Cardinal invariants and the generalized Baire spaces

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11th Young set theory workshop
Bernoulli Center. Laussane, June 25, 2018

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Contents

Section 1

Classic cardinal invariants and their generalizations

Motivation

"Cardinal invariants are simply the smallest cardinals $\leq \mathfrak{c}$ for which various results, true for $\aleph_0$, become false..."

Andreas Blass, *Combinatorial cardinal characteristics of the continuum*, 2010

Classical cardinal invariants of the Baire space ω^ω have been extensively studied and understood. Moreover, it is possible to directly abstract several definitions from ω to an arbitrary uncountable cardinal κ.

The generalized Baire spaces

Let κ be an uncountable regular cardinal satisfying $\kappa^{<\kappa} = \kappa$. The generalized Baire space is just the set of functions κ^κ endowed with the topology generated by the sets of the form: $[s] = \{f \in \kappa^\kappa : f \supseteq s\}$ for $s \in \kappa^{<\kappa}$.

Denote NWD_κ to be the collection of nowhere dense subsets of $\kappa^{<\kappa}$ with respect to this topology, recall that a set $A \subseteq \kappa^\kappa$ is *nowhere dense* if for every $s \in \kappa^{<\kappa}$ there exists $t \supseteq s$ such that $[t] \cap A = \emptyset$.

Then it we define the generalized κ-*meager* sets in κ^κ to be κ-unions of elements in NWD_κ and denote $\mathcal{M}_\kappa$ to be the κ-ideal that κ-meager sets determine (here κ-ideal means an ideal that in addition is closed under unions of size $\leq \kappa$).

It is well known that the Baire category theorem can be lifted to this context, i.e. it holds that the intersection of κ-many open dense sets is open (Friedman, Hyttinen, Kulikov, 2014).

The beginning

Since 1995, with the paper *"Cardinal invariants above the continuum "* from Cummings and Shelah, the study of the invariants associated to these spaces and their interactions has been developing.

It is also important to mention that the study of these spaces has been also approached from the point of view of Descriptive Set Theory (Friedman, Hyttinen, Kulikov, Motto Ros, Moreno) and Topology (Korch).

Some cardinal invariants

Definition

If f, g are functions in κ^κ, we say that $f <^* g$, if there exists an $\alpha < \kappa$ such that for all $\beta > \alpha$, $f(\beta) < g(\beta)$. In this case, we say that g eventually dominates *f.*

Definition

Let $\mathfrak{F}$ be a family of functions from κ to κ.

- $\mathfrak{F}$ is dominating, *if for all $g \in \kappa^\kappa$, there exists an $f \in \mathfrak{F}$ such that $g <^* f$.*
- $\mathfrak{F}$ is unbounded, *if for all $g \in \kappa^\kappa$, there exists an $f \in \mathfrak{F}$ such that $f \not<^* g$.*

The unbounding and dominating numbers

Definition

- *The unbounding number:*

$$\mathfrak{b}(\kappa) = \text{min}\{|\mathfrak{F}|: \mathfrak{F} \textit{ is an unbounded family of functions in } \kappa^\kappa\}$$

- *The dominating number:*

$$\mathfrak{d}(\kappa) = \text{min}\{|\mathfrak{F}|: \mathfrak{F} \textit{ is a dominating family of functions in } \kappa^\kappa\}$$

Cardinal invariants associated to an ideal

Let $\mathcal{I}$ be a κ-ideal (closed under κ-sized unions) on κ^κ:

Definition

- *The additivity number:*

$$\text{add}(\mathcal{I}) = \text{min}\{|\mathscr{J}|\colon \mathscr{J} \subseteq \mathcal{I} \textit{ and } \bigcup \mathscr{J} \notin \mathcal{I}\}.$$

- *The covering number:*

$$\text{cov}(\mathcal{I}) = \text{min}\{|\mathscr{J}|\colon \mathscr{J} \subseteq \mathcal{I} \textit{ and } \bigcup \mathscr{J} = \kappa^\kappa\}.$$

Definition

- *The cofinality number:*

$$\text{cof}(\mathcal{J}) = \min\{|\mathscr{J}|\colon \mathscr{J} \subseteq \mathcal{J} \textit{ and for all } M \in \mathcal{J} \textit{ there is a } J \in \mathscr{J} \textit{ with } M \subseteq J\}.$$

- *The uniformity number:*

$$\text{non}(\mathcal{J}) = \min\{|Y|\colon Y \subset X \textit{ and } Y \notin \mathcal{J}\}.$$

Cichón's diagram

Cichón's diagram summarizes the provable ZFC relationships between some cardinal invariants related to the σ-ideals of meager and null sets (with respect to the standard product measure) on the Baire space. Is there a straightforward generalization of (a) these ideals and (b) the corresponding diagram?

Cichoń's Diagram on the Baire space ω^ω

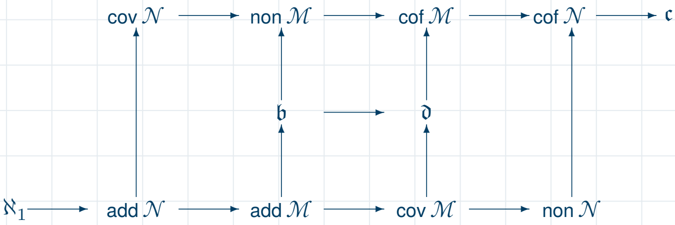
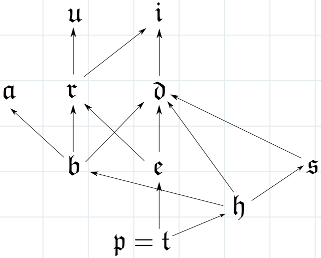


Figure 1: Cichón's diagram

Hasse's diagram

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Why cardinal invariants of these spaces? I

There are some remarkable differences between the countable and the uncountable cases that make this study interesting and present new challenges for future research. Here some examples:

- **Expected bounds**: Typically, classical invariants take values in the interval $[\aleph_1, \mathfrak{c}]$. However, in the uncountable for instance, the generalization of the splitting number $\mathfrak{s}(\kappa)$ can be $\leq \kappa$, and actually large cardinals are necessary to have the expected inequality $\mathfrak{s}(\kappa) \geq \kappa^+$ (Suzuki, 1998). Also, Ben-Neria and Gitik found the optimal large cardinal assumption to get $\mathfrak{s}(\kappa) > \kappa^+$, 2014. Specifically:

Why cardinal invariants of these spaces? II

Theorem

Let κ, λ be regular uncountable cardinals such that $\kappa^+ < \lambda$. $\mathfrak{s}(\kappa) = \lambda$ is equiconsistent to the existence of a measurable cardinal κ with $o(\kappa) = \lambda$.

- **New** ZFC **results**: Some examples
  - Raghavan and Shelah showed that, for uncountable κ, the inequality $\mathfrak{s}(\kappa) \leq \mathfrak{b}(\kappa)$ holds whereas in the countable case, there are two different forcing extensions in which inequalities $\mathfrak{s} < \mathfrak{b}$ and $\mathfrak{b} > \mathfrak{s}$ hold respectively.
  - They also proved that, if $\kappa > \beth_\omega$ then $\mathfrak{d}(\kappa) \leq \mathfrak{r}(\kappa)$. Recently, Fischer and Soukup showed (among others) that the same conclusion can be obtained under the hypothesis cf$(\mathfrak{r}(\kappa)) \leq \kappa$.

Why cardinal invariants of these spaces? III

- **Roitman's problem**: It asks whether from $\mathfrak{d} = \aleph_1$ it is possible to prove that $\mathfrak{a} = \aleph_1$.
  - So far, Shelah gave the best approximation to an answer to this problem: he developed the method of template iteration forcing to give a model in which the inequality $\mathfrak{d} < \mathfrak{a}$ is satisfied, yet in his model the value of $\mathfrak{d}$ is $\aleph_2$; the question that is still open asks if it is possible to find such a model but in addition having $\mathfrak{d} = \aleph_1$.
  - In the uncountable in contrast, Blass, Hyttinen and Zhang (2007) proved in ZFC that for uncountable regular κ Roitman's problem can be solved on the positive, i.e. if $\mathfrak{d}(\kappa) = \kappa^+$, then $\mathfrak{a}(\kappa) = \kappa^+$.

Why cardinal invariants of these spaces? IV

- **Global results:** Cummings and Shelah used an Easton-like iteration to prove the following:

Theorem

Assume GCH, if $\kappa \to (\beta(\kappa), \delta(\kappa), \mu(\kappa))$ is a class function from the class of all regular cardinals to the class of cardinal numbers, with $\kappa^+ \leq \beta(\kappa) = \mathrm{cf}(\beta(\kappa)) \leq \mathrm{cf}(\delta(\kappa)) \leq \delta(\kappa) \leq \mu(\kappa)$ and $\mathrm{cf}(\mu(\kappa)) > \kappa$ for all κ. Then, there exists a class forcing $\mathbb{P}$, preserving all cardinals and cofinalities, such that in the generic extension $\mathfrak{b}(\kappa) = \beta(\kappa)$, $\mathfrak{d}(\kappa) = \delta(\kappa)$ and $\mu(\kappa) = 2^\kappa$.

Why cardinal invariants of these spaces? V

- **More cardinal invariants via combinatorial characterizations:** In the countable case the following holds:

Theorem (Bastoszyński)

Let f and g be two functions in ω^ω. We say that f and g are eventually different *if there is $n \in \omega$, such that for all $m \geq n$ $f(m) \neq g(m)$ (and write $f \neq^* g$), then:*
non $\mathcal{M} = \min\{|\mathcal{F}|\colon (\forall g \in \omega^\omega)(\exists f \in \mathcal{F})\neg(f \neq^* g)\}$.
cov $\mathcal{M} = \min\{|\mathcal{F}|\colon (\forall g \in \omega^\omega)(\exists f \in \mathcal{F})(f \neq^* g)\}$.

Then, if we define for arbitrary uncountable regular κ:

- nm$(\kappa) = \min\{|\mathcal{F}|\colon (\forall g \in \kappa^\kappa)(\exists f \in \mathcal{F})\neg(f \neq^* g)\}$.
- cv$(\kappa) = \min\{|\mathcal{F}|\colon (\forall g \in \kappa^\kappa)(\exists f \in \mathcal{F})(f \neq^* g)\}$.

Why cardinal invariants of these spaces? VI

The following holds:

Proposition

- $\mathfrak{b}(\kappa) \leq \text{nm}(\kappa) \leq \text{non}(\mathcal{M}_\kappa)$.
- $\text{cov}(\mathcal{M}_\kappa) \leq \text{cv}(\kappa) \leq \mathfrak{d}(\kappa)$.

Moreover, if κ is strongly inaccessible, the corresponding cardinals coincide.

- **Club versions:**
  - Cummings and Shelah defined the "club" versions of $\mathfrak{d}(\kappa)$ and $\mathfrak{b}(\kappa)$, namely given $f, g \in \kappa^\kappa$, we say that $f <^*_{\text{cb}} g$ (g *club dominates* f), if there exists a club C on κ so that, for every $\alpha \in C$, $f(\alpha) < g(\alpha)$ and defined $\mathfrak{b}_{\text{cb}}(\kappa)$ and $\mathfrak{d}_{\text{cb}}(\kappa)$ accordingly. They proved:

Why cardinal invariants of these spaces? VII

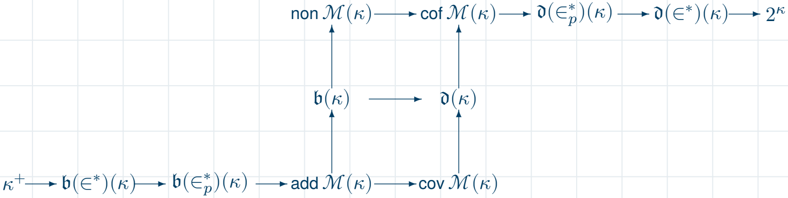
Theorem

$\mathfrak{b}_{\text{cb}}(\kappa) = \mathfrak{b}(\kappa)$, $\mathfrak{d}_{\text{cb}}(\kappa) \leq \mathfrak{d}(\kappa)$ and if κ is regular and $> \beth_\omega$, $\mathfrak{d}_{\text{cb}}(\kappa) = \mathfrak{d}(\kappa)$.

- *The pseudointersection number:* $\mathfrak{p}(\kappa)$ is defined as the minimum size of a family $\mathcal{F}$ of subsets of κ with the *strong intersection property* (i.e. for every $\mathcal{F}' \subseteq \mathcal{F}$, $|\mathcal{F}'| < \kappa$, $\bigcap \mathcal{F}'$ is unbounded) and no pseudointersection of size κ (i.e. no set $X \in [\kappa]^\kappa$ such that $X \subseteq^* F$, for all $F \in \mathcal{F}$).

Raghavan and Shelah proved also that $\mathfrak{s}(\kappa) \leq \mathfrak{p}_{\text{cb}}(\kappa) \leq \mathfrak{b}(\kappa)$ where $\mathfrak{p}_{\text{cb}}(\kappa)$ is the minimum size of a family of clubs without a pseudointersection of size κ. Recently, in joint work with Fischer and Soukup, we have proved that there is a model where $\mathfrak{p}(\kappa) < \mathfrak{p}_{\text{cb}}(\kappa)$.

Some results



Cichoń's Diagram on the uncountable for κ strongly inaccessible.

κ-Sacks forcing

Let κ be strongly inaccessible. Conditions in $\mathbb{S}_\kappa$ are κ-closed subtrees $T \subseteq 2^{<\kappa}$ such that $\forall s \in T, \exists t \in T, s \subseteq t$ splitting and the limit of splitting nodes is also splitting. Also $T \leq S$ if $T \subseteq S$.

- It has good fusion properties.
- It has the generalized h-Sacks property where $h \in \kappa^\kappa$ is defined by $h(\alpha) = 2^{|\alpha|}$, i.e. given $S \in \mathbb{S}_\kappa$ and $\dot{f}$ an $\mathbb{S}_\kappa$-name for an element in κ^κ, there are $T \leq S$ and $F : \kappa \to [\kappa]^{<\kappa}$ h-slalom ($|F(\alpha)| \leq h(\alpha)$) such that $T \Vdash \dot{f}(\alpha) \in F(\alpha)$ for all $\alpha < \kappa$.

The iteration of $\mathbb{S}_\kappa$ with κ-support of length κ^{++} has:

- Good fusion, so cardinals $\leq \kappa^+$ are preserved.
- The generalized Sacks property and, as a consequence $\mathfrak{d}(\in^*)(\kappa)$ as well as the other cardinals in the extended diagram are equal to κ^+.

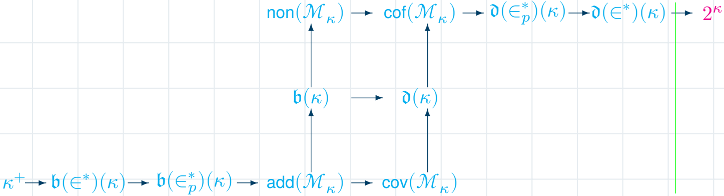


Figure 7: Effect of the iteration of $\kappa-$Sacks forcing

κ-Miller Forcing

Let $\mathcal{F}$ be a κ-complete filter on κ. Define $\mathbb{MI}^\kappa_{\mathcal{F}}$ to be the following forcing notion: Conditions in $\mathbb{MI}^\kappa_{\mathcal{F}}$ are κ-closed sub-trees T of the set of increasing sequences in $\kappa^{<\kappa}$, such that every node can be extended to a $\mathcal{F}$-splitting node.

Also we want that if $\alpha < \kappa$ is limit, $u \in \kappa^\alpha$, and for arbitrarily large $\beta < \alpha$, $u \restriction \beta$ $\mathcal{F}$-splits in T, then u $\mathcal{F}$-splits in T;

Properties of κ-Miller forcing

- It has good fusion which implies $\geq \kappa^+$ are preserved.
- $\mathbb{MI}^\kappa_\mathcal{C}$, where $\mathcal{C}$ is the club filter, adds a Cohen subset of κ.
- $\mathbb{MI}^\kappa_\mathcal{F}$ generically adds an unbounded function over $\kappa^\kappa \cap V$.
- The product $\mathbb{MI}^\kappa_\mathcal{F} \times \mathbb{MI}^\kappa_\mathcal{F}$ adds a κ-Cohen function.
- $\mathbb{MI}^\kappa_\mathcal{U}$ has the pure decision property when $\mathcal{U}$ is an ultrafilter. i.e. if $T \in \mathbb{MI}^\kappa_\mathcal{U}$ and φ is a formula in the forcing language, there is $S \leq T$ with the same stem such that S decides φ i.e. $S \Vdash \varphi$ or $S \Vdash \neg\varphi$.

The generalized ultrafilter number

Let κ be an uncountable cardinal.

Definition

$\mathfrak{u}(\kappa) = \min\{|\mathcal{B}| \colon \mathcal{B}$ *is a base for a uniform ultrafilter on* $\kappa\}$.

Uniform means that all the sets in the ultrafilter have size κ. Also, if $\mathcal{U}$ is an ultrafilter on κ, $\mathcal{B} \subseteq \mathcal{U}$ is a base if given $F \in \mathcal{U}$, there is $B \in \mathcal{B}$ such that $B \subseteq^* F$.

Theorem (Brooke-Taylor, Fischer, Friedman, M.)

Suppose κ is a supercompact cardinal, κ^* is a regular cardinal with $\kappa < \kappa^* \leq \Gamma$ and Γ is a cardinal that satisfies $\Gamma^\kappa = \Gamma$. Then there is a forcing extension in which cardinals have not been changed satisfying:

$$\kappa^* = \mathfrak{u}(\kappa) = \mathfrak{b}(\kappa) = \mathfrak{d}(\kappa) = \mathfrak{a}(\kappa) = \mathfrak{s}(\kappa) = \mathfrak{r}(\kappa) = \mathrm{cov}(\mathcal{M}_\kappa) = \mathrm{add}(\mathcal{M}_\kappa) = \mathrm{non}(\mathcal{M}_\kappa) = \mathrm{cof}(\mathcal{M}_\kappa) \text{ and } 2^\kappa = \Gamma.$$

If in addition $(\Gamma)^{<\kappa^*} \leq \Gamma$ then we can also provide that $\mathfrak{p}(\kappa) = \mathfrak{t}(\kappa) = \mathfrak{h}_{\mathcal{W}}(\kappa) = \kappa^*$ where $\mathcal{W}$ is a κ-complete ultrafilter on κ.

Section 2

The independence number

Independent families

Definition (Notation)

Let $\mathcal{A}$ be a family of infinite subsets of ω:

- *We denote* $\mathsf{FF}(\mathcal{A})$ *the family of finite partial functions from* $\mathcal{A}$ *to* 2. *Given* $h \in \mathsf{FF}(\mathcal{A})$, $\mathcal{A}^h = \bigcap\{A^{h(A)} : A \in \mathcal{A} \cap \mathrm{dom}(h)\}$, *where* $A^{h(A)} = A$ *if* $h(A) = 0$ *and* $A^{h(A)} = \omega \ \ A$ *otherwise.*
- *We refer to* $\{\mathcal{A}^h : h \in \mathsf{FF}(\mathcal{A})\}$ *as the family of Boolean combinations of* $\mathcal{A}$ *associated to* h.

Definition

A family $\mathcal{A} \subseteq [\omega]^\omega$ *is called* independent *if for for every* $h \in \mathsf{FF}(\mathcal{A})$, *the set* $\mathcal{A}^h$ *is infinite. An independent family* $\mathcal{A}$ *is said to be* maximal independent *if it is not properly contained in another independent family.*

An example

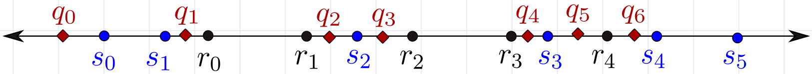
Put $\mathcal{C} = [\mathbb{Q}]^{<\omega}$ and for any real $r \in \mathbb{R}$, look at the set:

$$X_r = \{F \in \mathcal{C} : |F \cap (-\infty, r)| \text{ is even}\}$$

Then, the family $\{X_r : r \in \mathbb{R}\}$ is independent: Let $r_0 < r_2 < ... < r_k$ and $s_0 < s_1 < s_2 < ... < s_l$ two sets of reals, then the set

$$\bigcap_{i \leq k} E_{r_i} \cap \bigcap_{j \leq l} (\omega \;\; E_{s_j})$$

is infinite. Why? Let's look at the following drawing:



In the figure, the set of rationals $\{q_0, q_1, ...\} \in \bigcap_{i \leq k} E_{r_i} \cap \bigcap_{j \leq l} (\omega \ \ E_{s_j})$

The independence number

Definition

$\mathfrak{i} = \min\{|\mathcal{A}|\colon \mathcal{A}$ *is a maximal independent family of subsets of* $\omega\}$.

$\mathfrak{i}$ is a cardinal invariant, in the sense that $\aleph_1 \leq \mathfrak{i} \leq \mathfrak{c}$, some lower bounds for it are the cardinal invariants $\mathfrak{d}$ and $\mathfrak{r}$.

How to add an independent real?

The following results are due to Brendle:

Lemma

Let $\mathcal{A}$ be an independent family. Then there is an ideal $\mathcal{J}_{\mathcal{A}}$ on ω with the following properties:

1. $\mathcal{J}_{\mathcal{A}} \cap \{\mathcal{A}^h : h \in \mathsf{FF}(\mathcal{A})\} = \emptyset$.
2. *For every $X \in [\omega]^\omega$ there is $h \in \mathsf{FF}(\mathcal{A})$ such that either $X \cap \mathcal{A}^h$ or $\mathcal{A}^h \ \ X$ belongs to $\mathcal{J}_{\mathcal{A}}$.*

Whenever $\mathcal{A}$ be an independent family and $\mathcal{J}_{\mathcal{A}}$ is an ideal satisfying properties (1) and (2) of the lemma above, we say that $\mathcal{J}_{\mathcal{A}}$ is an *independence diagonalization ideal* associated to $\mathcal{A}$.

The forcing

Definition

Let $\mathcal{A}$ be an independent family and let $\mathcal{J}_{\mathcal{A}}$ be an independence diagonalization ideal associated to it. The poset $\mathbb{B}(\mathcal{J}_{\mathcal{A}})$ consists of all pairs (s, E) where $s \in [\omega]^{<\omega}$, $E \in [\mathcal{J}_{\mathcal{A}}]^{<\omega}$ with extension relation defined as follows: $(t, F) \leq (s, E)$ if and only if $t \supseteq s$, $F \supseteq E$ and $(t \setminus s) \cap \bigcup E = \emptyset$.

This poset $\mathbb{B}(\mathcal{J}_{\mathcal{A}})$ is σ-centered, so it preserves cardinals. Additionally it has the following *weakly diagonalization property*:

Lemma

Let G be a $\mathbb{B}(\mathcal{J}_{\mathcal{A}})$ generic filter. Then $x_G := \bigcup\{s : \exists F(s, F) \in G\}$ is an infinite subset of ω such that in $V[G]$, $\mathcal{A} \cup \{x_G\}$ is independent, while for every $Y \in ([\omega]^\omega \backslash \mathcal{A}) \cap V$, the family $\mathcal{A} \cup \{x_G, Y\}$ is not independent.

As a corollary we obtain:

Theorem

(GCH) *Let $\kappa < \lambda$ be regular uncountable cardinals. There is a ccc generic extension in which $\mathfrak{i} = \mathfrak{d} = \kappa < \mathfrak{c} = \lambda$.*

A generic maximal independent family

Shelah constructed a maximal independent family, which remains a witness to $\mathfrak{i} = \aleph_1$ in a model of $\mathfrak{u} = \aleph_2$. With Fischer, we showed that, over a model of GCH for example, his construction naturally gives rise to the existence of a countably closed, $\aleph_2$-cc poset $\mathbb{P}$, which generically adjoins a maximal independent family, which turns to be Sacks indestructible.

Lemma

Let $\mathcal{A}$ be an independent family and let $\mathcal{D}(X)$ to be the set of all functions $h \in \mathsf{FF}(\mathcal{A})$ for which $X \cap \mathcal{A}^h$ is finite, then:

$$\begin{aligned} \mathrm{id}(\mathcal{A}) &= \{X \subseteq \omega : \forall h \in \mathsf{FF}(\mathcal{A}) \exists h' \supseteq h (\mathcal{A}^{h'} \cap X) \textit{ is finite}\} \\ &= \{X \subseteq \omega : \mathcal{D}(X) \textit{ is dense in } \mathsf{FF}(\mathcal{A})\} \end{aligned}$$

is an ideal on ω, to which we refer as the independence density ideal associated to $\mathcal{A}$. Here when we say "dense" in $\mathsf{FF}(\mathcal{A})$, we mean dense respect to the inclusion relation.

The poset

Definition

Let $\mathbb{P}$ be the poset of all pairs $(\mathcal{A}, A)$ where $\mathcal{A}$ is a countable independent family, $A \in [\omega]^\omega$ such that for all $h \in \mathsf{FF}(\mathcal{A})$ the set $\mathcal{A}^h \cap A$ is infinite. The extension relation on $\mathbb{P}$ is given by: $(\mathcal{B}, B) \leq (\mathcal{A}, A)$ if and only if $\mathcal{B} \supseteq \mathcal{A}$ and $B \subseteq^* A$.

Proposition

Let G be $\mathbb{P}$-generic over V. Then $\mathcal{A}_G = \bigcup\{\mathcal{A} : \exists A \in [\omega]^\omega \text{ with } (\mathcal{A}, A) \in G\}$ is a maximal independent family.

Theorem

The generic maximal independent family adjoined by $\mathbb{P}$ *over a model of* CH *and* $2^{\aleph_0} = \aleph_1$ *remains maximal after the countable support iteration of Sacks forcing* $\mathbb{S}$ *of length* ω_2.

Comparing the two ideals I

- Let $\mathcal{A}$ be an independent family. Then $\text{id}(\mathcal{A}) \subseteq \mathcal{J}_{\mathcal{A}}$.
- If $\mathcal{A}$ is an independent family which is not maximal, then $\text{id}(\mathcal{A}) \subsetneq \mathcal{J}_{\mathcal{A}}$.

Definition

An independent family $\mathcal{A}$ is said to be densely maximal *if for every $X \in [\omega]^{\omega} \backslash \mathcal{A}$ and every $h \in \text{FF}(\mathcal{A})$, there is $h' \in \text{FF}(\mathcal{A})$ for which either $X \cap \mathcal{A}^{h'}$ of $\mathcal{A}^{h'} \backslash X$ is finite.*

Proposition

If $\mathcal{A}$ is densely maximal independent, then $\mathcal{J}_{\mathcal{A}} \subseteq \text{id}(\mathcal{A})$ and so $\mathcal{J}_{\mathcal{A}} = \text{id}(\mathcal{A})$.

Comparing the two ideals II

Proposition

The maximal family that turned to be Sacks indestructible is densely independent.

Corollary

A densely maximal independent family $\mathcal{A}$ such that the dual filter of its diagonalization ideal $\text{id}(\mathcal{A})$ *is generated by a Ramsey filter and the co-finite sets remains maximal after the countable support iteration of Sacks forcing, as well as after the countable support product of Sacks forcing.*

The generalized case

Definition

Let $\mathcal{A}$ be a family of unbounded subsets of κ of size $\geq \kappa$:

- *We call* $\mathrm{BF}_\kappa(\mathcal{A})$ *the family of functions from $\mathcal{A}$ to 2 with domain of size $< \kappa$.*
- *Given $h \in \mathrm{BF}_\kappa(\mathcal{A})$, $\mathcal{A}^h = \bigcap\{A^{h(A)} : A \in \mathcal{A} \cap \mathrm{dom}(h)\}$, where $A^{h(A)} = A$ if $h(A) = 0$ and $A^{h(A)} = \kappa \ A$ otherwise. We also refer to* $\{\mathcal{A}^h : h \in \mathrm{BF}_\kappa(\mathcal{A})\}$ *is the family of* generalized Boolean combinations *of $\mathcal{A}$.*

Definition

A family $\mathcal{A} \subseteq [\kappa]^\kappa$ such that $|\mathcal{A}| \geq \kappa$ is called κ-independent *if for for every $h \in \mathrm{BF}_\kappa(\mathcal{A})$, the set $\mathcal{A}^h$ is unbounded on κ. A κ-independent family $\mathcal{A}$ is said to be* κ-maximal independent *if it is not properly contained in another κ-independent family.*

A generalization of Brendle's result

Fischer and Shelah have characterized filters associated to an independent family that can be used to diagonalize an independent family of subsets of ω by using Mathias forcing with respect to such filters. Now, we aim to generalize their results:

Definition

Let $\mathcal{A}$ be a κ-independent family. A κ-complete filter $\mathcal{F}$ is called a diagonalization filter *for the family $\mathcal{A}$ if the following hold:*

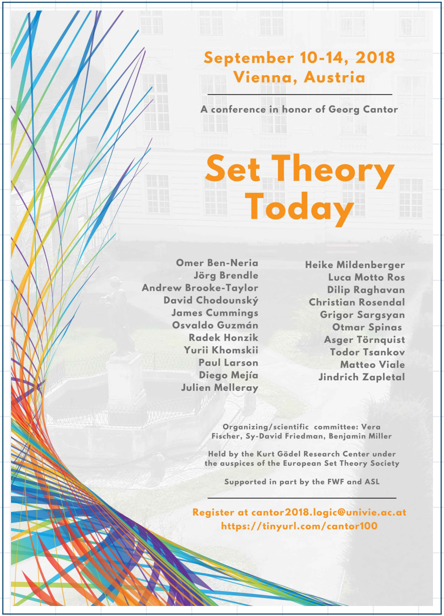
1. *For every $F \in \mathcal{F}$ and $h \in \mathsf{BF}_\kappa(\mathcal{A})$, $|F \cap \mathcal{A}^h| = \kappa$.*
2. $\mathcal{F} \cap \{\mathcal{A}^h : h \in \mathsf{BF}_\kappa(\mathcal{A})\} = \emptyset$.

In addition, a maximal diagonalization filter *is a κ-complete filter that is maximal with respect to properties (1) and (2), i.e. there is no κ-complete filter $\mathcal{F}' \supset \mathcal{F}$ satisfying these properties.*

Finally...

In recent work with V. Fischer we have used the results above to decide the value of $\mathfrak{i}(\kappa)$ in the model where $\mathfrak{u}(\kappa)$ was small. Namely, we also decide it to be κ^*.

Thanks!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