PS introduction to mathematical logic

Exercises week 3 *

October 21, 2016

1 Validity

1. Let $\varphi(u, v_1, v_2, \ldots, v_n)$ be a formula in the language $\mathcal{L}$ with free variables $u, v_1, v_2, \ldots, v_n$ only, and let x, y variables that do not appear in φ. Prove or disprove the validity of each of the following formulas:

   (a) $\forall u \varphi(u, v_1, v_2, \ldots, v_n) \to \exists u \varphi(u, v_1, v_2, \ldots, v_n)$.

   (b) $\forall u \varphi(u, v_1, v_2, \ldots, v_n) \to \forall x \forall y \varphi(F(x, y), v_1, v_2, \ldots, v_n)$ where F is a function symbol in $\mathcal{L}$.

   (c) $\exists u \varphi(u, v_1, v_2, \ldots, v_n) \to \exists x \exists y \varphi(F(x, y), v_1, v_2, \ldots, v_n)$ where F is a function symbol in $\mathcal{L}$.

2. Let $\mathcal{L} = \{P, F, c\}$ where P is 2-ary relational, F is 2-ary functional and c is a constant.

   (a) Find two different interpretations for $\mathcal{L}$ in a model whose universe is the set of natural numbers, $P(x, y)$ has the interpretation $x \leq y$, and such that the following is valid in both interpretations: $\forall x(P(F(x, c), x) \wedge P(x, F(x, c)))$.

   (b) Does the following sentence hold in both interpretations? $\forall x \forall y(F(x, y) = F(y, x))$.

   (c) If the answer to the question above was yes, find a third interpretation for which the sentence does not hold. If your answer was no, find an interpretation for which it does.

3. Let φ be the following formula:

$$(\forall x P(x, y) \wedge \forall x \forall y \forall z(P(x, y) \wedge P(y, z) \to P(x, z)) \wedge \forall x \forall y(P(x, y) \vee P(y, x))) \to (\exists x \forall y P(x, y)).$$

   Prove that any finite model (a model with a finite universe) is a model of φ but there exists infinite models that do not satisfy φ.

*All the exercises are taken from *The incompleteness phenomenon*, Goldstern-Judah.

2 Proof Systems

1. Using the axioms, give a complete derivation for each of the following formulas:

   (a) $\forall x(\varphi \to \psi) \to (\varphi \to \forall x\psi)$ where x is not free in φ.

   (b) $\varphi(t) \to \exists x\varphi(x)$ where t is free for x in φ.

   (c) $\forall x\varphi \to \exists x\varphi$.

   (d) $\forall x(\varphi \wedge \psi) \leftrightarrow (\forall x\varphi \wedge \forall x\psi)$.

2. Prove or disprove: (here you may use the generalization theorem, deduction, etc.)

   (a) $\vdash \exists x(\varphi \to \forall x\varphi)$.

   (b) $\vdash (\forall x\varphi \vee \forall x\psi) \to \forall x(\varphi \vee \psi)$.

   (c) $\vdash (\exists x\varphi \wedge \exists x\psi) \to \exists x(\varphi \wedge \psi)$.

   (d) $\{\varphi(x), \forall y(\varphi(y) \to \forall z\psi(z))\} \vdash \forall x\psi(x)$.

   (e) $\exists x\forall y\varphi \to \forall y\exists x\varphi$.

3. Suppose that $\Gamma \vdash \varphi$ and let P be a relation symbol that does not appear in Γ or in φ. Is there a proof of φ from Γ in which P does appear? Is there a proof of φ from Γ that includes only relation symbols that appear both in φ and Γ? Prove your answers!.