PS introduction to mathematical logic

Exercises week 11

December 15, 2016

1. Which of axioms $1, 2, 4, 5$ are true for the binary relation E on the domain D in the following examples:

   (a) $D = \{a\}; E = \emptyset$.
   (b) $D = \{a\}; E = \{(a, a)\}$.
   (c) $D = \{a, b\}; E = \{(a, b), (b, a)\}$.
   (d) $D = \{a, b, c\}; E = \{(a, b), (b, a), (a, c), (b, c)\}$.
   (e) $D = \{a, b, c\}; E = \{(a, b), (a, c)\}$.
   (f) $D = \{0, 1, 2, 3\}; E = \{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}$.
   (g) $D = \{a, b, c\}; E = \{(a, b), (b, c)\}$.

2. Prove that there is no set X such that $\mathcal{P}(X) \subseteq X$.

3. Say we have defined a "pair "$[(x, y)]$ in some way, and assume that we can prove $[(x, y)] = [(x', y')] \to x = x' \wedge y = y'$. Prove that $\{x : \exists y([(x, y)] \in R)\}$ and $\{y : \exists x([(x, y)] \in R)\}$ exists for all sets R.

4. Working from the axioms of ZF without the axiom of infinity, prove that the following are equivalent:

   (a) Axiom of infinity
   (b) $\exists w[\emptyset \in w \wedge \forall y \in w(\{y\} \in w)]$.

5. Prove that for all α, β and γ ordinals:

   (a) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$,
   (b) $\alpha^{\beta+\gamma} = \alpha^{\beta} \cdot \alpha^{\gamma}$,
   (c) $(\alpha^{\beta})^{\gamma} = \alpha^{\beta \cdot \gamma}$.

6. (a) Show that $(\omega + 1) \cdot 2 \neq \omega \cdot 2 + 1 \cdot 2$.
   (b) Show that $(\omega \cdot 2)^2 \neq \omega^2 \cdot 2^2$.