PS introduction to mathematical logic

Exercises week 2 *

October 13, 2016

1 More about sentencial logic.

1. Prove or disprove the following statements.

   (a) If $\Gamma \Rightarrow \alpha$ or $\Gamma \Rightarrow \beta$ then $\Gamma \Rightarrow (\alpha \vee \beta)$.

   (b) If $\Gamma \Rightarrow \alpha$ and $\Gamma \Rightarrow \beta$ then $\Gamma \Rightarrow (\alpha \wedge \beta)$.

   (c) If $\Gamma \Rightarrow (\alpha \vee \beta)$ then $\Gamma \Rightarrow \alpha$ or $\Gamma \Rightarrow \beta$.

   (d) If $\Gamma \Rightarrow (\alpha \wedge \beta)$ then $\Gamma \Rightarrow \alpha$ and $\Gamma \Rightarrow \beta$.

2. Call a formula α a dual n-clause, if α is of the form $\beta_1 \wedge \ldots \wedge \beta_n$ where each β_i is either A_i or $(\neg A_i)$. A formula is in n-disjunctive normal form (dnf) is and only if it is of the form $\gamma_1 \vee \ldots \vee \gamma_k$, where each γ_j is a dual n-clause. Show that for each formula α: Either $\neg\alpha$ is a tautology, or there is a formula $\bar{\alpha}$ in disjunctive normal form with $\alpha \Leftrightarrow \bar{\alpha}$.

2 First order logic.

1. For each one of the following sets of formulas give an example of a model that satisfies this set of formulas. Try to describe all finite models satisfying the formulas.

   (a) i. $R(x, y) \wedge R(y, z) \rightarrow R(x, z)$.

   ii. $R(x, y) \wedge R(x, z) \wedge R(y, w) \wedge R(z, w) \rightarrow R(y, z) \vee R(z, y) \vee (y = z)$.

   (b) i. $R(x, y) \wedge R(y, z) \rightarrow R(x, z)$.

   ii. $R(x, z) \wedge R(y, z) \rightarrow R(x, y) \vee R(y, x) \vee (x = y)$.

   iii. $R(c, x)$.

*All the exercises are taken from *The incompleteness phenomenon*, Goldstern-Judah.

2. In each of the following cases find an appropriate first order language and a formula such that there are models that satisfy the formula, and every model that satisfies the formula has the property that:

   (a) the model is a finite set with exactly n elements (for a given n).

   (b) the model is a dense linear ordering (like the rationals $\mathbb{Q}$).

   (c) the models is a field.

   (d) the model is a field of characteristic 3.

3. If x and y are distinct variables, σ and τ closed $\mathcal{M}$-terms and μ is any $\mathcal{M}$-term, show that:
$$\mu(x/\sigma)(y/\tau) = \mu(x/\sigma, y/\tau) = \mu(y/\tau)(x/\sigma).$$

4. $R(x, y)$ is an order relation in a model $\mathcal{M}$ if the following formulas are valid in the model:

   (a) $\neg(R(u, u))$.

   (b) $R(u, v) \to \neg R(v, u)$.

   (c) $R(u, v) \wedge R(v, w) \to R(u, w)$.

   Given a formula $\varphi(x)$, we define a subset $A_\varphi \subseteq M$ by:
$$A_\varphi := \{a \in M : \mathcal{M} \models \varphi(x/a)\}.$$
   Similarly, if we have a formula with two free variables $\varphi(x, y)$, we define $A_\varphi \subseteq M \times M$, a subset of the set of pairs from M, by:
$$A_\varphi := \{(a, b) \in M \times M : \mathcal{M} \models \varphi(x/a, y/b)\}.$$
   We call A_φ the set characterized by φ. What are the sets characterized by the following formulas:

   (a) R is an order relation in the model $\mathcal{M}$ and $\varphi(u, v) = \neg R(u, v)$.

   (b) R is an order relation in the model $\mathcal{M}$ and $\varphi(u, v) = \neg R(u, v) \wedge \neg R(v, u)$.

   (c) R is an order relation in the model $\mathcal{M}$ which is a tree, (i.e. the following formula is valid in the model $R(u, v) \wedge R(u, w) \to R(v, w) \vee R(w, v) \vee (v = w))$ and $\varphi(u, v) = \neg R(v, u)$.