PS introduction to mathematical logic

Exercises week 7

November 17, 2016

1. In this exercise you will show that if two models are elementary equivalent and finite, then they are isomorphic. Fix a language $\mathcal{L}$ of first order logic, and write $\mathcal{L}(c)$ for the language which has all function, relation and constant symbols from $\mathcal{L}$, and an additional constant symbol c. If $m \in M$, then write $(\mathcal{M}, m)$ for the model $\mathcal{M}'$ of $\mathcal{L}(c)$ which is an expansion of $\mathcal{M}$, and in which the constant symbol c is interpreted as m.

   (a) Assume that $\mathcal{M}$ is a finite model, and assume that $\mathcal{M}$ is elementarily equivalent to $\mathcal{N}$. Show that $\mathcal{N}$ is also finite, with the same number of elements as $\mathcal{M}$.

   (b) Assume that $\mathcal{M}$ and $\mathcal{N}$ are two elementarily equivalent finite models. Show that for any $m \in M$ there is $n \in N$ such that $(\mathcal{M}, m)$ and $(\mathcal{N}, n)$ are isomorphic.

   (c) Assume that $\mathcal{M}$ is a finite model in which each element is the interpretation of some constant symbol. Assume that $\mathcal{N}$ is elementarily equivalent to $\mathcal{M}$. Show that $\mathcal{M}$ and $\mathcal{N}$ are isomorphic. (Hint: Any isomorphism f must satisfy $f(c^{\mathcal{M}}) = c^{\mathcal{N}}$ for any constant symbol c.)

   (d) Assume that $\mathcal{M}$ and $\mathcal{N}$ are as in (a). Show that $\mathcal{M}$ and $\mathcal{N}$ are isomorphic. (Hint: Use (b) inductively to arrive at a situation where (c) applies.)

2. Assume that $\mathcal{M}$ and $\mathcal{N}$ are two models and $\mathcal{M} \equiv \mathcal{N}$. Show that there are natural numbers $n_1, n_2, \ldots, n_k$ such that the number of isomorphisms between $\mathcal{M}$ and $\mathcal{N}$ is $n_1! \cdot n_2! \cdot \ldots \cdot n_k!$. (Hint: Define an equivalence relation $\sim$ on M by : $a \sim b$ if and only if for all formulas $\varphi(x)$, $\mathcal{M} \models \varphi(a) \leftrightarrow \varphi(b)$. Let $n_1, n_2, \ldots, n_k$ be the sizes of the equivalence classes.)

3. Let $\mathcal{A} \subseteq \mathcal{B}$ be $\mathcal{L}$-structures. Suppose that for every finite sequence $a_1, a_2, \ldots, a_m \in A$ and every $b \in B$ there is an automorphism of $\mathcal{B}$ that fixes each element of $a_1, a_2, \ldots a_m$ and moves b into A. Show that $\mathcal{A} \preceq \mathcal{B}$.

4. Let K be a field and let $\mathcal{L}$ the first order language of vector spaces over K: the non-logical symbols of $\mathcal{L}$ are a constant 0, a binary function symbol $+$, and an unary function symbol F_a for each $a \in K$. Given a K-vector space V, we regard V as an $\mathcal{L}$-structure in the obvious way: 0 is interpreted by the identity element of V, $+$ is interpreted by the addition of V, and each F_a is interpreted by the operation of scalar multiplication by a. Suppose that $W \subseteq V$ are infinite dimensional K-vector spaces. Use the previous exercise to prove $W \preceq V$. Use this result to show that any two infinite K-vector spaces are elementarily equivalent.

5. Let Γ be a consistent theory all of whose countable models are isomorphic. Show that Γ is complete.

6. Show that any two countable saturated models are isomorphic. (Hint: If $f : \mathcal{M} \to \mathcal{N}$ is an isomorphism, then for all $a_1, a_2, \ldots \in \mathcal{M}$ we have :

$$\begin{gathered} \text{type}_{\mathcal{M}}(a_1) = \text{type}_{\mathcal{N}}(f(a_1)) \\ \text{type}_{\mathcal{M}}(a_2) = \text{type}_{\mathcal{N}_{f(a_1)}}(f(a_2)) \\ \vdots \end{gathered}$$

Use a back-and-forth argument.)