PERFECT TREES AND LARGE CARDINALS

κ is measurable iff
there is $j : V \to M$ with critical point κ

κ is λ-hypermeasurable iff in addition
$H(\lambda) \subseteq M$

κ is λ-supercompact iff in addition
$M^\lambda \subseteq M$

(Measurable = κ^+-hypermeasurable =
κ-supercompact.)

Question: Suppose κ is a large cardinal and G is P-generic over V. Is κ still a large cardinal in $V[G]$?

Lifting method (Silver):

Given $j : V \to M$ and P-generic G over V.

Let P^* be $j(P)$.

Find P^*-generic G^* over M s.t. $j[G] \subseteq G^*$.

Then $j : V \to M$ lifts to $j^* : V[G] \to M[G^*]$.

If G^* belongs to $V[G]$ then j^* is $V[G]$-definable, so κ is still measurable (and maybe more) in $V[G]$.

Singular cardinal hypothesis

SCH: The GCH holds at singular, strong limit cardinals

Prikry: Con(GCH fails at a measurable) $\rightarrow$ Con(not SCH)

Silver: Con(κ is κ^{++}-supercompact) $\rightarrow$ Con(GCH fails at a measurable)

Easy fact: GCH fails at measurable κ $\rightarrow$ GCH fails at measure-one $\alpha < \kappa$.

So for Silver's theorem, must violate GCH not only at κ, but also below κ.

Silver's strategy: Iterated Cohen forcing

Cohen$(\alpha, \alpha^{++}) = \alpha^{++}$-product of α-Cohen forcing (with supports of size $< \alpha$)

P_0 is trivial
$P_{\alpha+1} = P_\alpha * \text{Cohen}(\alpha, \alpha^{++})$, α inaccessible
$P_{\alpha+1} = P_\alpha$, otherwise
Inverse limits at singular ordinals, direct limits otherwise

$P =$ Direct limit of P_α, $\alpha \in$ Ord.

P preserves cofinalities and forces not GCH at each inaccessible.

Assume GCH in V.
Let $j : V \to M$ witness κ^{++}-supercompactness.
Let G be P-generic.
Want generic G^* for $P^* = j(P)$, $j[G] \subseteq G^*$.

Write $P^* = P^*(< j(\kappa)) * P^*(j(\kappa)) * P^*(> j(\kappa))$.

1. (Below $j(\kappa)$) Easy to build generic $G^*(< j(\kappa))$ containing $j[G(< \kappa)] = G(< \kappa)$.

2. (At $j(\kappa)$, key step) Using supercompactness, the conditions in $j[G(\kappa)] \subseteq P^*(j(\kappa))$ have a common lower bound *(master condition)* p. Choose $G^*(j(\kappa))$ to include p.

3. (Above $j(\kappa)$) Using distributivity of $P(> \kappa)$, easy to show that $j[G(> \kappa)]$ *generates* a generic $G^*(> j(\kappa))$.

So $G^* = G^*(< j(\kappa)) * G^*(j(\kappa)) * G^*(> j(\kappa))$ contains $j[G]$, as desired.

Woodin: Can replace κ^{++}-supercompactness with κ^{++}-hyperstrength in the Silver strategy.

Subtle argument:
Derived measure: Use both $j : V \to M$ and its derived measure embedding $j_0 : V \to M_0$.
Leaving the universe: Force a generic $G_0^*(j_0(\kappa))$ over $V[G]$. κ is measurable in $V[G][G_0^*(j_0(\kappa))]$.
Generic modification: Use $G_0^*(j_0(\kappa))$ to obtain a generic $G^{*'}(j(\kappa))$ for $P^*(j(\kappa))$, which must be modified to get the desired generic $G^*(j(\kappa))$.

A new strategy: Iterated Sacks forcing

Let α be inaccessible.

α-Sacks: α-closed, binary trees of height α, with CUB-many splitting levels.

In the Silver strategy, replace Cohen(α, α^{++}) by Sacks(α, α^{++}), the α^{++}-product of α-Sacks (with supports of size α).

Assume GCH in V.
Let $j : V \to M$ witness κ^{++}-hypermeasurability.
Let G be generic for $P =$ iterated Sacks(α, α^{++}).
Let $P^* = j(P)$.
We want a P^*-generic G^* s.t. $j[G] \subseteq G^*$.

The construction of G^* is now easy.
Do not need the derived measure, leaving the universe or generic modification.

α-Sacks has a weak form of α^+-closure called α-*fusion*:

Write $S \leq_i T$ iff $S \leq T$ and S has the same i-th splitting level as T. Then any sequence $T_0 \geq_0 T_1 \geq_1 T_2 \geq_2 \cdots$ of length α has a lower bound.

α-Sacks is α-closed and α^{++}-cc.
α-fusion implies that α^+ is preserved.

If G is α-Sacks generic then $G = \{T \mid f \in [T]\}$ for some unique $f : \alpha \to 2$. We also say that f is α-Sacks generic.

Tuning fork lemma (F - Katie Thompson) Suppose $j : V \to M$ with critical point κ and G is κ-Sacks generic. Then the intersection of the trees in $j[G]$ consists of exactly two $f_0, f_1 : j(\kappa) \to 2$, which agree below κ and disagree at κ. Moreover each f_i is $j(\kappa)$-Sacks generic over M.

Reason: The splitting levels of $j(T)$, $T \in G$, form CUB subsets $j(C)$ of $j(\kappa)$. The intersection of the $j(C)$'s is $\{\kappa\}$. (We assume that j is given by an extender ultrapower.)

There is a version of the Tuning Fork Lemma for Sacks(κ, κ^{++}), giving:

Theorem 1. (F - Thompson) Assume GCH. Suppose $j : V \to M$ witnesses that κ is κ^{++}-hypermeasurable and G is generic for the iteration of Sacks(α, α^{++}), α inaccessible. Then j lifts to $j^* : V[G] \to M[G^*]$, witnessing the failure of GCH at the measurable cardinal κ.

Using a result of Gitik, we also get:

Con($o(\kappa) = \kappa^{++}$) $\leftrightarrow$
Con(GCH fails at a measurable)

The Tree Property and Large Cardinals

κ-Aronszajn tree = κ-tree with no κ-branch

TP(κ): There is no κ-Aronszajn tree.

GCH holds at $\kappa \rightarrow$ TP(κ^{++}) fails

Question: What is the consistency strength of TP(κ^{++}), κ measurable?

Lemma (F - Natasha Dobrinen) Assume GCH, κ is regular, λ is weakly compact, $\kappa < \lambda$ and G is generic for Sacksit(κ, λ) = the λ-iteration of κ-Sacks (with supports of size κ). Then in $V[G]$, $\lambda = \kappa^{++}$ and TP(κ^{++}) holds.

Using a version of the Tuning Fork Lemma, we get:

Theorem 2. (F - Dobrinen) Assume GCH and $j : V \to M$ witnesses that κ is λ-hypermeasurable, where λ is weakly compact and greater than κ. Let G be generic for the iteration of Sacksit(α, λ_α), α an inaccessible limit of weakly compacts, λ_α the least weakly compact above α. Then in $V[G]$, κ is measurable and TP(κ^{++}) holds.

The upper bound given by Theorem 2 is nearly optimal:

Con(κ is weakly compact hypermeasurable) $\to$
Con(TP(κ^{++}), κ measurable) $\to$
Con(κ is $<$ weakly compact hypermeasurable)

Easton's theorem and large cardinals

Easton: Con(GCH fails at all regulars)

Question: What is the consistency strength of GCH fails at all regulars and there is a measurable cardinal?

We saw:
Con(κ^{++}-hypermeasurable) $\rightarrow$
Con(GCH fails at a measurable)

The same proof yields:
Con(κ^{++}-hypermeasurable) $\rightarrow$
Con(GCH fails at all regulars *except* at α^{+}, α^{++} when α is inaccessible)

Using Sacks(α, α^{++}) at inaccessibles and Cohen(α, α^{++}) elsewhere, one gets:

Theorem 3. (F - Radek Honzík) Assume GCH. There is a forcing P such that if G is P-generic then GCH fails at all regulars in $V[G]$. Moreover, if κ is κ^{++}-hypermeasurable in V, then κ remains measurable in $V[G]$.

One can also replace κ^{++}-hypermeasurable by $o(\kappa) = \kappa^{++}$, the optimal hypothesis.

Global Domination

So far: *Large cardinal preservation*

Now: *Internal consistency*

φ is *internally consistent* iff φ holds in an inner model (assuming large cardinals).

$\text{ICon}(\varphi)$ = φ is internally consistent.

Consistency result:
Con(ZFC + large cardinals) $\rightarrow$ Con(ZFC $+\varphi$)

Internal consistency result:
ICon(ZFC + large cardinals) $\rightarrow$ ICon(ZFC $+\varphi$)

Examples:

(a) (Easton) Con(ZFC) $\rightarrow$
Con(ZFC + GCH fails at all regulars)
(b) (F - Ondrejović) ICon(ZFC + $0^{\#}$ exists) $\rightarrow$ ICon(ZFC + GCH fails at all regulars)

(F - Dobrinen)
(a) Con(ZFC + proper class of ω_1-Erdős cards) $\rightarrow$ Con(ZFC + Global costat of ground model)
(b) ICon(ZFC + ω_1-Erdős hyperstrong with a sufficiently large measurable above) $\rightarrow$ ICon(ZFC + Global costat of ground model)

(a) Con(ZFC) $\rightarrow$ Con(ZFC + no L-inaccessible)
(b) $\sim$ ICon(ZFC + no L-inaccessible)

Internal consistency strength: What large cardinals are needed to prove ICon(φ)?

An application of perfect trees to internal consistency strength:

$d(\kappa) =$ dominating number for $f : \kappa \to \kappa$

$\kappa < d(\kappa) \leq 2^\kappa$

Global Domination: $d(\kappa) < 2^\kappa$ for all κ.

Cummings-Shelah: Con(ZFC) $\to$
Con(ZFC + Global Domination)

Proof uses Cohen$(\alpha, \alpha^{++}) *$ Hechlerit(α, α^{+}) for all regular α and gives:

ICon(ZFC + κ^{+}-supercompact +
measurable above) $\to$
ICon(ZFC + Global Domination)

Replacing Cohen$(\alpha, \alpha^{++}) *$ Hechlerit(α, α^{+}) with Sacks(α, α^{++}) for inaccessible α gives:

(F - Thompson)
ICon(ZFC + $0^{\#}$ exists) $\rightarrow$
ICon(ZFC + Global Domination *except* at α^{+}, α inaccessible)

And with Cohen(α^{+}, α^{+++}) followed by an interlacing of Hechlerit(α^{+}, α^{++}) with Sacksit(α, α^{++}) for inaccessible α, we get:

Theorem 4. (F - Thompson)
ICon(ZFC + $0^{\#}$ exists) $\rightarrow$
ICon(ZFC + Global Domination)

Conclusion

For large cardinal preservation and internal consistency, Sacks is better than Cohen!