PS introduction to mathematical logic

Exercises week 4 and 5

October 27, 2016

1 The compactness theorem

All the exercises below are applications of the compactness theorem for first order logic. I will give first an example of a typical argument that hopefully will give you ideas for the exercises:

Proposition 1. *If T is a first order theory that has arbitrarily large finite models, then T has an infinite model.*

Proof. Consider, for each $n \in \mathbb{N}$ the first order formula $\varphi_n = \exists x_1 \exists x_2 \ldots \exists x_n \bigwedge_{i<j\leq n} x_i \neq x_j$, then clearly $\mathcal{M} \models \varphi_n$ if and only if M has at least n elements. Let $T' = T \cup \{\varphi_n : n \geq 2\}$, note that if we can prove that T' is satisfiable then we have the proposition. Using the *compactness theorem* it is enough to show that T' is finitely satisfiable. Let then $\Gamma \subseteq T'$ be a finite set, using the hypothesis we know that T has models of arbitrarily large size, then for each n, there exists $\mathcal{M}$ such that the size of $\mathcal{M}$ is at least n and $\mathcal{M} \models T$, in particular $\mathcal{M} \models \Gamma$. Hence the compactness theorem gives us a model $\mathcal{N} \models T'$, so $\mathcal{N} \models \varphi_n$ for all n and this obviously implies that $\mathcal{N}$ is infinite. □

Corollary 2. *There is not a first order sentence φ, such that $\mathcal{M} \models \varphi$ if and only if $\mathcal{M}$ is infinite.*

Proof. Argue by contradiction and suppose there is such formula φ and consider the theory $T = \{\neg\varphi\}$, clearly this theory has finite models of arbitrarily large size. Namely, if M has exactly n elements, then $\mathcal{M} \models T$, then using the proposition T has an infinite model, which is clearly a contradiction. □

1.1 Exercises

1. We say that a class of $\mathcal{L}$-structures $\mathcal{K}$ [1] is an elementary class if there is an $\mathcal{L}$-theory T such that:

$$\mathcal{M} \in \mathcal{K} \leftrightarrow \mathcal{M} \models T$$

[1] $\mathcal{M}$ is an $\mathcal{L}$-structure if $\mathcal{M}$ is a model in the language $\mathcal{L}$

We also say that T axiomatizes $\mathcal{K}$. Decide if the following classes are elementary. Show the class is elementary by giving an axiomatization or prove that it is not probably by using the Compactness Theorem.

(a) Let $\mathcal{L} = \{E\}$ where E is a binary relation symbol.

   i. Let $\mathcal{K}$ be the class of all equivalence relations.

   ii. Let $\mathcal{K}$ be the class of all equivalence relations where each class has size 2.

   iii. Let $\mathcal{K}$ be the class of equivalence relations where each class is finite.

   iv. Let $\mathcal{K}$ be the class of equivalence relations with infinitely many infinite classes.

(b) Let $\mathcal{L} = \{E\}$ where E is a binary relation symbol. We say that an $\mathcal{L}$-structure $\mathcal{M}$ is a graph if $E^{\mathcal{M}}$ is symmetric.

   i. Let $\mathcal{K}$ be the class of connected graphs (A graph G is connected if every two vertices a and b have a finite path of edges connecting them).

   ii. Let $\mathcal{K}$ be the class of acyclic graphs.

   iii. Let $\mathcal{K}$ be the class of bipartite graphs. [Recall that a graph is bipartite if we can partition the edges into two sets A and B such that there every edge has one vertex in A and one vertex in B. Hint: a graph is bipartite if and only if there are no cycles of odd length.]

2. Show that there exists an ordered field $\mathbb{F}$ that is not Archimedean. Remember that this meand that given $a, b \in \mathbb{F}$ such that $a, b > 0$, there exists a natural number n such that $na \geq b$.

3. If T is a set of sentences with an infinite model, then T has an uncountably infinite model.

4. Let $\mathcal{L}$ be the language of ordering. Then there is no set Γ of sentences whose models are exactly the well-ordering structures. Remember, a linear ordering $\leq$ on a set P is a well-ordering if every non-empty subset of P has a least element.