Generic Absoluteness

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Abstract

We explore the consistency strength of $\mathbf{\Sigma^1_3}$ and $\mathbf{\Sigma^1_4}$ absoluteness, for a variety of forcing notions.

Introduction

Shoenfield's absoluteness theorem states that a Σ^1_2 predicate is true of a real in a ground model exactly if it is true of the same real in any forcing extension. However, this is not true for Σ^1_3 predicates. Indeed, if we add a Cohen real to L, then the sentence "There exists a non-constructible real" is Σ^1_3 and, while failing in L, it holds in the generic extension. In this paper we shall mainly investigate the strength of generic absoluteness for Σ^1_3 predicates. The consistency strength of Σ^1_3 absoluteness under ccc forcing extensions is just ZFC ([B 3]). But by extending the class of ccc forcing extensions, we obtain stronger absoluteness properties. Our aim in this paper is to explore the large cardinal strength of these properties.

Our notation is largely standard. We write $\ell h(s)$ for the length of a sequence s. If X is a set, $[X]^{<\omega}$ is the set of all finite subsets of X. We denote by KP Kripke–Platek set theory including the axiom of infinity. Transitive models of KP are called *admissible sets*.

We call the set-theoretical universe $\mathbf{\Sigma^1_n}$*-absolute* with respect to some generic extension if each Σ^1_n or Π^1_n predicate true of a real in the ground model is true of the same real in the extension. We call it Σ^1_n*-absolute* if each Σ^1_n or Π^1_n sentence (without parameters) holding in the ground model holds in the extension. (These definitions should be contrasted with the "2-step" absoluteness of [W], where it is required that absoluteness hold not only between V and $V[G]$, but between $V[G]$ and $V[G][H]$ for successive generic extensions $V \subseteq V[G] \subseteq V[G][H]$.)

We use Solovay's almost-disjoint coding ([J-S]), which we shall review in the next section. We shall also use the method of Baumgartner, Harrington, and Kleinberg ([B-H-K]) to shoot a club through an arbitrary stationary subset of ω_1, while preserving ω_1.

Almost disjoint coding

The following account of Solovay's almost-disjoint coding is due to A. Mathias, whom we thank for letting us include it here.

First, let $\langle s_i \mid i \in \omega\rangle$ be a recursive enumeration of ${}^{<\omega}2$, the set of finite sequences of 0's and 1's, such that each such sequence is enumerated before any of its proper extensions. For any subset a of ω, let $\tilde{a} : \omega \longrightarrow 2$ be the characteristic function of a. Fix a recursive partition of ω into infinitely many infinite pieces X_i ($i \in \omega$). For $a \subseteq \omega$ and $i \in \omega$, define

$$\begin{aligned} f^a &=_{df} \{j \mid \tilde{a} \restriction \ell h(s_j) = s_j\} \\ f^a_i &=_{df} \{j \mid \tilde{a} \restriction \ell h(s_j) = s_j \text{ and } \ell h(s_j) \in X_i\} \end{aligned}$$

Thus each f^a is a member of a well-known perfect family of pairwise almost disjoint infinite subsets of ω, and is the disjoint union of the infinitely many infinite sets f^a_i.

For subsets a and b of ω, set $b \odot a =_{df} \{i \in \omega \mid b \cap f^a_i \text{ is finite}\}$.

Secondly, let $A \subseteq \mathcal{P}(\omega)$ and let $\pi : A \longrightarrow \mathcal{P}(\omega)$. Define the graph of π by

$$G =_{df} \{\langle a, i\rangle | a \in A, \text{ and } i \in \pi(a)\}.$$

We shall use Solovay's coding to add a set $b \subseteq \omega$ by a c.c.c. forcing such that

$$\forall a \in A \quad b \odot a = \pi(a).$$

A condition will be a pair $\langle s, g\rangle$ where $s \in [\omega]^{<\omega}$, and $g \in [G]^{<\omega}$, and the partial ordering is given by

$$\langle t, h\rangle \leqslant \langle s, g\rangle \text{ iff } s \subseteq t,\ g \subseteq h, \text{ and } \forall \langle a, i\rangle \in g\ \ (t \cap f^a_i \subseteq s).$$

Any two conditions with the same first part are compatible, so this forcing is c.c.c. The first part s of a condition describes a finite subset of the set b to be added; to place $\langle a, i\rangle$ into the second part is to give a promise that no further elements of b will be in f^a_i. By standard density arguments, $b \cap f^a_i$ will be finite whenever $i \in \pi(a)$, and infinite (since the family of sets f^a_i is pairwise almost disjoint) whenever $i \notin \pi(a)$. Thus this forcing achieves what is promised.

ω_1 inaccessible to reals

Theorem 1 *Suppose that $\omega_1 = \omega_1^L$. Then Σ^1_3-absoluteness fails for some forcing that preserves ω_1.*

Proof: For each countably infinite ordinal α let g_α be the $<_L$-first function mapping ω onto α. For each n, fix λ_n such that $\{\alpha \mid g_\alpha(n) = \lambda_n\}$ is stationary.

Consider a fixed n. By [B-H-K], we may add, preserving ω_1, a club subset C^n of that set. By Solovay, we may add a real b such that whenever $a \in L$ is a subset of ω that codes a ordinal, $b \odot a$ codes the first member of C^n that is strictly larger than that ordinal.

Lemma 1 *Let $M = L_\eta[b]$ be a countable admissible set such that:*

$$M \models \texttt{every ordinal is constructibly countable}.$$

Then $\eta \in C^n$.

Proof: Let $\theta < \eta$, and let $c \in L_\eta$ be a subset of ω that codes θ. Then $b \odot c$ codes a member ζ of C^n greater than θ. $b \odot c \in M$, and so by admissibility, ζ can be recovered inside M, and so is less that η. As C^n is closed, $\eta \in C^n$. $\square$

Let $\varphi(\eta, b)$ be the formula:

$$L_\eta[b] \models KP + \texttt{every ordinal is constructibly countable.}$$

Our construction has added a real b such that

$$\forall \theta_1 < \theta_2 < \theta_3 < \omega_1[(\varphi(\theta_1, b) \wedge \varphi(\theta_2, b) \wedge \varphi(\theta_3, b)) \Rightarrow L_{\theta_3}[b] \models g_{\theta_1}(n) \equiv g_{\theta_2}(n)].$$

Translated into codes of countable admissible sets, that is a Π^1_2 assertion $\vartheta(b, n)$ about b and the natural number n. So the statement $\exists b\, \vartheta(b, n)$ is a Σ^1_3 sentence. If Σ^1_3-absoluteness holds, then this sentence will be true in the ground model, witnessed by B say. Consider $L_{\omega_1}[B]$: it is admissible and believes that every ordinal is constructibly countable. There is, therefore, a club D^n, namely $\{\eta < \omega_1 \mid L_\eta[B] \prec_{\Sigma_\omega} L_{\omega_1}[B]\}$, lying in the ground model, such that each $\eta \in D^n$ satisfies the predicate $\varphi(\eta, B)$.

If for each n we can find such a club D^n, then the intersection $\bigcap_{n<\omega} D^n$ will be a club, D say. But then for $\theta_1 < \theta_2$, both in D, for every n, $g_{\theta_1}(n) = g_{\theta_2}(n)$, an absurdity. Thus there is an n for which Σ^1_3 absoluteness fails for the sentence $\exists b\, \vartheta(b, n)$. $\square$

The above argument relativises easily to show that for each $a \subseteq \omega$, $\Sigma^1_3(a)$-absoluteness for ω_1-preserving forcing implies that $\omega_1 > \omega_1^{L[a]}$. Hence,

Corollary 1 $\mathbf{\Sigma^1_3}$*-absoluteness for ω_1-preserving forcing implies that ω_1 is inaccessible to reals, i.e., for every $a \subseteq \omega$, $\omega_1 > \omega_1^{L[a]}$.*

Proper and semi-proper forcing

We could try to strengthen the previous result by restricting the class of forcing notions to those that preserve stationary subsets of ω_1, or even to semi-proper forcing.

The notion of *semi-proper* forcing is due to Shelah and generalizes his own weaker notion of *proper* forcing, itself a generalization of ccc and σ-closed forcing notions.

Thus, semi-proper posets include all proper posets, plus other well-known forcing notions, like Prikry forcing.

A forcing notion $\mathbb{P}$ is *semi-proper* if for some large-enough regular cardinal λ (e.g., larger than $2^{2^{|\mathbb{P}|}}$), there is a club $C \subseteq [H(\lambda)]^\omega$ such that for all $N \in C$ and all $p \in N \cap \mathbb{P}$, there is a $q \leq p$ which is $(\mathbb{P}, N)$-semi-generic, i.e., for all $\mathbb{P}$-names τ in N, if $\Vdash_{\mathbb{P}}$ "$\tau \in \omega_1^V$", then $q \Vdash_{\mathbb{P}}$ "$\tau \in N$".

Hence, semi-proper forcing preserves ω_1. In fact it preserves stationary subsets of ω_1 (see [F-M-S]).

$\mathbf{\Sigma^1_3}$-absoluteness for semi-proper forcing does not imply that ω_1 is inaccessible in L. This follows from results of Goldstern-Shelah [G-S] and Bagaria [B 2]. Let us call a regular cardinal κ *reflecting* if for every $a \in H(\kappa)$ and every first-order formula $\varphi(x)$, if for some cardinal λ, $H(\lambda) \models \varphi(a)$, then there exists a cardinal $\delta < \kappa$ such that $H(\delta) \models \varphi(a)$.

Notice that if κ is reflecting, then it must be inaccessible. If κ is reflecting, then κ is reflecting in L. The consistency strength of a reflecting cardinal is below a Mahlo.

Suppose κ is reflecting. It follows from [G-S] that there is an ω_1-preserving iteration of length κ over L of semi-proper forcing notions that forces the bounded semi-proper forcing axiom. But in [B 2] it is shown that the bounded semi-proper forcing axiom implies $\mathbf{\Sigma^1_3}$-absoluteness with respect to all semi-proper forcing extensions.

However, even for the more restricted class of proper forcing notions, $\mathbf{\Sigma^1_3}$-absoluteness implies that either ω_1 or ω_2 is inaccessible in L.

Recall that a poset $\mathbb{P}$ is *proper* if for some large-enough regular cardinal λ, there is a club $C \subseteq [H(\lambda)]^\omega$ such that for all $N \in C$ and all $p \in N \cap \mathbb{P}$, there is a $q \leq p$ which is $(\mathbb{P}, N)$-generic, i.e., for all $\mathbb{P}$-names τ in N, if $\Vdash_{\mathbb{P}}$ "τ is an ordinal", then $q \Vdash_{\mathbb{P}}$ "$\tau \in N$".

All ccc and all σ-closed posets are proper. Also, properness is preserved by countable-support iteration. In particular, any finite iteration of ccc and σ-closed posets is proper.

The following result follows from work in [Ba]; we thank B. Veličković for calling it to our attention.

Theorem 2 *Suppose $\mathbf{\Sigma^1_3}$-absoluteness holds for proper forcing. Then, either ω_1 is inaccessible in L, or ω_2 is.*

Proof: If ω_2 is not inaccessible in L, then there is a Kurepa tree T in L (i.e., a tree of height ω_1, with countable levels, and $\aleph_2$-many branches) which remains Kurepa in V (see [J], 24). Further, T is Δ_1-definable over L_{ω_1}.

Let $\mathbb{Q}_1 = \mathbb{Q}_0 * Coll(\omega_1, \omega_2)$, where $\mathbb{Q}_0$ is the ccc forcing for adding ω_2 Cohen reals, and $Coll(\omega_1, \omega_2)$ is the σ-closed poset for collapsing ω_2 to ω_1 with countable conditions.

Claim: Every branch of T in $V^{\mathbb{Q}_1}$ is already in V.

Proof of Claim: Since $\mathbb{Q}_0$ has property K (i.e., every uncountable subset of $\mathbb{Q}_0$ contains an uncountable pairwise compatible set) it adds no new branches to T (see [Ba], 8.5). Thus, it will be enough to show that every branch of T in $V^{\mathbb{Q}_1}$ is already in $V^{\mathbb{Q}_0}$.

But since $V^{\mathbb{Q}_0} \models 2^{\aleph_0} > \aleph_1$, and since $Coll(\omega_1, \omega_2)$ is σ-closed, no new branches to T are added by $Coll(\omega_1, \omega_2)$ (see [Ba], 8.6). This proves the Claim.

Let $\mathbb{Q}_2 = \mathbb{Q}_1 * \mathbb{P}_T$, where $\mathbb{P}_T$ is the forcing that specializes T. Namely, let $\{b_\alpha : \alpha < \omega_1\}$ be an enumeration, in $V^{\mathbb{Q}_1}$, of all the branches of T.

Let

$$b'_\alpha = b_\alpha - \bigcup_{\beta < \alpha} b_\beta$$

and let $s_\alpha = min(b'_\alpha)$, $\alpha < \omega_1$.

Let
$$T' = \bigcup_{\alpha<\omega_1} \{t \in b_\alpha : s_\alpha < t\}$$
Then $S =_{df} T - T'$ is a subtree of T without any uncountable branches.

$\mathbb{P}_T$ is the poset of all functions p from a finite subset of S into ω, such that $p(s) \neq p(t)$ whenever $s < t$, ordered by reversed inclusion. $\mathbb{P}_T$ is ccc (see [Ba], 8.2). If $g : S \longrightarrow \omega$ is a $\mathbb{P}_T$-generic function, then the function $h : T \longrightarrow \omega$ defined by: $h(t) = g(s_\alpha)$, if $t \in b_\alpha$ and $s_\alpha < t$, satisfies: for every $s \leq t, u$, if $h(s) = h(t) = h(u)$, then t and u are comparable. We call such a h a *specializing function*. Note that if T has a specializing function, then T has at most $\aleph_1$-many branches. For if b is a branch, there is $s \in b$ such that the set $\{t \in b : h(t) = h(s)\}$ is uncountable. But then, $b = \{t \in T : t \leq s$ or $\exists u\ (t \leq u \wedge h(u) = h(s)\}$. i.e., b is determined by s. Since there are only $\aleph_1$-many such s, there are at most $\aleph_1$-many branches.

Now suppose ω_1 is not inaccessible in L. So, $\omega_1 = \omega_1^{L[x]}$, for some $x \subseteq \omega$.

Let $\mathbb{P} = \mathbb{Q}_2 * \mathbb{Q}_3$, where $\mathbb{Q}_3$ codes $h : T \longrightarrow \omega$ into a $y \subseteq \omega$, by almost-disjoint coding relative to the reals in $L[x]$. Thus, $\mathbb{P}$ is a four-step iteration of ccc and σ-closed posets, hence $\mathbb{P}$ is proper.

In $V^{\mathbb{P}}$ the following is true:
$$\exists y \subseteq \omega(L[x,y] \models y \texttt{ codes a specializing map } h : T \longrightarrow \omega)$$
But since T is Δ_1-definable over L_{ω_1}, in $V^{\mathbb{P}}$ the following holds:
$$\exists y \forall M(M \texttt{ is a transitive well-founded model of ZF} \wedge x, y \in M \rightarrow$$
$$M \models \texttt{``}y \texttt{ codes a specializing map } h : T^M \rightarrow \omega\texttt{''})$$
This is a $\Sigma^1_3(x)$ sentence. By $\boldsymbol{\Sigma}^1_3$-absoluteness, it holds in V. So, in V,
$$\exists y \subseteq \omega(L[x,y] \models y \texttt{ codes a specializing map } h : T \longrightarrow \omega)$$
And this contradicts the fact that T has $\aleph_2$-many branches. $\Box$

Remark. The conclusion of the previous Theorem can be stengthened to: either ω_1 is Mahlo in L or ω_2 is inaccessible in L. For if ω_1 is inaccessible but not Mahlo in L, then we may add, by almost-disjoint coding, an $x \subseteq \omega$ such that $\omega_1 = \omega_1^{L[x]}$ (see the proof of Theorem 6 below).

A reflecting cardinal in L

Recall that a regular cardinal κ is *reflecting* if for every $a \in H(\kappa)$ and every first-order formula $\varphi(x)$, if for some cardinal $\lambda, H(\lambda) \models \varphi(a)$, then there exists a cardinal $\delta < \kappa$ such that $H(\delta) \models \varphi(a)$. As reflecting cardinals are strongly inaccessible, this is seen to be equivalent to: $V_\kappa \prec_{\Sigma_2} V$.

If we allow all set-forcing extensions, even those that do not preserve ω_1, then ω_1 becomes a reflecting cardinal in L.

The following result is due to Feng-Magidor-Woodin [F-M-W], and independently to the second author. For completeness, we give a proof.

Theorem 3 *The following are equiconsistent:*

1. $\mathbf{\Sigma^1_3}$*-absoluteness for set forcing.*
2. *There exists a reflecting cardinal.*

Proof: Assume that V is $\mathbf{\Sigma^1_3}$-absolute for set forcing. Let $\kappa = \omega_1$. We first show that κ is inaccessible in L and that $L_\kappa = (V_\kappa)^L$ is Σ_2-elementary in L. For suppose $\kappa = (\lambda^+)^L$. Let x_0 be a real coding λ and let φ be the sentence:

$$\exists x \subseteq \omega (x \texttt{ codes an ordinal } \alpha > \lambda \wedge \alpha \texttt{ is an } L\texttt{-cardinal})$$

φ is a force-able Σ^1_3 sentence with x_0 as a parameter, so it is true, so κ is not the L-successor cardinal to κ. A contradiction.

Now suppose ψ is Σ_2 with a parameter x_1 from L_κ, ψ true in L. Let θ be the sentence

$$\exists x \subseteq \omega (x \texttt{ codes } L_\alpha \wedge L_\alpha \models \psi \wedge x_1 \in L_\alpha \wedge \alpha \texttt{ is an } L\texttt{-cardinal})$$

θ is a force-able Σ^1_3 sentence with x_1 as parameter, so it is true. Thus, there is a countable α such that $L_\alpha \models \psi$, $x_1 \in L_\alpha$, and α is an L-cardinal. So, $L_\kappa \models \psi$, since $L_\alpha \prec_{\Sigma_1} L_\kappa$.

Start now with a regular κ, $V_\kappa \prec_{\Sigma_2} V$, and force with the Levy collapse $Coll(\omega, <\kappa)$. Let G be $Coll(\omega, <\kappa)$-generic over V.

We claim that $V[G]$ is $\mathbf{\Sigma^1_3}$-absolute with respect to set forcing.

For suppose $V[G][H] \models \varphi$, where $V[G][H]$ is a set generic extension of $V[G]$ and φ is a $\mathbf{\Sigma^1_3}$ sentence with parameter a real $x_0 \in V[G]$. For every $\alpha < \kappa$, let $G(<\alpha)$ denote $G \cap Coll(\omega, <\alpha)$. Choose $\alpha < \kappa$ so that $x_0 \in V[G(<\alpha)]$. Then,

$$V \models \texttt{There exists a forcing } \mathbb{Q} \texttt{ such that } Coll(\omega, <\alpha) * \mathbb{Q} \Vdash \varphi(\dot{x_0})$$

where $\dot{x_0}$ is a $Coll(\omega, <\alpha)$-term for x_0. By Σ_2-elementarity, V_κ satisfies the same sentence. As any $\mathbb{Q} \in V_\kappa^{Coll(\omega,<\alpha)}$ can be embedded in $Coll(\omega, <\beta)$ for some β, we get $\Vdash_{Coll(\omega,<\beta)} \varphi(\dot{x_0})$. So, $V[G(<\beta)] \models \varphi(x_0)$, hence $V[G] \models \varphi$. $\square$

Remark. The above proof shows that Σ^1_3-absoluteness for set forcing is consistent relative to the consistency of ZFC. For, if we choose any cardinal κ such that $V_\kappa \prec_{\Sigma_2} V$, κ not necessarily regular, we then obtain the desired absoluteness when κ is Levy-collapsed to ω_1 (using $Coll(\omega, <\kappa)$).

A refinement of the previous argument yields the following:

Theorem 4 *Suppose that* $\mathbf{\Sigma^1_3}$*-absoluteness holds for* ω_1*-preserving set forcing. Then* ω_2 *is* ω_1*-reflecting in* L*: For every* $a \in H(\omega_1) \cap L$ *and every first-order formula* $\varphi(x)$*, if for some* L*-cardinal* λ*,* $H(\lambda)^L \models \varphi(a)$*, then there exists an* L*-cardinal* $\delta < \omega_2$ *such that* $H(\delta)^L \models \varphi(a)$*.*

Proof. Assume $\mathbf{\Sigma^1_3}$-absoluteness for ω_1-preserving set forcing, and suppose that φ is a formula with parameters from $H(\omega_1) \cap L$ which holds in $H(\lambda)^L$ for some L-cardinal λ. We are done if λ is less than ω_2. Otherwise, by Levy-collapsing λ to ω_1 (using $Coll(\omega_1, \lambda)$), we can produce $X \subseteq \omega_1$ coding $H(\lambda)^L$; thus any ZF^- model M containing X satisfies:

$$(*) \qquad \varphi \texttt{ holds in some } H(\lambda)^L, \ \lambda \texttt{ an } L\texttt{-cardinal.}$$

By choosing λ to be a singular cardinal and using the fact that we may assume that $0^\#$ does not exist, we can in addition require that in $V[X]$, ω_2 is a successor L-cardinal. This is enough to guarantee that by an ω_1-closed almost-disjoint forcing we can produce $Y \subseteq \omega_1$ coding $H(\omega_2)$, in the sense that every subset of ω_1 in $V[Y]$ belongs to $L[Y]$. One more ω_1-preserving forcing "reshapes" Y, in the sense that it produces $Z \subseteq \omega_1$ such that $Y \in L[Z]$ and Z is "reshaped", meaning that every countable α is in fact countable in $L[Z \cap \alpha]$. (Z is produced by forcing with countable $z \subseteq \beta < \omega_1$, satisfying the latter at all $\alpha \leq \beta$.)

Now force $W \subseteq \omega_1$ using countable $w : \alpha \to 2$ with the properties that $w(2\gamma) = Z(\gamma)$ for $2\gamma < \alpha$ and for each $\beta \leq \alpha$, if M is a ZF^- model containing $w \restriction \beta$ and $\beta = \omega_1$ of M then M satisfies $(*)$. This forcing is ω_1-distributive using the fact that all subsets of ω_1 belong to $L[Z]$ and the fact that $(*)$ holds in any ZF^- model containing Z.

Now using the fact that W is reshaped, code it by a real R preserving ω_1, via almost-disjoint coding. Then R satisfies the Π^1_2 formula (with parameters from $H(\omega_1) \cap L$):

$$\forall M (M \models ZF^- \texttt{ and } R \in M \texttt{ and } M \models \texttt{``}\omega_1 \texttt{ exists''} \longrightarrow$$

$$M \models \texttt{``}\varphi \texttt{ holds in } H(\lambda)^L \texttt{ for some } L\texttt{-cardinal } \lambda\texttt{''})$$

By our absoluteness assumption, we may suppose that R belongs to the ground model. Apply this property to the ZF^- model $M = L_{\omega_2}[R]$ and we see that there is an L-cardinal $\delta < \omega_2$ such that $H(\delta)^L$ satisfies φ, as desired. $\square$

Remark. The previous result implies that under the hypothesis of $\mathbf{\Sigma^1_3}$ absoluteness for ω_1-preserving forcing, either ω_1 is reflecting in L or many L-cardinals greater than ω_1 must be collapsed. For, if $\lambda \geq \omega_1$ is least such that $H(\lambda)^L \models \varphi$ for some φ with parameters from $H(\omega_1) \cap L$, then λ^+ of L, λ^{++} of L, ... must all be less than ω_2. We do not know if the previous result holds with "ω_1-preserving" replaced by "preserving stationary subsets of ω_1".

Class forcing

We next show that $\mathbf{\Sigma^1_3}$-absoluteness for class-forcing is false.

Theorem 5 *Suppose M is a model of ZFC. Then there is a class-generic extension N of M and a $\mathbf{\Sigma^1_3}$ sentence φ with real parameters from M such that φ is true in N and false in M.*

Proof: By Jensen's Coding Theorem, M can be extended to a model of the form $L[r]$, r a real. Then, by the relativisation to r of a result of Beller-David (see [Da]), the latter model can be extended to $L[s]$, s a real, which is minimal; i.e., this model satisfies the statement:

$(\star)$ `For all ordinals` α, $L_\alpha[s]$ `is not a model of` ZF.

But notice that this is a Π^1_2-property of s and, therefore, if $\mathbf{\Sigma^1_3}$-absoluteness for class-forcing holds, there is a real s in M such that M satisfies $(\star)$.

In particular, M satisfies $s^\sharp$ does not exist. But, under this hypothesis, it is shown in [F] that there is a class forcing extension of M in which some $\mathbf{\Sigma^1_3}$ sentence φ with parameter s holds, where φ is false in M. $\square$

Σ^1_4-absoluteness

A Mahlo cardinal in L

In terms of consistency strength, $\mathbf{\Sigma^1_4}$-absoluteness is much stronger than $\mathbf{\Sigma^1_3}$-absoluteness. Indeed, $\mathbf{\Sigma^1_3}$-absoluteness for random forcing, plus Σ^1_4-absoluteness for Cohen forcing, already implies that ω_1 is inaccessible to reals ([B 1]).

Recall that a poset is σ-centered if it can be partitioned into countably many classes so that for every finite collection $p_1, ..., p_n$ of conditions, all in the same class, there exists p such that $p \leq p_1, ..., p_n$. We have the following:

The following result is implicit in the work of Jensen and Solovay [J-S], although in its present form is due to A. Mathias. We thank him for calling it to our attention.

Theorem 6 *Suppose that Σ^1_4-absoluteness holds for σ-centered forcing and ω_1 is inaccessible to reals. Then ω_1 is a Mahlo cardinal in L.*

Proof: The argument is due to Jensen [J-S], and was his first step towards coding the universe by a real.

Suppose that C is a constructible club of countable ordinals, each singular in L. By almost-disjoint coding, a σ-centered forcing notion, we may add a real b such that whenever a is a real in the ground model that codes an ordinal, $b \odot a$ is a real coding the next greater element of C. Note that $\omega_1 = \omega_1^{L[b]}$: for, working inside $L[b]$, we may define a sequence of codes of ordinals by setting c_0 to be some constructible code of ω, and given c_ν we set $c_{\nu+1} = b \odot c_\nu$. At a limit stage λ, writing γ_ν for the ordinal coded by c_ν, we take c_λ to be the first code of $\bigcup_{\nu<\lambda} \gamma_\nu$ in the inner model $L[\langle c_\nu \mid \nu < \lambda \rangle]$, "first" meaning first in the canonical well-ordering of that model definable from $\langle c_\nu \mid \nu < \gamma \rangle$. That c_λ exists follows from the fact that each γ_ν lies in C, and therefore so does γ_λ, which is therefore singular in L; so in the inner model $L[\langle c_\nu \mid \nu < \gamma \rangle]$ it is countable, being singular and the limit of countable ordinals. This construction evidently will continue for $\theta = \omega_1^{L[b]}$ steps. But if $\theta < \omega_1$, we shall have $\theta \in C$, and so is singular in L, contradicting its regularity in $L[b]$.

The sentence $\exists b(\omega_1 = \omega_1^{L[b]})$ is Σ^1_4: it says that

$$\exists b \forall x\ (\ x\ \texttt{codes an ordinal}\ \longrightarrow$$

$$\exists y (y \in L[b]\ \texttt{and}\ y\ \texttt{codes the same ordinal as}\ x)).$$

Hence, this sentence is true in the ground model, contrary to hypothesis. □

R. Bosch has shown ([B-B]) that if G is generic over V for the Levy collapse $Coll(\omega, < \kappa)$, κ a Mahlo cardinal, then $V[G]$ is absolute for all predicates definable from reals and ordinals, under σ-centered posets. Therefore, $\mathbf{\Sigma^1_4}$-absoluteness for σ-centered posets is equiconsistent with the existence of a Mahlo cardinal.

A weakly-compact cardinal

By allowing all ccc posets, ω_1 becomes a weakly-compact cardinal in L.

Theorem 7 *The following are equiconsistent:*

1. $\mathbf{\Sigma^1_4}$*-absoluteness with respect to ccc forcing extensions.*
2. *There exists a weakly compact cardinal.*

Proof: 1 ⇒ 2: We already know that ω_1 must be inaccessible in L. If it is not weakly-compact, then in L there is an Aronszajn tree T on ω_1 such that for every model M of ZFC, if $M \models$ "T has a branch of length ω_1^V", then $M \models$ "$cf(\omega_1^V) = \omega$" (see [D]). For every sequence $\langle d_\alpha : \alpha < \omega_1 \rangle$ of distinct reals, there is a ccc poset for coding the sequence along the levels of T (see [H-S]). i.e., there is a ccc poset such that if G is generic for this poset over V, then in $V[G]$ there is a real c such that $\langle d_\alpha : \alpha < \omega_1 \rangle \in L[T, c]$. Since $T \in L$,

$$V[G] \models L[c]\ \texttt{has uncountably many reals}$$

But the sentence

$$\exists x \in \omega^\omega (L[x]\ \texttt{has uncountably many reals})$$

is Σ^1_4. So, by Σ^1_4-absoluteness it holds in V, contradicting the inaccessibility of ω_1 to reals in V.

2 ⇒ 1: This direction follows from a result of Kunen (see [H-S]) which states that if κ is weakly-compact and G is $Coll(\omega, < \kappa)$-generic over V, then the $L(\mathbb{R})$ of $V[G]$ is an elementary substructure of the $L(\mathbb{R})$ of any ccc forcing extension of $V[G]$. □

We finish with the following result, independently observed by K. Hauser, which corrects a claim from [F-M-W]:

Theorem 8 *The following are equiconsistent:*

1. *$\mathbf{\Sigma^1_4}$-absoluteness for set forcing.*

2. *Every set has a sharp and there exists a reflecting cardinal.*

Proof: Assume $\mathbf{\Sigma^1_4}$ absoluteness for set forcing and suppose that some set x does not have a sharp. Then for some singular cardinal κ, $x \in H(\kappa), \kappa^+ = (\kappa^+)^{L[x]}$ and hence $H(\kappa^+)$ can be coded into $L[R]$ for some real R using a set forcing. (The only need for class forcing is to reshape; however by our hypothesis on x, any subset of κ^+ is reshaped in $L[x]$.) So the Σ^1_4 sentence:

`For some real` R`, every real is constructible from` R

is true in a set-generic extension and hence true in V. This contradicts $\mathbf{\Sigma^1_3}$-absoluteness for Cohen forcing.

As every set has a sharp, we have Martin-Solovay absoluteness and therefore every set-generic extension of V is $\mathbf{\Sigma^1_3}$ absolute for set forcing. Now the proof that ω_1 is reflecting in L assuming $\mathbf{\Sigma^1_3}$ absoluteness for set forcing (Theorem 3) shows that ω_1 is reflecting in the least inner model closed under #'s, assuming $\mathbf{\Sigma^1_4}$-absoluteness for set forcing.

Conversely, if V is closed under #'s for sets and κ is reflecting, then as in the proof that a reflecting cardinal gives $\mathbf{\Sigma^1_3}$ absoluteness for set forcing (Theorem 3), Levy collapsing κ to ω_1 (via $Coll(\omega, < \kappa)$) yields $\mathbf{\Sigma^1_4}$ absoluteness for set forcing. $\square$

Remark. The above argument shows that $\mathbf{\Sigma^1_4}$-absoluteness for ω_1-preserving set forcings implies that every set has a #. In addition, it shows that the following are equiconsistent: (a) Σ^1_4 absoluteness for set forcing + ω_1 is inaccessible to reals; (b) Every set has a sharp (see the Remark at the end of Theorem 3).

Open questions

1. What is the consistency strength of $\mathbf{\Sigma^1_3}$ absoluteness for set-forcing notions that preserve ω_1? that preserve stationary subsets of ω_1? that are proper?

2. What is the consistency strength of $\mathbf{\Sigma^1_4}$ absoluteness for set forcings that preserve stationary subsets of ω_1? that are proper?

3. Is Σ^1_3 absoluteness for class forcing consistent? By [F], it implies the existence of $0^\#$.

4. What is the consistency strength of $\mathbf{\Sigma^1_n}$ absoluteness for set forcing when n is greater than 4?

   It is shown in [H] that $\mathbf{\Sigma^1_n}$-absoluteness for all n is equiconsistent with the existence of ω strong cardinals.

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