PS introduction to mathematical logic

Exercises week 8

November 24, 2016

1. Let DLO^- be the theory containing only the following formulas:

   - $\forall x(\neg x < x)$.
   - $\forall x \forall y \forall z(x < y < z \to x < z)$.
   - $\forall x \forall y(x = y \vee x < y \vee y < x)$.
   - $\forall x \exists y x < y$.
   - $\forall x \forall y(x < y \to \exists z(x < z < y))$.

   Show that DLO^- has exactly two non-isomorphic countable models. Does this contradict the theorem that states that a complete theory cannot have exactly two non-isomorphic models?.

2. Let $\mathcal{L}_0$ be the language with only one binary relation symbol $\leq$. Let $\mathcal{L} \supset \mathcal{L}_0$ be the language that in addition has countably many constant symbols $c_1, c_2, \ldots$. Let Γ_0 be the DLO, the theory of dense linear orders. Find a complete theory $\Gamma \supseteq \Gamma_0$ in the language $\mathcal{L}$ which has no countable saturated model.

3. Consider the language which has only countably many constant symbols $c_1, c_2, \ldots$ (no function or relation symbols). Let $\Gamma = \{c_1 \neq c_2, c_2 \neq c_3, c_1 \neq c_3, \ldots\}$.

   (a) Show that Γ has countably many non-isomorphic models. (Hint: For any model $\mathcal{M}$, consider the set $M \setminus \{c_1^{\mathcal{M}}, c_2^{\mathcal{M}}, \ldots\}$.)

   (b) Show that there is a countable model of Γ into which all countable models can be elementarily embedded.

   (c) Conclude that Γ is a complete theory.

   (d) Describe the atomic and the saturated countable model of Γ.

4. Prove that $\mathcal{M}$ is atomic and saturated if and only if every countable model is isomorphic to $\mathcal{M}$.

5. Show that no infinite well ordering is ω-saturated.

6. Find non-isomorphic countable models $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M}$ can be elementarily embedded into $\mathcal{N}$ and $\mathcal{N}$ can be elementarily embedded into $\mathcal{M}$.

7. Fix a theory Γ with an infinite model. Let $\mathcal{M}$ be a model. Show that the following are equivalent.

   (a) $\mathcal{M}$ is a prime model of Γ, i.e., $\mathcal{M}$ can be elementarily embedded into every model of Γ.

   (b) $\mathcal{M}$ can be elementarily embedded into every countable model of Γ.

   Show that the above conditions imply that Γ is complete.