(\gamma, x) \in F(s)$. We call $\pi_2 : (\alpha \times V_{\lambda+1})^{<\lambda} \to (V_{\lambda+1})^{<\lambda}$ the projection. For every $t \in (V_{\lambda+1})^{<\lambda}$, say $t = \langle x_\xi : \xi < \nu \rangle$, we define $c(t)$ by induction as a sequence in $(\alpha \times V_{\lambda+1})^{<\lambda}$ such that $\pi_2(c(t)) \subseteq t$: $c(t) = \langle (\gamma_\xi, x_\xi) : \xi < \bar{\nu} \rangle$, where

$$\gamma_\xi = \min\{\gamma : (\gamma, x_\xi) \in F(c(t \upharpoonright \xi))\},$$

so that $\bar{\nu}$ is the smallest one such that there is no γ such that $(\gamma, x_{\bar{\nu}}) \in F(c(t) \upharpoonright \bar{\nu})$. Let $G : (V_{\lambda+1})^{<\lambda} \to \mathcal{P}(V_{\lambda+1}) \setminus \emptyset$ defined as

$$G(t) = \{x \in V_{\lambda+1} : (m(c(t)), x) \in F(c(t))\},$$

then by $\mathsf{DC}_\lambda(V_{\lambda+1})$ there exists $g : \lambda \to V_{\lambda+1}$ such that for every $\beta < \lambda$ $g(\beta) \in G(g \upharpoonright \beta)$. Now let $f : \lambda \to (\alpha \times V_{\lambda+1})$ be defined by induction, $f(\beta) = (m(f \upharpoonright \beta), g(\beta))$. We want to prove that $f(\beta) \in F(f \upharpoonright \beta)$ for every $\beta < \lambda$.

We prove by induction that $f \upharpoonright \beta = c(g \upharpoonright \beta)$. Suppose that for every $\xi < \beta$, $f \upharpoonright \xi = c(g \upharpoonright \xi)$. By definition $c(g \upharpoonright \beta) = \langle (\gamma_\xi, g(\xi)) : \xi < \bar{\beta} \rangle$, with

$$\gamma_\xi = \min\{\gamma : (\gamma, g(\xi)) \in F(c(g \upharpoonright \xi))\},$$

so $(\gamma_\xi, g(\xi)) \in F(c(g \upharpoonright \xi))$. But $g(\xi) \in G(g \upharpoonright \xi)$, so by definition of G,

$$(m(c(g \upharpoonright \xi)),\ g(\xi)) \in F(c(g \upharpoonright \xi)),$$

therefore $\gamma_\xi = m(c(g \upharpoonright \xi))$ and

$$f(\xi) = (m(f \upharpoonright \xi), g(\xi)) = (m(c(g \upharpoonright \xi)), g(\xi)) = (\gamma_\xi, g(\xi)) = c(g \upharpoonright \beta)(\xi).$$

So $f \upharpoonright \beta = c(g \upharpoonright \beta)$ and, since for every ξ, $(\gamma_\xi, g(\xi)) \in F(c(g \upharpoonright \xi))$, $f(\beta) \in F(f \upharpoonright \beta)$.

Finally, let $X \in L(V_{\lambda+1})$. Let α be such that $\Phi''(\alpha \times V_{\lambda+1}) \supseteq X$, and let $F : (X)^{<\lambda} \to \mathcal{P}(X) \setminus \emptyset$. For every $t = \langle (\gamma_\xi, x_\xi) : \xi < \nu \rangle \in (\alpha \times V_{\lambda+1})^{<\lambda}$, we call $c(t) = \langle \Phi(\gamma_\xi, x_\xi) : \xi < \bar{\nu} \rangle \in X^{<\lambda}$, where $\bar{\nu}$ is the minimum such that $\Phi(\gamma_{\bar{\nu}}, x_{\bar{\nu}}) \notin X$. Then we define $G : (\alpha \times V_{\lambda+1})^{<\lambda} \to \mathcal{P}(\alpha \times V_{\lambda+1}) \setminus \emptyset$,

$$G(t) = \{(\gamma, x) : \Phi(\gamma, x) \in F(c(t))\},$$

and by $\mathsf{DC}_\lambda(\alpha \times V_{\lambda+1})$ we find $g : \lambda \to (\alpha \times V_{\lambda+1})$ such that $g(\beta) \in G(g \upharpoonright \beta)$ for every $\beta < \lambda$. Then $f = \Phi \circ g$ is as we wanted, because for every $\beta < \lambda$, $\Phi(g(\beta)) \in F(c(g \upharpoonright \beta))$, and as above we can prove that $c(g \upharpoonright \beta) = f \upharpoonright \beta$.

So we have $\mathsf{DC}_\lambda(X)$ for every $X \in L(V_{\lambda+1})$, that is exactly DC_λ.

□

Now we have a sufficient understanding of the structure of $L(V_{\lambda+1})$ for starting a study on its elementary embeddings, that is essential for an analysis of the hypothesis between I1 and I0.

Fix until the end of the chapter a $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$. Define

$$U_j = \{X \subseteq V_{\lambda+1} : X \in L(V_{\lambda+1}), j \upharpoonright V_\lambda \in j(X)\}.$$

Then U_j is a normal non-principal $L(V_{\lambda+1})$-ultrafilter in $V_{\lambda+1}$, and we can construct the ultraproduct $Ult(L(V_{\lambda+1}), U_j)$. Note that for every $f, g : V_{\lambda+1} \to L(V_{\lambda+1})$

$$\begin{aligned} [f] = [g] &\text{ iff } \{x \in V_{\lambda+1} : f(x) = g(x)\} \in U_j \\ &\text{ iff } j \upharpoonright V_\lambda \in j(\{x \in V_{\lambda+1} : f(x) = g(x)\}) = \\ &\qquad = \{x \in V_{\lambda+1} : j(f)(x) = j(g)(x)\} \\ &\text{ iff } j(f)(j \upharpoonright V_\lambda) = j(g)(j \upharpoonright V_\lambda), \end{aligned}$$

and in the same way $[f] \in [g]$ iff $j(f)(j \upharpoonright V_\lambda) \in j(g)(j \upharpoonright V_\lambda)$, so

$$Ult(L(V_{\lambda+1}), U_j) \cong \{j(f)(j \upharpoonright V_\lambda) : \mathrm{dom}\, f = V_{\lambda+1}\}.$$

Let $i : L(V_{\lambda+1}) \to Ult(L(V_{\lambda+1}), U_j)$ be the natural embedding of the ultraproduct, then for every $a \in L(V_{\lambda+1})$, $i(a) = [c_a]$ corresponds in the equivalence to $j(c_a)(j \upharpoonright V_\lambda) = j(a)$, so we can suppose $i = j$. Is j an elementary embedding from $L(V_{\lambda+1})$ to $Ult(L(V_{\lambda+1}), U_j)$? Since we don't have AC, the answer is not immediate because we possibly don't have Łos' Theorem.

We will prove Łos' Theorem for this case, and this will imply that j is an elementary embedding. It is clear that the only real obstacle is to prove that for every formula φ and $f_1, \dots, f_n \in L(V_{\lambda+1})$ such that $\mathrm{dom}\, f_i = V_{\lambda+1}$

$$\begin{aligned} &Ult(L(V_{\lambda+1}), U_j) \vDash \exists x\ \varphi([f_1], \dots, [f_n]) \\ &\qquad\qquad \text{iff } \{x \in V_{\lambda+1} : L(V_{\lambda+1}) \vDash \exists y\ \varphi(f_1(x), \dots, f_n(x))\} \in U_j. \end{aligned}$$

The direction from left to right is immediate: if $[g]$ witness the left side, then $g(x)$ witness the right side. For the opposite direction, we need a sort of U_j-choice, i.e. we need to find a function g such that

$$\{x \in V_{\lambda+1} : L(V_{\lambda+1}) \vDash \varphi(g(x), f_1(x), \dots, f_n(x))\} \in U_j.$$

We re-formulate this considering $f : V_{\lambda+1} \to L(V_{\lambda+1}) \setminus \emptyset$,

$$f(x) = \{y \in L(V_{\lambda+1}) : L(V_{\lambda+1}) \vDash \varphi(y, f_1(x), \dots, f_n(x))\}.$$

Lemma 0.7. *Let $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ and U_j as above. Then for every $F : V_{\lambda+1} \to L(V_{\lambda+1}) \setminus \emptyset$ there exists $H : V_{\lambda+1} \to L(V_{\lambda+1}) \setminus \emptyset$ such that $\{x \in V_{\lambda+1} : H(x) \in F(x)\} \in U_j$.*

Proof. First we consider the case $\forall a \in V_\lambda \ F(a) \subseteq V_{\lambda+1}$. We have to define H such that $j(H)(j \restriction V_\lambda) \in j(F)(j \restriction V_\lambda)$. Fix a $b \in j(F)(j \restriction V_\lambda)$, and define

$$H(k) = \begin{cases} c & \text{if } k : V_\lambda \prec V_\lambda,\ k(k) = j \text{ and } k(c) = b \\ 0 & \text{otherwise} \end{cases}$$

Note that

$$K_b := \{k \in V_{\lambda+1} \mid k : V_\lambda \prec V_\lambda, k(k) = j, \exists c \ k(c) = b\} \in U_j,$$

because

$$j(K_b) = \{k \in V_{\lambda+1} \mid k : V_\lambda \prec V_\lambda, k(k) = j(j), \exists c \ k(c) = j(b)\}$$

and $j \restriction V_\lambda \in j(K_b)$ (with $c = b$), so $\{x \in V_{\lambda+1} : H(x) \neq \emptyset\} \in U_j$. Then

$$j(H)(k) = \begin{cases} c & \text{if } k : V_\lambda \prec V_\lambda,\ k(k) = j(j) \text{ and } k(c) = j(b) \\ 0 & \text{otherwise} \end{cases}$$

so $j(H)(j \restriction V_\lambda) = b \in j(F)(j \restriction V_\lambda)$.

For the more general case $\forall a \in V_\lambda \ F(a) \subseteq L(V_{\lambda+1})$ fix $\Phi : Ord \times V_{\lambda+1} \twoheadrightarrow L(V_{\lambda+1})$ definable and define

$$\hat{F}(a) = \{x \in V_{\lambda+1} : \exists \gamma \ \Phi(\gamma, x) \in F(a)\}.$$

Then there exists $\hat{H}$ such that $\{a \in V_{\lambda+1} : \hat{H}(a) \in \hat{F}(a)\} \in U_j$. Let $\gamma_a = \min\{\gamma : \Phi(\gamma, \hat{H}(a)) \in F(a)\}$. Therefore $H(a) = \Phi(\gamma_a, \hat{H}(a))$ is as desired. □

Therefore, calling $\mathcal{Z} = \{j(f)(j \restriction V_\lambda) : f \in L(V_{\lambda+1}), \mathrm{dom}(F) = V_{\lambda+1}\}$, $j : L(V_{\lambda+1}) \to \mathcal{Z}$ is an elementary embedding, and $\mathcal{Z} \cong L(V_{\lambda+1})$. Let k_U be the inverse of the collapse of $\mathcal{Z}$. We've seen in the proof of the previous Lemma that for every $b \in V_{\lambda+1}$ there exists h such that $j(h)(j \restriction V_\lambda)$, so $V_{\lambda+1} \subseteq \mathcal{Z}$ and $k_U : L(V_{\lambda+1}) \prec \mathcal{Z}$. Moreover, if R is a pwo in $V_{\lambda+1}$, then $R = \{a \in V_{\lambda+1} : j(a) \in j(R)\}$, and since $j(a), j(R) \in \mathcal{Z}$ and $V_{\lambda+1} \subseteq \mathcal{Z}$ we have that R is not collapsed, so $\Theta \subseteq \mathcal{Z}$ and $\mathrm{crt}(k_U) > \Theta$.

Theorem 0.8 (Woodin, [1]). *For every $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ there exist a $L(V_{\lambda+1})$-ultrafilter U in $V_{\lambda+1}$ and $j_U, k_U : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ such that j_U is the elementary embedding from U, $j = j_U \circ k_U$ and $j \restriction L_\Theta(V_{\lambda+1}) = j_U \restriction L_\Theta(V_{\lambda+1})$.*

Definition 0.9. *Let $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$. We say that j is* weakly proper *if $j = j_U$.*

Lemma 0.10 (Woodin, [1]). *For every $j_1, j_2 : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$, if $j_1 \restriction V_\lambda = j_2 \restriction V_\lambda$ then $j_1 \restriction L_\Theta(V_{\lambda+1}) = j_2 \restriction L_\Theta(V_{\lambda+1})$.*

Proof. We can suppose that j_1 and j_2 are weakly proper. By the usual analysis of the ultraproduct, we have that every strong limit cardinal with cofinality bigger than Θ is a fixed point for j_1 and j_2, so $I = \{\eta : j_1(\eta) = j_2(\eta) = \eta\}$ is a proper class. Let $M = H^{L(V_{\lambda+1})}(I \cup V_{\lambda+1})$. Since $V_{\lambda+1} \subseteq M$ we have that $\Theta \subseteq M$, so k^*, the inverse of the collapse, has domain $L(V_{\lambda+1})$. If k^* is not the identity, then $\mathrm{crt}(k^*) > \Theta$. But in that case $\mathrm{crt}(k^*)$ is a strong limit cardinal with cofinality bigger than Θ, so $\mathrm{crt}(k^*) \in I$, and this is a contradiction, because $I \subseteq \mathrm{ran}(k^*)$ and $\mathrm{crt}(k^*) \notin \mathrm{ran}(k^*)$. So k^* is the identity and $L(V_{\lambda+1}) = H^{L(V_{\lambda+1})}(I \cup V_{\lambda+1})$.

Therefore every element of $L_\Theta(V_{\lambda+1})$ is definable with parameters in $I \cup V_{\lambda+1}$. Let $A \in L(V_{\lambda+1}) \cap V_{\lambda+2}$, $A = \{x \in V_{\lambda+1} : L(V_{\lambda+1}) \vDash \varphi(\eta, a)\}$ with $\eta \in I$ and $a \in V_{\lambda+1}$. Then $j_1(A) = \{x \in V_{\lambda+1} : L(V_{\lambda+1}) \vDash \varphi(j_1(\eta), j_1(a))\} = \{x \in V_{\lambda+1} : L(V_{\lambda+1}) \vDash \varphi(\eta, j_2(a))\} = j_2(A)$. But every element of $L_\Theta(V_{\lambda+1})$ is definable from an ordinal $\alpha < \Theta$ and an element of $V_{\lambda+1}$, α is definable from some pwo in $L(V_{\lambda+1}) \cap V_{\lambda+2}$, so $j_1 \restriction L_\Theta(V_{\lambda+1}) = j_2 \restriction L_\Theta(V_{\lambda+1})$. □

In other words, we can group together all the elementary embeddings from $L(V_{\lambda+1})$ to itself depending on their behaviour on V_λ. Between all the elementary embeddings that share the same $j \restriction V_\lambda$, there is one (and only one) that come from an ultraproduct, and it is the weakly proper one. All the others are equal on $L_\Theta(V_{\lambda+1})$, but outside can differ, for example shifting indiscernibles, if there are any.

References

[1] H. Woodin, *An AD-like axiom*. Unpublished.