MAXIMAL ALMOST DISJOINT FAMILIES AT SINGULAR CARDINALS

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ABSTRACT. We study the almost disjointness number for singular cardinals and based on the results of Erdös and Shelah in [EH75] and Kojman, Kubiś and Shelah in [KKS04], we present a model in which the inequality $\mathfrak{a}(\lambda) < \mathfrak{a}$ for λ a singular cardinal of countable cofinality holds. We also present some results dealing with the concept of destroying the maximality of a given maximal almost disjoint family at a singular cardinal λ by using forcing. Additionally, we present a preservation result of madness when changing the cofinality of a given large cardinal κ.

1. Introduction

Maximal almost disjoint families have been for a long time the object of study in different areas in mathematics, for instance set theory and general topology. Particularly, within set theory the study of maximal almost disjoint families at ω and their possible sizes has been a fruitful area of research which has led to the discovery of groundbreaking techniques in forcing theory, which themselves have been crucial to solve important open questions regarding sizes of special subsets of the real line.

On the other hand, in the last years special interest has been given to the study of cardinal characteristics of the generalized Baire spaces (the spaces of functions κ^κ for κ an uncountable cardinal) and its properties. Many similarities with and also some remarkable differences to the classical case ($\kappa = \omega$) have been established.

In particular, the generalized almost disjoint number $\mathfrak{a}(\kappa)$ has been studied in the cases when κ is a regular cardinal and some consistency results have been proved by assuming κ to be a large cardinal (for instance κ being supercompact). An example is the proof of $\text{Con}(\mathfrak{a}(\kappa) < 2^\kappa)$ in [Bro+17], more involved results can be found in [RS17].

This paper deals with the study of maximal almost disjoint families on the space λ^λ when λ is a singular cardinal and their possible sizes. The study of cardinal characteristics at singulars has turned out be quite interesting, mostly because of the remarkable differences between this and the regular case. Likewise, the use of the beautiful theory of *possible cofinalities (pcf)* of Shelah in order to get new bounds for these cardinals has been crucial.

As background results on the theory of almost disjointness at singulars the work of Erdös and Hechler in [EH75] is crucial. In their paper, they introduced the concept of almost disjointness

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for families of subsets of a singular cardinal λ and proved many interesting results: for instance, if λ is a singular cardinal of cofinality $\kappa < \lambda$ and there is an almost disjoint family at κ of size γ, then there is a maximal almost disjoint family at λ of the same size.

This paper is organized as follows: In Section 3 we present the basic definitions and a summary of the relevant existing results on maximal almost disjoint families for singular cardinals and motivate the upcoming sections. Section 4 has an outline of the basics on pcf theory and a couple applications which are relevant for the results in this paper.

Section 5 includes the main results of this paper, namely the construction of a generic extension in which the inequality $\mathfrak{a}(\lambda) < \mathfrak{a}$ holds for λ a singular cardinal of countable cofinality. The model combines the classical technique of Brendle to get a model in which $\mathfrak{b} < \mathfrak{a}$ (here $\mathfrak{a}$ and $\mathfrak{b}$ correspond to the classical almost disjointness and bounding numbers at ω) together with the use of Příkrý type forcings which change the cofinality of a given large cardinal κ to be countable and, at the same time control the size of the power set of this given cardinal (see Section 3).

Additionally, there are two results regarding both a technique for destroying maximality of a maximal almost disjoint family at a singular and also a preservation result which shows that a mad family at a measurable cardinal κ can be preserved to be maximal after forcing with Příkrý forcing.

Finally, Section 6 presents a discussion of some open problems and future possible lines of research in this subject.

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3. Mad families for singular cardinals

First and foremost, we introduce the main definitions regarding maximal almost disjoint families. Let κ be an infinite cardinal:

Definition 1.

(1) We say that a family $\mathcal{A} \subseteq [\kappa]^\kappa$ is κ-almost disjoint if and only if $\bigcup \mathcal{A}$ has cardinality κ and the intersection of any two distinct elements of $\mathcal{A}$ has cardinality strictly less than κ.

(2) The family $\mathcal{A}$ is said to be *maximal* κ-almost disjoint if for every $X \in [\kappa]^\kappa$ there exists $A \in \mathcal{A}$ such that the set $X \cap A$ has cardinality κ.

Throughout this paper we will focus on the study of maximal λ-almost disjoint families, when λ is a singular cardinal. The uncountable regular case has been studied in the last years and some important references for the reader, which include some results on maximal κ-almost disjoint families and their possible sizes can be found in [BHZ07; Bro+17].

Because it is relevant for the upcoming results, we mention a few comments on the regular case: Let κ be a regular cardinal and $\mathcal{A}$ be a maximal κ-almost disjoint family, then its size must be at least κ^+. Moreover, if we define the *generalized almost disjoint number* as follows:

Definition 2 (The almost disjointness number at κ)**.**

$$\mathfrak{a}(\kappa) = \min\{|\mathcal{A}| \colon \mathcal{A} \text{ is maximal } \kappa\text{-almost disjoint family of subsets of } \kappa\}$$

Then $\kappa^+ \leq \mathfrak{a}(\kappa) \leq 2^\kappa$ and it is possible to find models in which the inequality $\kappa^+ < \mathfrak{a}(\kappa) < 2^\kappa$. However, it is still open how the spectrum of such families behaves, we define:

Definition 3 (The madness spectrum)**.** Let κ be a cardinal, the madness spectrum at κ is defined as follows:

$\mathrm{MAD}(\kappa) = \{\gamma : \gamma \text{ is a cardinal and there exists a maximal } \kappa\text{-almost disjoint family } \mathcal{A} \subseteq [\kappa]^\kappa \text{ of size } \gamma\}$

When studying the *madness spectrum* at a fixed cardinal κ, the main question of interest is the following:

Question 4. Let $\Gamma \subseteq [\kappa^+, 2^\kappa]$ be a set of cardinals, is it possible to find a model in which the set $\mathrm{MAD}(\kappa)$ coincides with Γ?

For the case $\kappa = \omega$ this question has been deeply studied. For instance, Hechler [Hec72] was interested in characterizing the sets of cardinals Γ which are the mad spectrum at $\kappa = \omega$ in some forcing extension of V. Turns out that under the assumption $V \models \mathrm{GCH}$ he was able to constructed a ccc forcing notion $\mathbb{P}$, such that $\Vdash_{\mathbb{P}} \Gamma = \dot{A}$ (where $\dot{A}$ is a $\mathbb{P}$-name for the mad spectrum), provided that:

(a) Γ is a set of uncountable cardinals.
(b) Γ is closed under singular limits.
(c) If $\mu \in \Gamma$ has $\mathrm{cf}(\mu) = \aleph_0$ then $\mu = \sup(\Gamma \cap \mu)$.
(d) $\max(\Gamma)$ exists and $\max(\Gamma)^{\aleph_0} = \max(\Gamma)$.
(e) $\aleph_1 \in \Gamma$.
(f) If μ is a cardinal and $\aleph_1 < \mu \leq |\Gamma|$ then $\mu \in \Gamma$.
(g) If $\mu \in \Gamma$ and $\mathrm{cf}(\mu) = \aleph_0$, then $\mu^+ \in \Gamma$.

Afterward, Spinas and Shelah proved in [SS15] that for every subset of cardinals Γ with properties (a), (b), (c), and (d) there exists some ccc forcing $\mathbb{P}_\Gamma$ with $\Vdash_{\mathbb{P}_\Gamma} \Gamma = \dot{A}$, provided that $\theta = \min(\Gamma)$ satisfies $\theta = \theta^{<\theta}$ (hence θ is regular) and $\max(\Gamma) < \theta = \max(\Gamma)$.

Now we turn to the singular case: Let λ be a singular cardinal, Erdös and Hechler studied the existence of maximal λ-almost disjoint families and their possible sizes in [EH75]. We list below their main results because they are crucial to this paper and they have motivated the questions we will work on:

Theorem 5 (Erdös-Hechler [EH75])**.**

(1) For every singular cardinal λ of cofinality κ and every cardinal $\mu < \lambda$ there exists a maximal λ-almost disjoint family of cardinality δ where $\mu \leq \delta \leq \mu^\kappa$.

(2) If λ is a a singular cardinal of cofinality κ and there exists a maximal κ-almost disjoint family of cardinality μ, then there exists a maximal λ-almost disjoint family of size μ.

The almost disjointness number $\mathfrak{a}(\lambda)$ at a singular λ of cofinality κ satisfies $\mathfrak{a}(\lambda) \geq \kappa^+$ (this is an easy consequence of a standard diagonalization argument for the regular case when proving $\mathfrak{a}(\kappa) > \kappa$). Also, as a consequence of item (2) in the Theorem 5 above we get the inequality $\mathfrak{a}(\lambda) \leq \mathfrak{a}(\kappa)$.

The next corollary also follows from Theorem 5 above and it shows in particular, that the situation in the singular case differs quite drastically in comparison with the regular case.

Corollary 6.

(1) If λ is a singular cardinal of cofinality κ and $\mu^\kappa < \lambda$ for every cardinal $\mu < \lambda$, then there exists a maximal λ-almost disjoint family of cardinality μ. Thus, if λ is a strong limit cardinal, then there exists a maximal λ-almost disjoint family of size λ.

(2) If λ is a singular cardinal of cofinality κ, then it is consistent with ZFC that there is a maximal λ-almost disjoint family of every cardinality $\mu \leq 2^\kappa$ except $\mu = \kappa$. Hence, it is consistent with ZFC that $\lambda < 2^\kappa$ and there exists maximal λ-almost disjoint families of cardinality λ.

After Erdös and Hechler, (around 30 years later) the paper [KKS04] from Kojman, Kubiś and Shelah appeared. In it, the authors studied and answered positively two open questions left by Erdös and Hechler regarding the consistency of $\lambda \in \mathrm{MAD}(\lambda)$ for all singular λ. In their results, they used the following auxiliary cardinal characteristics:

Definition 7 (The bounding numbers)**.**

- If κ is a regular cardinal and $\mathcal{F} \subseteq \kappa^\kappa$, we say that $\mathcal{F}$ is unbounded if for all $f \in \mathcal{F}$ there is no $g \in \kappa^\kappa$ so that $f \leq^* g$. Here $f \leq^* g$ if and only if $\exists \beta < \kappa$ such that $\forall \alpha \geq \beta$, $f(\alpha) \leq g(\alpha)$. The bounding number at κ is defined by:

$$\mathfrak{b}(\kappa) = \min\{|\mathcal{F}| \; \mathcal{F} \subseteq \kappa^\kappa \text{ is unbounded }\}$$

- More generally, if $\mathcal{F} \subseteq \prod_{i\in\gamma} \kappa_i$ for $(\kappa_i : i \in \gamma)$ a sequence of cardinals we say that $\mathcal{F}$ is unbounded in $\prod_{i\in\gamma} \kappa_i$ if for all $f \in \mathcal{F}$ there is no $g \in \prod_{i\in\gamma} \kappa_i$ so that $f \leq^* g$. Analogously we say that $\mathcal{G} \subseteq \prod_{i\in\gamma} \kappa_i$ is unbounded if for all $f \in \mathcal{G}$ there is no $g \in \prod_{i\in\gamma} \kappa_i$ so that $f \leq^* g$, so we can define $\mathfrak{b}(\prod_{i\in\gamma} \kappa_i)$ similarly.
- If λ is a singular cardinal of cofinality κ, define, the *bounding number* at λ as follows:

$$\mathfrak{b}(\lambda) = \sup\{\mathfrak{b}(\prod_{i<\kappa} \mu_i, \leq^*)) : (\mu_i : i < \kappa) \text{ is a sequence of regular cardinals with limit } \lambda\}.$$

The following inequalities correspond to the main results in their paper and will be crucial in the upcoming sections:

Theorem 8 (Kojman, Kubiś and Shelah, [KKS04])**.** *Let λ be a singular cardinal of cofinality κ. Then, the following hold:*

(1) $\mathfrak{a}(\lambda) \geq \min\{\mathfrak{b}(\lambda), \mathfrak{b}(\kappa)\}$.
(2) If $\mathfrak{a}(\lambda) \leq \mu < \mathfrak{b}(\lambda)$*, then there is a maximal* λ*-almost disjoint family of size* μ*, i.e.* $\mu \in \mathrm{MAD}(\lambda)$.

4. A bit of PCF theory

In this section, we present the main definitions and relevant results (for the purposes of this paper) of the famous theory of *possible cofinalities* (pcf) developed by Saharon Shelah. For an extended reference on this subject, we refer the reader to [She94].

Definition 9. Consider an infinite set A and an ideal $\mathcal{I}$ on A, and an indexed set $\{\gamma_a : a \in A\}$ of limit ordinals. A *scale* in $\prod_{a\in A} \gamma_a$ is a $<_{\mathcal{I}}$-increasing transfinite sequence $(f_\alpha : \alpha < \lambda)$ of functions in $\prod_{a\in A} \gamma_a$ which is $<_{\mathcal{I}}$ cofinal in $\prod_{a\in A} \gamma_a$. Here $f <_{\mathcal{I}} g$ means that the set $\{a \in A : f(a) > g(a)\} \in \mathcal{I}$.

Definition 10.

(1) Let A be a set of regular cardinals and $\mathcal{D}$ be an ultrafilter on A. $\prod A = \prod_{a\in A}\{a : a \in A\}$ denotes the product $\{f : \mathrm{dom}(f) = A \text{ and } f(a) \in A\}$. The ultraproduct $\prod A/\mathcal{D}$ is linearly ordered by the classic order, i.e $f <_{\mathcal{D}} g$ if and only if $\{a \in A : f(a) < g(a)\} \in \mathcal{D}$. Finally, denote its cofinality $\mathrm{cf}(\mathcal{D}) = \mathrm{cf}(\prod A/\mathcal{D})$, i.e. the minimum size of a cofinal set of functions (with respect to the order $<_{\mathcal{D}}$) on $\prod A/\mathcal{D}$.
(2) Let A be a set of regular cardinals, then

$$\mathrm{pcf}(A) = \{\mathrm{cf}(\mathcal{D}) : \mathcal{D} \text{ is an ultrafilter on } A\}.$$

The following Proposition summarizes the main properties of the set $\mathrm{pcf}(A)$:

Proposition 11. Let A be a set of regular cardinals. Then:

- $\mathrm{pcf}(A)$ is a set of regular cardinals.
- $\mathrm{pcf}(A) \supseteq A$.
- If $A_1 \subseteq A_2$ then $\mathrm{pcf}(A_1) \subseteq \mathrm{pcf}(A_2)$.
- $\mathrm{pcf}(A_1 \cup A_2) = \mathrm{pcf}(A_1) \cup \mathrm{pcf}(A_2)$.
- $|\mathrm{pcf}(A)| \leq 2^{2^{|A|}}$.
- $\sup(\mathrm{pcf}(A)) \leq |\prod A|$.

The following are crucial properties of this set:

- If A is an interval of cardinals, i.e. a set of the form $\{\delta_1 \leq \gamma \leq \delta_2 : \gamma \text{ is a cardinal}\}$ and $2^{|A|} < \min(A)$, then $\mathrm{pcf}(A)$ is an interval of cardinals as well.
- If $|\mathrm{pcf}(A)| < \min(A)$, then $\mathrm{pcf}(\mathrm{pcf}(A)) = \mathrm{pcf}(A)$.
- If A is an interval of cardinals without greatest element and $(\min(A))^{|A|} < \sup(A)$, then $\sup(A)^{|A|} = \max(\mathrm{pcf}(A))$.

As a consequence of the former Proposition, the following Theorem holds:

Theorem 12 (Shelah)**.** *If $\aleph_\omega$ is a strong limit cardinal, then:*

$$\max(\text{pcf}(\{\aleph_n : n \in \omega\})) = 2^{\aleph_\omega}.$$

The next Theorem is the *fundamental theorem of the pcf theory*:

Theorem 13 (Existence of generators)**.** *If A is a set of regular cardinals such that $2^{|A|} < \min(A)$, then there exists a set $\{B_\lambda \subseteq A : \lambda \in \text{pcf}(A)\}$ such that for every $\lambda \in \text{pcf}(A)$:*

- $\lambda = \max(\text{pcf}(B_\lambda))$.
- $\lambda \notin \text{pcf}(A - B_\lambda)$.
- $\prod\{a : a \in B_\lambda\}$ *has a scale modulo J_λ where J_λ is the ideal generated by the sets B_ν for $\nu < \lambda$.*

If we look for a moment at the case $\kappa = \aleph_\omega$, the Theorem above implies that one can get a better bound on $2^{\aleph_\omega}$ if $\aleph_\omega$ is a strong limit: Namely $2^{\aleph_\omega} < \aleph_{(2^{\aleph_0})^+}$.

Theorem 14 (An upper bound for pcf(A))**.** *If λ is a singular cardinal of uncountable cofinality, then there exists a closed unbounded set $C \subseteq \lambda$ such that $\text{cf}((\prod_{\alpha \in C}(\alpha^+)/\mathcal{D}) = \lambda^+$ for every ultrafilter $\mathcal{D}$ concentrating on end-segments of C.*

Finally, from the results above the famous bound for the power set of $\aleph_\omega$ (when it is a strong limit) $2^{\aleph_\omega} < \aleph_{\omega_4}$ follows.

4.1. The SCH and the bounding number at singulars.

The *Singular cardinal hypothesis* states that for every singular cardinal λ, if $2^{\text{cf}(\lambda)} < \lambda$, then $\lambda^{\text{cf}(\lambda)} = \lambda^+$. The following two Propositions are mentioned in [KKS04], we include them here as well as their proof for self-containment purposes. Throughout this section, λ is a singular cardinal of cofinality κ.

Proposition 15. If SCH fails at λ, then $\mathfrak{b}(\lambda) > \lambda^+$.

Proof. If SCH fails at λ, we know that $2^\kappa < \lambda$ and $\lambda^\kappa > \lambda^+$. Now, let A be an interval of regular cardinals $(\lambda_i : i < \kappa)$ converging to λ such that $\lambda_0 > 2^\kappa$.

Notice that $\text{pcf}(A) = \{\mathfrak{b}(\prod_{i<\kappa} \lambda_i, \leq_I) : I \subseteq \mathcal{P}(\kappa) \text{ is a proper ideal}\}$ and since $2^{|A|} = 2^\kappa < \min(A) = \lambda_0$ by Proposition 11 we have that pcf(A) is an interval of regular cardinals and has a maximum.

On the other hand, by the existence of generators (Theorem 13), for every $\delta \in \text{pcf}(A)$ there exists a pcf generator B_δ such that: If $\mathcal{J}_{<\delta}$ is the ideal generated by $\{B_\theta : \theta \in \text{pcf}(A) \wedge \theta < \delta\}$, then $\delta = \mathfrak{b}(\prod_{i<\kappa} \lambda_i, \leq_{\mathcal{J}_{<\delta}})$. In particular, for $\delta = \max(\text{pcf}(A))$ we get $\max(\text{pcf}(A)) = \max(\text{pcf}(B_{\max(\text{pcf}(A))})) = \mathfrak{b}(\prod_{i<\kappa} \lambda_i, \leq_{\mathcal{J}_{<\max(\text{pcf}(A))}})$.

Also, since the ideal $\mathcal{J}_{<\max \text{pcf}(A)}$ is proper and it is generated by κ-many sets, then there is an unbounded set $B \subseteq \kappa$ such that $\mathcal{J}_{<\max \text{pcf}(A)} \restriction B$ is contained in the ideal of bounded subsets of κ.

Thus, if $\max(\text{pcf}(A)) = \mathfrak{b}(\prod_{i<\kappa} \lambda_i, \leq_{\mathcal{J}_{<\max(\text{pcf}(A))}})$, then $\mathfrak{b}(\prod_{i<\kappa} \lambda_i, \leq^*) = \max(\text{pcf}(A))$ and finally, since $\max(\text{pcf}(A)) = \lambda^\kappa > \lambda^+$ we get $\mathfrak{b}(\lambda) > \lambda^+$. □

Proposition 16. $\mathfrak{b}(\lambda)$ cannot be changed by ccc forcing.

Proof. This is a consequence of the fact that if $\mathbb{P}$ is a ccc forcing notion every new function in λ^κ is bounded (modulo $<^*$) by a function from the ground model and so $\mathfrak{b}(\prod_{i<\kappa}\mu_i, \leq^*)$ is preserved after forcing with $\mathbb{P}$. This implies then that $\mathfrak{b}(\lambda)$ is preserved as well. □

The following result shows that there are forcing extensions of the universe, in which the value of $\mathfrak{b}(\aleph_\omega)$ can take its maximum.

Theorem 17. *For every $\beta < \omega_1$ it is consistent (from large cardinal axioms) that $2^{\aleph_\omega} = \mathfrak{b}(\aleph_\omega) = \aleph_{\omega+\beta+1}$.*

Proof sketch: Let V be a model in which $\aleph_\omega$ is a strong limit and $2^{\aleph_\omega} = \max(\mathrm{pcf}(\{\aleph_n : n \in \omega\})) = \aleph_{\omega+\beta+1}$. For more details on how such models are built see next section.

In V the ideal $J_{<\max(\mathrm{pcf}(\{\aleph_n:n\in\omega\}))}$ is proper and its generated by countably many sets, therefore there is an infinite $B \subseteq \omega$ so that $J_{<\max(\mathrm{pcf}(\{\aleph_n:n\in\omega\}))} \upharpoonright B$ is contained in the ideal of finite subsets of B.

Since $\mathfrak{b}(\prod_{n\in\omega}\aleph_n, \leq_{J_{<\max(\mathrm{pcf}(\{\aleph_n:n\in\omega\}))}}) = \aleph_{\omega+\beta+1}$, it follows that $\mathfrak{b}(\prod_{n\in\omega}\aleph_n, \leq^*) = \aleph_{\omega+\beta+1}$ and so $\mathfrak{b}(\aleph_\omega) = \aleph_{\omega+\beta+1}$. □

4.2. **Příkrý type forcings and singulars.**

In the sketch of the proof of Theorem 17 above one starts with a model V in which $\aleph_\omega$ is a strong limit and

$$2^{\aleph_\omega} = \max(\mathrm{pcf}(\{\aleph_n : n \in \omega\})) = \aleph_{\omega+\beta+1}.$$

Such models are usually obtained as *Příkrý-type forcing extensions.*

In order to give the reader a flavor of how such models are built, we present now the basics of two of the forcings leading to such generic extensions. For more details see Section 4 in [Git10].

Suppose that κ is a strong cardinal and $m \in \omega$. Fix a $(\kappa, \kappa+m)$-extender E over κ for some $m \in \omega$. Recall that given κ and γ two cardinals with $\kappa \leq \gamma$. Then, a set $E = \{E_a : a \in [\gamma]^{<\omega}\}$ is called a (κ, γ)-extender if the following properties are satisfied:

- Each E_a is a κ-complete non-principal ultrafilter on $[\kappa]^{<\omega}$. Furthermore at least one E_a is not κ^+-complete and for each $\alpha \in \kappa$ at least one E_a contains the set $\{s \in [\kappa]^{|a|} : \alpha \in s\}$.
- (Coherence) The E_a are coherent (so that the ultrapowers $\mathrm{Ult}(V, E_a)$ form a directed system).
- (Normality) If f is such that $\{s \in [\kappa]^{|a|} : f(s) \in \max s\} \in E_a$, then for some $b \supseteq a$, $\{t \in \kappa^{|b|} : (f \circ \pi_{ba})(t) \in t\} \in E_b$.
- (Wellfoundedness) The limit ultrapower $\mathrm{Ult}(V, E)$ is wellfounded (here $\mathrm{Ult}(V, E)$ denotes the direct limit of the ultrapowers $\mathrm{Ult}(V, E_a)$).

Let $j : V \to M \simeq \mathrm{Ult}(V, E)$, $\mathrm{crit}(j) = \kappa$, $M \supseteq V_{\kappa+m}$, be the canonical embedding.

Definition 18. Let E be a (κ, λ)-extender, we define the order $\leq_E$ by: $\alpha \leq_E \beta$ if and only if $\alpha \leq \beta$ and for some $f \in \kappa^\kappa$, $j(f)(\beta) = \alpha$.

If $\alpha \leq_E \beta$ this implies $\mathcal{U}_\alpha \leq_{\mathrm{RK}} \mathcal{U}_\beta$ [1], where U_α is a κ-complete ultrafilter over κ defined by setting $X \in \mathcal{U}_\alpha$ if and only if $\alpha \in j(X)$.

Definition 19 (Nice systems of ultrafilters)**.** Let $\bar{U} = (\mathcal{U}_\alpha : \alpha < \lambda)$ be a sequence of ultrafilters. Given $\beta \leq_E \alpha < \lambda$, fix a projection $\pi_{\alpha\beta} : \kappa \to \kappa$ satisfying $j(\pi_{\alpha\beta})(\alpha) = \beta$. We say that the sequence $\bar{U}$ is a *nice system of ultrafilters* if the following hold:

(1) $(\lambda, \leq_E)$ is a κ^{++}-directed partial ordering.
(2) $(\mathcal{U}_\alpha : \alpha < \lambda)$ is a Rudin-Keisler commutative sequence of κ-complete ultrafilters over κ with projections $(\pi_{\alpha\beta} : \alpha, \beta < \lambda, \beta \leq_E \alpha)$.
(3) For every $\alpha < \lambda$, $\pi_{\alpha\alpha}$ is the identity on a fixed set $\bar{X}$ which belongs to every $\mathcal{U}_\beta$ for $\beta < \lambda$.
(4) (Commutativity) For every $\alpha, \beta, \gamma < \lambda$ such that $\alpha \geq_E \beta \geq_E \gamma$, there is a $Y \in \mathcal{U}_\alpha$ so that for every $\nu \in Y$
$$\pi_{\alpha\gamma}(\nu) = \pi_{\beta\gamma}(\pi_{\alpha\beta}(\nu)).$$
(5) For every $\alpha < \beta$, $\gamma < \lambda$, if $\gamma \geq_E \alpha, \beta$ then
$$\{\nu < \kappa : \pi_{\gamma\alpha}(\nu) < \pi_{\gamma\beta}(\nu)\} \in \mathcal{U}_\gamma.$$
(6) $\mathcal{U}_\kappa$ is a normal ultrafilter.
(7) $\kappa \leq_E \alpha$ when $\kappa \leq \alpha < \lambda$.
(8) (Full commutativity at κ) For every $\alpha, \beta < \lambda$ and $\nu < \kappa$, if $\alpha \geq_E \beta$, then $\pi_{\alpha\kappa}(\nu) = \pi_{\beta\kappa}(\pi_{\alpha\beta}(\nu))$.
(9) (Independence of the choice of projection to κ) For every α, β, $\kappa \leq \alpha$, $\beta < \lambda$ and $\nu < \kappa$
$$\pi_{\alpha\kappa}(\nu) = \pi_{\beta\kappa}(\nu).$$
(10) Each $\mathcal{U}_\alpha$ is a P-point ultrafilter, i.e. for every $f \in \kappa^\kappa$, if f is not constant modulo $\mathcal{U}_\alpha$, then there is a $Y \in \mathcal{U}_\alpha$ such that for every $\nu < \kappa$, $|Y \cap f^{-1}\{\nu\}| < \kappa$.

Definition 20.

- Denote $\pi_{\alpha\kappa}(\nu)$ by ν^0, where $\kappa \leq \alpha < \lambda$ and $\nu < \kappa$. By a $\circ$-increasing sequence of ordinals we mean a sequence $(\nu_1, \dots, \nu_n)$ of ordinals below κ so that:
$$\nu_1^0 < \nu_2^0 \dots < \nu_n^0$$
- Let $\nu < \kappa$ and $(\nu_1, \dots, \nu_n)$ be a $\circ$-increasing sequence of ordinals, we say that ν is *permitted* for $(\nu_1, \dots, \nu_n)$ if $\nu^0 > \max\{\nu_i^0 : 1 \leq i \leq n\}$.

Theorem 21. *Assume that there is a strong cardinal κ and that* GCH *holds in the ground model. Let also $2 \leq m < \omega$. Then there is a generic extension in which $2^{\aleph_n} = \aleph_{n+1}$ for every $n < \omega$ and $2^{\aleph_\omega} = \aleph_{\omega+m}$.*

Proof. Let $(\mathcal{U}_\alpha : \alpha < \lambda)$ and $(\pi_{\alpha\beta} : \alpha, \beta < \lambda, \beta \leq_E \alpha)$ be a nice system of ultrafilters with $\lambda = \kappa^{+m}$ and $f_\lambda : \kappa \to \kappa$ defined by $f_\lambda(\nu) = \nu^{+m}$. The model will be obtained as a generic extension given by the following posets:

[1] Here RK denotes the Rudin-Keisler order on ultrafilters over κ, i.e. $\mathcal{U} \leq_{\mathrm{RK}} \mathcal{W}$ if and only if $\exists f \in \kappa^\kappa$ such that $\mathcal{U} = f(\mathcal{W})$

The preliminary forcing:

Let's call the preliminary forcing $\mathbb{Q}$, this forcing was indeed used to add λ-many Příkrý sequences to κ, and thus after forcing with it one gets a cardinal preserving generic extension in which $\mathrm{cf}(\kappa) = \omega$ and additionally $\kappa^\omega \geq \lambda$.

Conditions $p \in \mathbb{Q}$ correspond to the collection of pairs of the form:

$$\{(\gamma, p^\gamma) : \gamma \in g \setminus \{\max(g), 0\}\} \cup \{(\max(g), p^{\max(g)}, T)\}$$

where:

(1) $g \subseteq \lambda$ is a set of cardinality $\leq \kappa$ which has a maximal element in the $\leq_E$-ordering and additionally $0 \in g$. Furthermore, $\mathrm{supp}(p)$ denotes g, $\mathrm{mc}(p)$ denotes $\max(g)$, T^p denotes T and p^{mc} denotes $p^{\max(g)}$.
(2) For $\gamma \in g$, p^γ is a finite $\circ$-increasing sequence of ordinals $< \kappa$.
(3) $(T. <_T)$ is a tree with trunk p^{mc} consisting of $\circ$-increasing sequences, all the splittings in T are required to be sets in $\mathcal{U}_{\mathrm{mc}(p)}$, i.e. for every $\eta \in T$, if $\eta \geq_T p^{\mathrm{mc}}$ then the set

$$\mathrm{Suc}_T(\eta) = \{\nu < \kappa : \eta^\frown(\nu) \in T\} \in \mathcal{U}_{\mathrm{mc}(p)}.$$

Also require that for nodes $\eta_1, \eta_2 \in T$ such that $\eta_2 \geq_T \eta_1 \geq_T p^{\mathrm{mc}}$

$$\mathrm{Suc}_T(\eta_1) \subseteq \mathrm{Suc}_T(\eta_2)$$

(4) For every $\gamma \in g$, $\pi_{\mathrm{mc}(p),\gamma}(\max(p^{\mathrm{mc}}))$ is not permitted for p^γ.
(5) For every $\nu \in \mathrm{Suc}_T(p^{\mathrm{mc}})$

$$|\{\gamma \in g : \nu \text{ is permitted for } p^\gamma\}| \leq \nu^0.$$

(6) $\pi_{\mathrm{mc}(p),0}$ projects p^{mc} onto p^0.

The order is given as follows: $p \leq^* q$ if and only if

(1) $\mathrm{supp}(p) \supseteq \mathrm{supp}(q)$.
(2) For every $\gamma \in \mathrm{supp}(q)$, p^γ is an end-extension of q^γ.
(3) $p^{\mathrm{mc}(q)} \in T^q$.
(4) For every $\gamma \in \mathrm{supp}(q)$:

$$p^\gamma \backslash q^\gamma = \pi_{\mathrm{mc}(q),\gamma}``((p^{\mathrm{mc}(q)} \backslash q^{\mathrm{mc}(q)}) \restriction \mathrm{length}(p^{\mathrm{mc}}) \backslash (i+1)),$$

where $i \in \mathrm{dom}(p^{\mathrm{mc}(q)})$ is the largest such that $p^{\mathrm{mc}(q)}(i)$ is not permitted for q^γ.
(5) $\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}$ projects $T^p_{p^{\mathrm{mc}}}$ into $T^q_{q^{\mathrm{mc}}}$.
(6) For every $\gamma \in \mathrm{supp}(q)$ and $\nu \in \mathrm{Suc}_{T^p}(p^{\mathrm{mc}})$, if ν is permitted for p^γ, then

$$\pi_{\mathrm{mc}(p),\gamma}(\nu) = \pi_{\mathrm{mc}(q),\gamma}(\pi_{\mathrm{mc}(p),\mathrm{mc}(q)}(\nu)).$$

The main forcing:

Now we introduce the forcing notion we are really interested in. The following poset is a modification of $\mathbb{Q}$ introduced above. It was introduced by ** to find a generic extension in which additionally $\aleph_\omega = \kappa$ and GCH holds below it (i.e. $2^{\aleph_n} = \aleph_{n+1}$).

The set of forcing conditions $\mathbb{P}$ consists of all elements p of the form:

$$\{(0,(\tau_1,\ldots,\tau_n),(f_0,\ldots,f_n),F)\} \cup \{(\gamma,p^\gamma) : \gamma \in g \setminus \{\max(g),0\}\} \cup \{(\max(g),p^{\max(g)},T)\},$$

where

(1) $n \in \omega$.
(2) $\{(0,(\tau_1,\ldots,\tau_n))\} \cup \{(\gamma,p^\gamma) : \gamma \in g \setminus \{\max(g),0\}\} \cup \{(\max(g),p^{\max(g)},T)\}$ is a condition in the preliminary forcing $\mathbb{Q}$.
(3) $f_0 \in \mathrm{Col}(\omega,\tau_1)$, $f_i \in \mathrm{Col}(\tau_i^{+m+1},\tau_{i+1})$ for $0 < i < n$ and $f_n \in \mathrm{Col}(\tau_n^{+m+1},\kappa)$.
(4) F is a function on the projection of $T_{p^{\mathrm{mc}}}$ by $\prod_{\mathrm{mc}(p),0}$ so that:

$$F((\nu_0,\ldots,\nu_{i-1})) \in \mathrm{Col}(\tau_{i-1}^{+m+1},\kappa).$$

The order is given as follows: $p \leq q$ if and only if

(1) $\{(0,p^0)\} \cup \{(\gamma,p^\gamma) : \gamma \in \mathrm{supp}(p)\backslash\{(\mathrm{mc}(p),0)\}\} \cup \{(\mathrm{mc}(p),p^{\mathrm{mc}},T^p)\}$ extends $\{(0,q^0)\} \cup \{(\gamma,q^\gamma) : \gamma \in \mathrm{supp}(q)\backslash\{(\mathrm{mc}(q),0)\}\} \cup \{(\mathrm{mc}(q),q^{\mathrm{mc}},T^q)\}$ in the sense of the preliminary forcing $\mathbb{Q}$.
(2) For every $i < \mathrm{length}(q^0) = n^q$, $f_i^p \geq f_i^q$.
(3) For every $\eta \in T_{p^0}^{p,0}$, $F^p(\eta) \supseteq F^q(\eta)$.
(4) For every i with $n^q \leq i < n^p$,

$$f_i^p \supseteq F^q((p^0\backslash q^0)|i+1).$$

(5) $\min(p^0\backslash q^0) > \sup(\mathrm{ran}(f_{n^q}))$.[2]

The poset $(\mathbb{P},\leq)$ is κ^{++}-cc and it satisfies the Příkrý condition, meaning that given a condition $p \in \mathbb{P}$ and φ a statement in the forcing language, there is a condition $q \leq^{**} p$ such that q decides φ. Here $q \leq^{**} p$ means $q \leq p$ and for every $\gamma \in \mathrm{supp}(q)$, $q^\gamma = p^\gamma$.

If G is a generic subset of $\mathbb{P}$ and $\alpha < \kappa^{+m}$, we define $G^\alpha = \bigcup\{p^\alpha : p \in G\}$. Then, the G^α's are Příkrý sequences of order type ω and if $\aleph_0 < \tau < \kappa$ and τ remains a cardinal in $V[G]$, then for some n and for some $m' \leq m$, $\tau = \kappa_n^{+m'+1}$. Thus, $V[G] \models 2^{\aleph_\omega} = \aleph_{\omega+m}$ and $2^{\aleph_n} = \aleph_{n+1}$.

□

5. Some inequalities

Theorem 5 implies that if λ is a singular cardinal of cofinality κ, then $\mathfrak{a}(\lambda) \leq \mathfrak{a}(\kappa)$. We are interested into the study of the relationship between these two cardinals. In particular, we would like to know if it is possible to have (consistently) $\mathfrak{a}(\lambda) < \mathfrak{a}(\kappa)$. Note that the consistency of $\mathfrak{a}(\lambda) = \mathfrak{a}(\kappa)$ is rather trivial when assuming large cardinals: indeed, it is possible to force the

[2] ran denotes the range of the function f_{n^q}

value of $\mathfrak{a}(\kappa)$ to be κ^+ when κ is a supercompact cardinal. To see the specifics of such a model see [Bro+17].

To start, we restrict ourselves to the case $\mathrm{cf}(\lambda) = \omega$ and λ being a strong limit cardinal. For the results to come, some basics on the arithmetic of cardinal characteristics on ω are needed, in particular we mention an important result regarding the cardinals $\mathfrak{a}$ and $\mathfrak{b}$.

5.1. **Brendle's model.** [See Theorem in [Bre97]]

Definition 22. Some special subsets of the reals:

- If f, g are functions from ω to ω, we say that $f \leq^* g$, if there exists an $n \in \omega$ such that for all $m > n$, $f(m) \leq g(m)$. In this case, we say that g *eventually dominates* f. Also $f <^* g$ if and only if $f \leq^* g$ and $\{n \in \omega : f(n) = g(n)\}$ is finite.
- Let $\mathcal{F} \subseteq \omega^\omega$, we say that $\mathcal{F}$ is *dominating*, if for all $g \in \omega^\omega$, there exists an $f \in \mathcal{F}$ such that $g \leq^* f$. $\mathcal{F} \subseteq \omega^\omega$ is *unbounded*, if for all $g \in \omega^\omega$ there exists an $f \in \mathcal{F}$ such that $f \not\leq^* g$.

Assume we have a model V of ZFC and a $<^*$-well-ordered sequence $\bar{F} = (f_\alpha : \alpha < \kappa)$ of strictly increasing functions from ω to ω such that:

$$V \models \mathfrak{b} = \kappa \wedge 2^\kappa = \kappa^+ \wedge (f_\alpha : \alpha < \kappa) \text{ is unbounded.}$$

Such a sequence can be gotten by adding Cohen reals. Now let $\mathcal{A}$ be a mad family in V, then there is a ccc forcing $\mathbb{P}(\mathcal{A})$ such that $\Vdash_{\mathbb{P}(\mathcal{A})} \mathcal{A}$ is not mad and $(f_\alpha : \alpha < \kappa)$ is still unbounded. The poset $\mathbb{P}(\mathcal{A})$ is actually Mathias' forcing with respect to a specific filter which preserves the unboundness of the family $\mathcal{F}$. We recall here the definition of this poset:

Definition 23 (Mathias-Příkrý forcing)**.** Let $\mathcal{F}$ be a filter on ω, Mathias-Příkrý forcing with respect to $\mathcal{F}$ is the poset $\mathbb{M}(\mathcal{F})$ consisting of pairs (s, F) so that $s \in [\omega]^{<\omega}$ and $F \in \mathcal{F}$. The order is given by: $(t, G) \leq (s, F)$ if and only if $t \supseteq s$, $G \subseteq F$ and $t \backslash s \subseteq G$.

If $\mathcal{A}$ is a mad family of subsets of ω, one can consider the ideal associated to it. Namely, the ideal $\mathcal{I}_\mathcal{A} = \{X \subseteq \omega : X \subseteq^* \bigcup_{A \in F} A$ such that $F \subseteq \mathcal{A}$ and $|F| < \omega\}$. If G is $\mathbb{M}(\mathcal{F}_\mathcal{A})$-generic over a ground model V (here $\mathcal{F}_\mathcal{A}$ is the dual filter to $\mathcal{I}_\mathcal{A}$), then Mathias-Příkrý's poset destroys the maximality of $\mathcal{A}$ by adding a set $x_G \subseteq^* F$ for all $F \in \mathcal{F}_\mathcal{A}$.

Brendle proved that, given a mad family $\mathcal{A}$ it is possible to choose a filter $\mathcal{G}_\mathcal{A}$ such that $\mathbb{M}(\mathcal{G}_\mathcal{A})$ destroys the maximality of $\mathcal{A}$ and additionally it keeps the family of reals $(f_\alpha : \alpha < \kappa)$ unbounded in the corresponding generic extension. One of his main results states the following:

Theorem 24. *Let κ be a regular uncountable cardinal. It is consistent with ZFC that $\mathfrak{b} \leq \kappa < \kappa^+ = \mathfrak{a} = \mathfrak{c}$.*

A simple bookkeeping argument ensures that one can run a finite support forcing iteration of Mathias-Příkrý's forcing with respect to filters of the form $\mathcal{G}_\mathcal{A}$ for all maximal almost disjoint families $\mathcal{A} \subseteq \omega^\omega$ such that $|\mathcal{A}| < \kappa^+$, while preserving the unboundedness of the sequence $\bar{F}$.

5.2. **A model in which $\mathfrak{a}(\lambda) < \mathfrak{a}$ for λ singular.** In this section, we will use the models of Theorem 21 as well as Brendle's model above (24) to prove the consistency of the existence of a singular cardinal λ of countable cofinality, for which the equality $\mathfrak{a}(\lambda) < \mathfrak{a}$ holds.

Theorem 25. *Let κ and Γ be two regular cardinals with $\Gamma \geq \kappa^{++}$ and assume that κ is $V_{\kappa+\delta}$-strong cardinal for a δ such that $\kappa^{+\delta} = \lambda$. Then, there is a generic cardinal preserving extension in which $\mathrm{cf}(\kappa) = \omega$, $\kappa^\omega \geq \lambda$ and $\mathfrak{a} = 2^{\aleph_0} = \delta^+ < 2^\kappa = \mathfrak{b}(\kappa)$.*

Proof. The proof of this result is quite similar to the proof of the result below, so we will present that one in all detail. Here we just point out that the only difference between the two arguments is that, instead of using the forcing $\mathbb{P}$ (like in the proof below), here one uses the preliminary forcing $\mathbb{Q}$ of Section 4.2. The rest of the argument follows through. □

Theorem 26. *Assume that there is a strong cardinal κ and that* GCH *holds in the ground model V. Let also $2 \leq m < \omega$ and δ be a regular cardinal such that $\aleph_1 < \delta < \aleph_{\omega+m}$. Then, there is a generic extension of V in which $2^{\aleph_\omega} = \aleph_{\omega+m}$, $\delta = \mathfrak{b} = \mathfrak{a}(\aleph_\omega)$ and $\mathfrak{a} = 2^{\aleph_0} = \delta^+ < 2^{\aleph_\omega} = \mathfrak{b}(\aleph_\omega)$.*

Proof. Start with a ground model V where GCH holds and there is a strong cardinal κ, let us use the poset $\mathbb{P}_m$ from Theorem 21 to get a generic extension $V^{\mathbb{P}_m}$ in which $2^{\aleph_n} = \aleph_{n+1}$ for every $n < \omega$ and $2^{\aleph_\omega} = \aleph_{\omega+m}$.

Moreover, in the generic extension $V^{\mathbb{P}_m}$ we have the following configuration of cardinals:

$$\aleph_1 \leq \mathfrak{b} \leq \mathfrak{a}(\aleph_\omega) \leq \mathfrak{a} \leq 2^{\aleph_0} \leq \kappa \leq 2^{\aleph_\omega} = \mathfrak{b}(\aleph_\omega)$$

Notice that in this model SCH fails and $\aleph_\omega$ is a strong limit, so Theorem 12 applies and $\max(\mathrm{pcf}\{\aleph_n : n \in \omega\}) = 2^{\aleph_\omega}$. If we mimic the argument from the proof of Theorem 17, we get that $\max(\mathrm{pcf}\{\aleph_n : n \in \omega\}) = 2^{\aleph_\omega} = \mathfrak{b}(\aleph_\omega)$. The other inequalities $\mathfrak{b} \leq \mathfrak{a}(\aleph_\omega)$ and $\mathfrak{a}(\aleph_\omega) \leq \mathfrak{a}$ follow from Theorems 5 and 8.

Now, since we want to prove that consistently $\mathfrak{a}(\kappa) < \mathfrak{a}$ we use the poset of Brendle of Theorem 24 for the cardinal δ, denote it by $\mathbb{R}_\delta$ and force with it over the model $V^{\mathbb{P}_m}$. Thus, we get a model in which $\mathfrak{b} = \delta < \mathfrak{a} = \delta^+$, moreover in $V^{\mathbb{P}_m} * \dot{\mathbb{R}}_\delta$ we have the following:

$$\aleph_1 < \delta = \mathfrak{b} \leq \mathfrak{a}(\aleph_\omega) \leq \mathfrak{a} = \delta^+ = 2^{\aleph_0} < \aleph_\omega \leq 2^{\aleph_\omega} = \mathfrak{b}(\aleph_\omega)$$

Note that since the poset $\mathbb{R}_\delta$ is ccc, the value of $\mathfrak{b}(\aleph_\omega)$ as well as the fact that $\kappa = \aleph_\omega$ are preserved. Hence, it is left to prove that in this extension $\mathfrak{a}(\aleph_\omega) \leq \mathfrak{b}$ or, in other words, to prove that there is a maximal $\aleph_\omega$-almost disjoint family of size $\mathfrak{b}$.

Look again to Theorem 5 and notice that since $\mathfrak{b} < \aleph_\omega$, there is an $\aleph_\omega$-mad family of cardinality δ where $\mathfrak{b} \leq \delta \leq \mathfrak{b}^{\aleph_0}$. In particular, there is a $\aleph_\omega$-mad of size $\mathfrak{b}$ which implies $\mathfrak{b} = \mathfrak{a}_{\aleph_\omega}$.

□

5.3. **Other simple inequalities.** Simpler models can witness another set of inequalities, we list them here so the reader has a full scope on the methods and the possible open questions: Let λ be a singular cardinal of cofinality κ

- A first natural question is whether consistently, are there models for which $\kappa^+ < \mathfrak{a}(\lambda) < 2^\lambda$?. For this, it is enough to notice that if we work in a model for which λ is a strong limit cardinal and $\mathfrak{b}(\lambda)$ is such that $\mathfrak{b}(\lambda) = 2^\lambda$ (like in the models above for $\lambda = \aleph_\omega$). It suffices to push the value of $\mathfrak{b}(\kappa)$ to be, for instance, 2^κ.

  Thus, if we start with a model in which GCH holds, δ is a measurable cardinal and $\Gamma > \delta$, we can force first with Příkrý forcing to change the cofinality of δ to be ω. After that we can add Γ-many generalized Cohen functions, since this poset is ccc, the cofinality of δ is still unchanged while $\mathfrak{b} = \Gamma = 2^{\aleph_0}$. Hence we get:

$$\aleph_1 < \mathfrak{b} = \mathfrak{a}(\aleph_\omega) = 2^{\aleph_0} < 2^{\aleph_\omega}.$$

- **Question**: Is it consistent to have $\mathfrak{b} < \mathfrak{a}(\aleph_\omega)$? The following section aims to answer this.

5.4. **Destroying maximality at a singular.** We are interested in using forcing to add an almost disjoint set to a given maximal almost disjoint family $\mathcal{A}$ in a ground model V, here $\mathcal{A} \subseteq [\lambda]^\lambda$ and λ is a singular cardinal of cofinality ω.

Throughout this section, we assume that $\mathcal{A}$ is a maximal λ-almost disjoint family and we consider the ideal associated to it, namely:

$$\mathcal{I}_\mathcal{A} = \{X \subseteq \lambda : \text{ There exists } F \subseteq_{\text{fin}} \mathcal{A} \text{ such that } X \subseteq^* \bigcup_{A \in F} A\}$$

Note that $\mathcal{I}_\mathcal{A}$ is a σ-complete ideal on λ. Let $(\lambda_n : n \in \omega)$ be a sequence of regular cardinals cofinal in λ. Define $S_n = \lambda_n \backslash \lambda_{n-1}$ for $n > 0$ and given $X \in \mathcal{A}$ define also $\mathcal{Z}_X = \{n \in \omega : S_n \cap X \neq \emptyset\}$. The family $\mathcal{J}_\mathcal{A} = \{\mathcal{Z}_\mathcal{X} : X \in \mathcal{A}\}$ generates a σ-complete ideal on ω. Indeed, just notice that $\mathcal{Z}_X \cup \mathcal{Z}_Y = \mathcal{Z}_{X \cup Y}$.

Claim 27. If z is an infinite subset of ω such that $|z \cap \mathcal{Z}_X| < \omega$ for all $X \in \mathcal{A}$, then there is a set $x_z \subseteq \lambda$ such that $|x_z \cap X| < \lambda$ for all $X \in \mathcal{A}$.

Proof. Let $x_z = \bigcup_{m \in z} S_m$ and $X \in \mathcal{A}$ arbitrary, notice first that $X \subseteq \bigcup_{m \in \mathcal{Z}_X} S_m$ and since $|z \cap \mathcal{Z}_X| < \omega$ we get that $|\bigcup_{m \in \mathcal{Z}_X} S_m \cap \bigcup_{m \in z} S_m| = |\bigcup_{m \in z \cap \mathcal{Z}_X} S_m| < \lambda$. This implies that $|X \cap \bigcup_{m \in z} S_m| < \lambda$ and so $|X \cap x_z| < \lambda$.

□

The immediate consequence of the lemma above is: if we want to destroy the maximality of a given maximal almost disjoint family $\mathcal{A} \subseteq [\lambda]^\lambda$ by adding a set $x \subseteq \lambda$ so that $|x \cap X| < \lambda$ for all $X \in \mathcal{A}$, it is enough to add a set $z \subseteq \omega$ with the property $|z \cap \mathcal{Z}_X| < \kappa$ for all $X \in \mathcal{A}$. Adding such a set can be achieved by using Mathias-Příkrý forcing with respect to the dual filter to the ideal $\mathcal{J}_\mathcal{A}$, call it $\mathcal{H}_\mathcal{A}$ (recall Definition 23).

If G is $\mathbb{M}(\mathcal{H}_{\mathcal{A}})$-generic over a model V, then the generic real $x_{\mathcal{H}_{\mathcal{A}}}$ has the property that $x_{\mathcal{H}_{\mathcal{A}}} \subseteq^* H$ for all $H \in \mathcal{H}_{\mathcal{A}}$.

Theorem 28. *Let $\delta \geq \aleph_{\omega+1}$ be a cardinal and V a ground model in which* GCH *holds. Then there is a ccc generic extension of V in which $\mathfrak{a}(\aleph_\omega) \geq \delta$.*

Proof. We will perform a finite support iteration of ccc forcings of length δ. First, we fix a bookkeeping surjective map $\pi : \delta \to \delta \times \aleph_{\omega+1}$ with the property that if $\pi(\alpha) = (\pi_0(\alpha), \pi_1(\alpha))$, then $\pi_1(\alpha) < \alpha$.

Perform a finite support iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha : \alpha \leq \delta)$ such that $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$ where $\mathbb{Q}_\alpha = \mathbb{M}(\dot{\mathcal{F}}_{\gamma,\beta})$ where $\pi(\alpha) = (\beta, \gamma)$ and $\dot{\mathcal{F}}_{\gamma,\beta}$ is the γ-th $\mathbb{P}_\alpha$-name for the dual filter to the ideal $\mathcal{J}_{\mathcal{A}}$ associated to a mad family $\mathcal{A} \subseteq [\aleph_\omega]^{\aleph_\omega}$ of size $< \delta$.

Then, in the final extension $V^{\mathbb{P}_\delta}$, if $\dot{\mathcal{A}}$ is a $\mathbb{P}_\delta$-name for a maximal $\aleph_\omega$-almost disjoint family, then there is some $\gamma < \delta$ such that $\dot{\mathcal{A}}$ is a $\mathbb{P}_\gamma$-name for the β-th maximal $\aleph_\omega$-almost disjoint family. Thus, if we put $\alpha = \pi(\beta, \gamma)$, in the α-th step in the iteration we are adding a set x killing the maximality of $\mathcal{A}$ via the Claim 27 above. □

Corollary 29. Let $\delta \geq \aleph_{\omega+1}$ be a cardinal and κ be a regular such that $\kappa < \delta$. Let also V be a ground model in which GCH holds. Then there is a ccc generic extension of V in which $\mathfrak{b} = \kappa$ while $\delta \leq \mathfrak{a}(\aleph_\omega)$.

Proof. This is basically Brendle's model construction: Start with a model in which GCH holds and then add Cohen reals $(f_\alpha : \alpha < \kappa)$ in order to force $\mathfrak{b} = \kappa$ and $2^\kappa = \kappa^+$. Afterward, using a bookkeeping argument we can iterate Mathias-Prikry forcing over all filters of the form $\mathcal{H}_{\mathcal{A}}$ (see section above) associated to maximal almost disjoint families $\mathcal{A} \subseteq [\aleph_\omega]^{\aleph_\omega}$ of size $< \delta$, such that the forcing $\mathbb{M}(\mathcal{H}_{\mathcal{A}})$ preserves the family $(f_\alpha : \alpha < \kappa)$ to be unbounded. □

The limitation of Theorem 28 above is that the iteration has to be long in order to be able to go through the list of all mad families of some determined size, which pushes also the value of the continuum $2^{\aleph_0}$ to be high as well. Indeed, in each step of the iteration we are adding a new subset of ω, so in the final model $2^{\aleph_0} \geq \delta$.

If, for instance we want to study the open question (in [KKS04]) whether $\mathfrak{a}(\aleph_\omega) = \aleph_\omega$ is consistent, we would have to come up with an idea of how to be able to get $\mathfrak{a}(\aleph_\omega) \geq \aleph_\omega$ and not so far up like in the Proposition above. We present a preservation result on this direction for specific case of Příkrý forcing.

5.5. **Příkrý forcing and mad families.** Within the theory of cardinal characteristics one important question which usually arises is whether a specific combinatorial object can be preserved after forcing. For instance, whether the maximality of a given maximal almost disjoint family can be preserved after forcing with some poset $\mathbb{P}$.

Definition 30 (Příkrý forcing)**.** Let λ be a measurable cardinal and $\mathcal{W}$ be a normal measure on λ. The poset $\mathcal{P}_{\mathcal{W}}$ has as conditions all pairs of the form (s, F) where $s \in [\lambda]^{<\lambda}$ and $F \in \mathcal{W}$ ordered as follows: $(s, F) \leq (t, G)$ if and only if s is an initial segment of t, $F \subseteq G$ and $s \backslash t \subseteq G$.

The poset $\mathcal{P}_{\mathcal{W}}$ has the following properties:

Proposition 31.

- $\mathcal{P}_{\mathcal{W}}$ is κ^+-cc.
- In the generic extension $\mathrm{cf}(\kappa) = \omega$.
- The Příkrý property: Let σ be a sentence of the forcing language and let (t, G) be a condition in $\mathcal{P}_{\mathcal{W}}$, then there is $F \subseteq G$ in $\mathcal{W}$ such that (t, F) decides σ.

We are interested into controlling the value of $\mathfrak{a}(\lambda)$, when λ is a singular cardinal of cofinality κ. Recall that $\kappa^+ \leq \mathfrak{a}(\lambda) \leq \mathfrak{a}(\kappa) \leq 2^\kappa \leq 2^\lambda$.

Theorem 32. *Let κ be a strongly compact cardinal in a universe V in which there is a κ-maximal independent family $\mathcal{A}$. Then there is a measure $\mathcal{W}$ on κ such that, when considering the Příkrý forcing with respect to $\mathcal{W}$, in the corresponding generic extension $V^{\mathcal{P}_{\mathcal{W}}}$:*

$$V^{\mathcal{P}_{\mathcal{W}}} \models \mathcal{A} \textit{ is a } \kappa\textit{-maximal almost disjoint family of subsets of } \kappa.$$

Proof. Let $\mathcal{A}$ be a κ-almost disjoint family and $\mathcal{I}_{\mathcal{A}}$ be the κ-ideal associated to $\mathcal{A}$, recall that $\mathcal{I}_{\mathcal{A}} = \{X \subseteq \kappa : \exists \Delta \subseteq_{\text{fin}} \mathcal{A} \text{ such that } X \subseteq^* \bigcup_{A \in \Delta} A\}$.

Also let us consider the dual filter $\mathcal{F}_{\mathcal{A}}$ associated to the ideal $\mathcal{I}_{\mathcal{A}}$ and use strong compactness of the cardinal κ to extend it to a κ-complete ultrafilter $\mathcal{U}_{\mathcal{A}}$.

Claim 33. There is a normal ultrafilter $\mathcal{W}$ on κ and a set $U_{\mathcal{A}} \in \mathcal{W}$ such that $\mathcal{W} \neq \mathcal{U}_{\mathcal{A}}$ and for some $A_{\mathcal{W}} \in \mathcal{A}$, $|A_{\mathcal{W}} \cap U_{\mathcal{A}}|= \kappa$.

Proof. First, notice that if $\mathcal{U}$ is an arbitrary non-principal ultrafilter, then it must be the case that if $\mathcal{A} \cap \mathcal{U} \neq \emptyset$, then $|\mathcal{A} \cap \mathcal{U}| \leq 1$. Take an arbitrary $A \in \mathcal{A}$ and consider a κ-complete filter $\mathcal{H}$ such that $A \in \mathcal{H}$, if we extend it to a κ-complete ultrafilter $\mathcal{G}_{\mathcal{A}}$ then it must be the case that $\mathcal{G}_{\mathcal{A}} \cap \mathcal{A} = \{A\}$.

Let $f \in \kappa^\kappa$ such that $[f]_{\mathcal{G}_{\mathcal{A}}} = \kappa$, then we know that the filter

$$\mathcal{W}_{\mathcal{A}} = \{X \subseteq \kappa : f^{-1}(X) \in \mathcal{G}_{\mathcal{A}}\}$$

is a normal ultrafilter on κ. Moreover, if we look at $f(A) \subseteq \kappa$, then clearly $f(A) \in \mathcal{W}_{\mathcal{A}}$ and since $\mathcal{A}$ is maximal, there exists $B_A \in \mathcal{A}$ such that $|\mathcal{B}_A \cap f(A)|= \kappa$. This normal ultrafilter has the desired property. ◻

Consider now the Příkrý generic extension obtained after forcing over V with the poset $\mathcal{P}_{\mathcal{W}_{\mathcal{A}}}$. In order to prove the result, we start with $\dot{X}$ to be a $\mathcal{P}_{\mathcal{W}}$-name for a subset of λ and a condition $(s, F) \in \mathcal{P}_{\mathcal{W}}$. It is then enough to find a condition $(t, G) \leq (s, F)$ and an element $A \in \mathcal{A}$ such that $(t, G) \Vdash \dot{X} \cap A$ is unbounded.

Given $\alpha < \kappa$, there is $G_\alpha \in \mathcal{W}$ such that (s, G_α) decides the statement "$\check{\alpha} \in \dot{X}$". Let $G = \triangle_{\alpha<\kappa} G_\alpha$ and consider the set $Y = \{\alpha < \kappa : (s, G) \Vdash \check{\alpha} \in \dot{X}\}$ and ask ourselves whether or not it belongs to $\mathcal{W}$.

- If $Y \in \mathcal{W}$, then given $\alpha < \kappa$, if we fix $H = F \cap Y$ we know that since $\mathcal{A}$ is maximal, there exists some $A_H \in \mathcal{A}$ such that $|A_H \cap H| = \kappa$. Thus take $\beta \geq \max\{\alpha, \sup(s)\}$, $\beta \in H \cap A_H$ and note that the condition $(s, H) \Vdash \beta \in \dot{X} \cap A_H$.
- If $\kappa \backslash Y \in \mathcal{W}$ and $\alpha < \kappa$, note that for all $A_{\mathcal{W}} \in \mathcal{A} \cap \mathcal{W}$, if we define $H = F \cap (\kappa \backslash Y) \cap (A_{\mathcal{W}})$, then $(s, H) \Vdash x_g \cap \dot{X} =^* \emptyset$ where x_g is $\mathcal{P}_{\mathcal{W}_{\mathcal{A}}}$-generic.
  Thus $(s, H) \Vdash (\kappa \setminus A_W) \cap \dot{X} = \emptyset$, so $(s, H) \Vdash \dot{X} \subseteq A_{\mathcal{W}}$.

□

Corollary 34. If $V \models \mathfrak{a}(\kappa) = \delta$, then $V^{\mathcal{P}_{\mathcal{W}}} \models \mathfrak{a}(\kappa) = \delta$

Proof. From the Theorem, one gets that $V^{\mathcal{P}_{\mathcal{W}}} \models \mathfrak{a}(\kappa) \leq \delta$.
From Erdös-Hechler, one has that there are no ω-mad families of size $< \mathfrak{a}(\kappa)$. □

6. Questions and final remarks

Theorem 5 assumes that λ is a singular cardinal of countable cofinality, so a natural question appears:

Question 35. Suppose λ is a singular cardinal and $\mathrm{cf}(\lambda) = \kappa > \omega$. Is it consistent that $\mathfrak{a}(\lambda) < \mathfrak{a}(\kappa)$?

While there are methods to change the cofinality of a given large cardinal to be uncountable, if we would like to follow the lines of our proof, we would like to have a model for which $\mathfrak{b}(\kappa) < \mathfrak{a}(\kappa)$. As far as we know, this is still an open problem on the theory of cardinal characteristics on the higher Baire spaces.

Kojman, Kubiś and Shelah have left open the following question in their paper, we also list it here because it is quite relevant and Theorem 28 suggests that one would have to develop new forcing techniques in order to get an answer:

Question 36. Is it consistent to have $\mathfrak{a}(\aleph_\omega) = \aleph_\omega$?

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