BPFA and Inner Models

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August 26, 2010

BPFA denotes the Bounded Proper Forcing Axiom. By a result due independently to Bagaria [1] and Stavi-Väänänen [7], this axiom is equivalent to the Σ_1 elementarity of $H(\omega_2)$ in the $H(\omega_2)$ of all proper forcing extensions. Goldstern-Shelah [4] proved that BPFA is equiconsistent with the existence of a "Σ_1 reflecting cardinal". In accordance with current terminology, we use the term "reflecting cardinal", i.e. a regular cardinal κ such that V_κ is Σ_2 elementary in V (this is equivalent to Goldstern-Shelah's definition of Σ_1 reflecting cardinal). Such a large cardinal hypothesis is not very strong; indeed if κ is reflecting then it is also reflecting in L.

In this paper we prove the following:

Theorem 1 *The following are equiconsistent:*
(a) There is a reflecting cardinal.
(b) BPFA holds and there is an inner model M not containing all reals with the correct ω_2 (i.e., such that ω_2 equals ω_2^M).

By "inner model" we mean a transitive class satisfying ZFC which is definable with parameters.

*The author is honoured to have been invited to contribute to this journal issue in honour of Professor Kakuda, who has been so important for the development of set theory in Japan. He also wishes to thank the Austrian Science Fund (FWF) for its generous support through Project Number P 19375-N18.

This theorem answers a question of Caicedo and Velickovic (see [2]), who proved that assuming BPFA, any inner model M satisfying BPFA with the correct ω_2 must contain all subsets of ω_1.

Perhaps more interesting than the result itself is its proof, which combines the methods of [3] with a new method of "diagonal iteration". The paper [3] provided a technique for adding closed unbounded subsets to ω_2 with finite conditions, using countable models as side conditions. (This technique was later independently discovered and applied by Mitchell [6].) The present proof uses this technique to convert a reflecting cardinal into ω_2 using finite conditions, and follows this with a variant, based on diagonal iteration, of the Goldstern-Shelah construction [4] of a model of BPFA.

We begin the proof. Our ground model is L, where κ is reflecting. Fix a Σ_1 definable coding $\# : [\kappa]^{\aleph_0} \to \kappa$ of countable subsets of κ by ordinals less than κ. (For example, we can take $\#(x)$ to be the rank of x in the canonical wellorder of L.) The forcing P consists of all pairs $p = (A, S)$, where:

1. A is a finite set of pairwise disjoint closed intervals $[\alpha, \beta]$ whose "left" endpoint α is an ordinal less than κ which is either a successor ordinal or a cardinal of cofinality at most ω_1. (We allow the one-point intervals $[\alpha, \alpha]$.) Let L_A denote the set of left endpoints of intervals in A.

2. S is a finite collection of countable x of the form $M \cap \kappa$ where M is Σ_1 elementary in L.

3. For each interval $I = [\alpha, \beta]$ in A and each $x \in S$:
 3a. If I intersects x then the endpoints of I belong to x.
 3b. If $I = [\alpha, \beta]$ does not intersect x and $\alpha < \sup(x)$ then α_x belongs to L_A, where α_x is the least element of x greater than α.

4. Let F_A be the set of all elements of L_A of cofinality ω_1, together with κ. For $x \in S$, the F_A-*height of* x is the least element of F_A greater than $\sup(x)$.
 4a. If x belongs to S and α belongs to F_A then $x \cap \alpha$ belongs to S.
 4b. Suppose that $x, y \in S$ have the same F_A-height. Then $\#(x) \in y$, $\#(y) \in x$ or $x = y$.

(A^*, S^*) extends (A, S) iff A^* contains A and S^* contains S. The following is proved just as in [3].

Lemma 2 *(1) If G is P-generic then the union of the L_A for (A, S) in G forms a club $C(G)$ in κ and the union of the S for (A, S) in G forms a stationary subset of $[\kappa]^{\aleph_0}$ in $L[G]$. P forces that κ equals ω_2.*
(2) P is proper. Indeed, in any universe in which M is sufficiently elementary[1] and contains κ as an element, any condition (A, S) with $M \cap \kappa \in S$ is (M, P)-generic.

P also satisfies the following factoring property: For any condition $p = (A, S)$ in P and α in $F_A \cap \kappa$ let $p(< \alpha)$ denote $(A(< \alpha), S(< \alpha))$ where $A(< \alpha) = \{I \in A \mid I \subseteq \alpha\}$ and $S(< \alpha) = \{x \cap \alpha \mid x \in S\}$. Also let $p[\alpha, \kappa)$ denote $(A[\alpha, \kappa), S[\alpha, \kappa))$ where $A[\alpha, \kappa) = \{I \in A \mid I \subseteq [\alpha, \kappa)\}$ and $S[\alpha, \kappa) = \{x \cap [\alpha, \kappa) \mid x \in S\}$. Then P below p factors as $(P(< \alpha)$ below $p(< \alpha)) \times (P[\alpha, \kappa)$ below $p[\alpha, \kappa))$ where $P(< \alpha)$ consists of all $q(< \alpha)$, $q \in P$ and $P[\alpha, \kappa)$ consists of all $q[\alpha, \kappa)$, $q \in P$, ordered in the obvious way. Moreover, $P[\alpha, \kappa)$ is proper, and indeed in any universe in which M is sufficiently elementary and contains α, κ as elements, any condition (A, S) with $M \cap [\alpha, \kappa) \in S$ is $(M, P[\alpha, \kappa))$-generic.

We will also need the following consequence of the factoring properties of P. Recall that for an inner model N and a forcing Q, we say that Q is *N-proper* iff for some parameter x and all sufficiently elementary countable M, if Q and x belong to M and $M \cap N$ belongs to N then every condition in $Q \cap M$ can be extended to a condition in Q which is (M, Q)-generic.

Lemma 3 *Suppose that G is P-generic. Let α be an element of $C(G)$ of uncountable cofinality. Suppose that $M[G]$ is sufficiently elementary in $L[G]$, M belongs to L and α, κ belong to M. Let R be a forcing in $L[G(< \alpha)]$ and r a condition in R which is $(M[G(< \alpha)], R)$-generic. Then r is also $(M[G], R)$-generic. In particular, if R is L-proper in $L[G(< \alpha)]$ then R is also L-proper in $L[G]$.*

Proof. It suffices to show that if r is $(M[G(< \alpha)], R)$-generic and H is R-generic over $L[G]$ containing r then $M[G][H]$ is Σ_1 elementary in $L[G][H]$, for this implies that $D \cap M[G][H]$ is predense below r for each dense D in $M[G][H]$. By hypothesis, $M[G(< \alpha)][H]$ is sufficiently elementary in $L[G(<$

[1]Here, and throughout the paper, "sufficiently elementary" can be taken to be "Σ_3 elementary".

$\alpha)][H]$. Now $M[G][H]$, $L[G][H]$ factor as $M[G(<\alpha)][H][G[\alpha,\kappa)]$, $L[G(<\alpha)][H][G[\alpha,\kappa)]$, respectively. By Lemma 2, $M[G(<\alpha)][H]$ is the union of sufficiently elementary $M_0[G(<\alpha)][H]$ where $M_0 \cap [\alpha,\kappa)$ belongs to S_0 for some (A_0, S_0) in $G[\alpha,\kappa)$; moreover, (A_0, S_0) is $(M_0[G(<\alpha)][H], P[\alpha,\kappa))$-generic. It follows that $M_0[G(<\alpha)][H][G[\alpha,\kappa)]$ is sufficiently elementary in $L[G(<\alpha)][H][G[\alpha,\kappa)]$ for such M_0 and therefore $M[G(<\alpha)][H][G[\alpha,\kappa)] = M[G][H]$ is Σ_1 elementary in $L[G(<\alpha)][H][G[\alpha,\kappa)] = L[G][H]$, as desired. $\square$

Fix a P-generic G. We want to use an iteration over $L[G]$ of length ω_2 which like the Goldstern-Shelah iteration forces BPFA, but which also preserves ω_2. The difficulty with the usual countable support iteration is that it allows all reals of $L[G]$, of which there are ω_2 many, to be coded into the first ω_1-many components of the generic, resulting in a collpase of ω_2. So instead we perform the following countable support "diagonal iteration" Q. The iteration Q will be L-proper (but with "diagonal support").

Let C denote $C(G)$, the generic club added by G. Also let D be the set of α in C of uncountable cofinality such that L_α is Σ_2 elementary in L and $\bar{D} \subseteq \kappa$ the closure of D together with $\{0\}$. For α in $\bar{D}$ we let $\alpha^+_{\bar{D}}$ denote the least element of $\bar{D}$ greater than α and set $\alpha^* = \alpha^+_{\bar{D}}$ unless α is a limit point of D of uncountable cofinality, in which case α^* equals α.

By induction on α in $\bar{D}$ we define the forcing Q_α in $L[G(<\alpha^*)]$ as follows:

Q_0 is trivial.

Suppose that Q_α is defined and belongs to $L[G(<\alpha^*)]$. If α is not a limit point of D of uncountable cofinality then $\alpha^* = \alpha^+_{\bar{D}}$ and we define Q_{α^*} to be the $Q_\alpha * \dot{Q}(\alpha)$ of $L[G(<\alpha^{**})]$, where $\dot{Q}(\alpha)$ is a Q_α-name for the trivial forcing. If $\alpha<\kappa$ is a limit point of D of uncountable cofinality then we define $Q_{\alpha^+_{\bar{D}}}$ to be the $Q_\alpha * \dot{Q}(\alpha)$ of $L[G(<(\alpha^+_{\bar{D}})^+_{\bar{D}})]$ where $\dot{Q}(\alpha)$ is a Q_α-name for the sum of all L-proper forcings in the $H(\alpha^+_{\bar{D}})$ of $L[G(<\alpha)]^{Q_\alpha}$.

For α a limit point of D (including κ) we take Q_α to be the direct limit of the Q_β for β in $\bar{D}\cap\alpha$ if α has uncountable cofinality and otherwise to be the inverse limit of the Q_β for β in $\bar{D}\cap\alpha$, taken in $L[G(<\alpha^*)]$.

Q is $Q_\kappa = Q_{\omega_2}$.

Lemma 4 *The forcing Q is L-proper in $L[G]$.*

Proof. We prove the following statement by induction on α in $\bar{D} \cup \{\kappa\}$:

$(*)$ Let $M[G]$ be a countable sufficiently elementary submodel of $L[G]$ where M belongs to L and κ belongs to M. Let $\gamma < \alpha$ belong to $M \cap \bar{D}$ and let q_γ be $(M[G(<\gamma^*)], Q_\gamma)$-generic. Also assume that $\dot{q}$ is a Q_γ-name in $L[G(<\gamma^*)]$ which is forced by q_γ to denote an element q of $Q_\alpha \cap M[G(<\alpha^*)]$ such that $q \restriction \gamma$ belongs to the Q_γ-generic $\dot{H}_\gamma$. Then there is an $(M[G(<\alpha^*)], Q_\alpha)$-generic condition q_α such that $q_\alpha \restriction \gamma$ equals q_γ and q_α forces that $\dot{q}$ belongs to the Q_α-generic $\dot{H}_\alpha$.

Note that any sufficiently elementary N in $L[G]$ which contains the parameter G as an element is of the form $M[G]$ where $M = N \cap L$. The lemma therefore follows from the special case of $(*)$ where $(\gamma, \alpha) = (0, \kappa)$, as it produces an $(M[G], Q)$-generic condition below any given condition in $Q \cap M[G]$.

$(*)$ is vacuous for $\alpha = 0$.

Suppose that $\alpha = \beta^+_{\bar{D}}$, β in $\bar{D}$. We first treat the case where γ equals β. Thus we are given q_β and $\dot{q}$ and we are looking for a Q_β-name $\dot{q}(\beta)$ in $L[G(<\alpha^*)]$ such that $q_\beta * \dot{q}(\beta)$ is $(M[G(<\alpha^*)], Q_\alpha)$-generic and forces $\dot{q}$ to belong to the Q_α-generic $\dot{H}_\alpha$. To describe $\dot{q}(\beta)$ fix a Q_β-generic H_β over $L[G(<\alpha^*)]$ and we specify the condition $\dot{q}(\beta)^{H_\beta} = q(\beta)$ in $\dot{Q}(\beta)^{H_\beta}$. If q_β does not belong to H_β then $q(\beta)$ is the trivial condition. Otherwise $\dot{q}^{H_\beta}$ is a condition in $M[G(<\alpha^*)] \cap (Q_\beta * \dot{Q}(\beta))$ whose restriction to β belongs to H_β. Write this condition as $(r_\beta, \dot{r}(\beta))$. As $M[G(<\alpha^*)]$ is sufficiently elementary in $L[G(<\alpha^*)]$ and H_β is a Q_β-generic over $L[G(<\alpha^*)]$ containing the $(M[G(<\beta^*)], Q_\beta)$-generic (and therefore by Lemma 3 $(M[G(<\alpha^*)], Q_\beta)$-generic) condition q_β, it follows that $M[G(<\alpha^*)][H_\beta]$ is sufficiently elementary in $L[G(<\alpha^*)][H_\beta]$. Moreover $\dot{Q}(\beta)^{H_\beta}$ is L-proper in $L[G(<\alpha^*)]$. As $\dot{r}(\beta)^{H_\beta}$ belongs to $M[G(<\alpha^*)][H_\beta]$ it follows that there is an $(M[G(<\alpha^*)][H_\beta], \dot{Q}(\beta)^{H_\beta})$-generic condition $q(\beta)$ extending it. This completes the description of the Q_β-name $\dot{q}(\beta)$. We claim that $q_\beta * \dot{q}(\beta)$ is $(M[G(<\alpha^*)], Q_\beta * \dot{Q}(\beta))$-generic. Indeed, if $H_\beta * H(\beta)$ is $Q_\beta * \dot{Q}(\beta)$-generic below $q_\beta * \dot{q}(\beta)$ then as q_β is $(M[G(<\alpha^*)], Q_\beta)$-generic we have that $M[G(<\alpha^*)][H_\beta]$ is sufficiently elementary in $L[G(<\alpha^*)][H_\beta]$; as

$\dot{q}(\beta)^{H_\beta}$ is $(M[G(<\alpha^*)][H_\beta], \dot{Q}(\beta)^{H_\beta})$-generic we get $M[G(<\alpha^*)][H_\beta][H(\beta)]$ sufficiently elementary in $L[G(<\alpha^*)][H_\beta][H(\beta)]$, as desired. Finally, we claim that $q_\beta * \dot{q}(\beta)$ forces $\dot{q}$ to belong to the Q_α-generic. Indeed, if $H_\beta * H(\beta)$ is $Q_\beta * \dot{Q}(\beta)$-generic containing $q_\beta * \dot{q}(\beta)$ then by hypothesis the restriction of $\dot{q}^{H_\beta}$ to β belongs to H_β and as $\dot{q}(\beta)^{H_\beta}$ was chosen to extend $\dot{q}^{H_\beta}$ it follows that $\dot{q}^{H_\beta}$ also belongs $H(\beta)$; so $\dot{q}^{H_\beta}$ belongs to $H_\beta * H(\beta)$, as desired.

Now suppose that γ is less than β. Then we first apply induction to get an $(M[G(<\beta^*)], Q_\beta)$-generic condition q_β such that q_β restricted to γ equals q_γ and q_β forces the restriction of $\dot{q}$ to β to belong to the Q_β-generic. Then apply the previous case to q_β and the Q_γ-name $\dot{q}$ (which can also be viewed as a Q_β-name) to obtain an $(M[G(<\alpha^*)], Q_\alpha)$-generic q_α whose restriction to β is q_β (and therefore whose restriction to γ is q_γ) and which forces $\dot{q}$ to belong to the Q_α-generic, as desired.

Suppose now that α is a limit point of $\bar{D}$. Let β be the supremum of $M \cap \alpha$. Note that as M belongs to L and $M[G(<\alpha^*)]$ is sufficiently elementary in $L[G(<\alpha^*)]$ it follows that $M[G(<\alpha^*)]$ belongs to $L[G(<\beta^*)]$. Choose $\gamma_0 < \gamma_1 < \cdots$ cofinal in β with $\gamma_0 = \gamma$ and each γ_{n+1} in D. Also let $(D_n \mid n \in \omega)$ be an enumeration of all dense subsets of Q_α which belong to $M[G(<\alpha^*)]$; we may choose the sequence of γ_n's in L and the sequence of D_n's in $L[G(<\alpha^*)]$.

We construct the desired $(M[G(<\alpha^*)], Q_\alpha)$-generic condition q_α as the limit of conditions q_{γ_n} in Q_{γ_n} where $q_{\gamma_{n+1}}$ restricted to γ_n is q_{γ_n} and q_{γ_n} is $(M[G(<\gamma_n^*)], Q_{\gamma_n})$-generic. Together with the q_{γ_n}'s we construct Q_{γ_n}-names $\dot{q}_n$ such that for each n, q_{γ_n} forces that $\dot{q}_n$ belongs to the Q_α-generic, that $\dot{q}_n$ belongs to $M[G(<\alpha^*)]$, that $\dot{q}_n$ extends $\dot{q}_{n-1}$, that $\dot{q}_n$ belongs to D_n (for $n > 0$) and that the restriction of $\dot{q}_n$ to γ_n belongs to the Q_{γ_n}-generic.

Set q_{γ_0} equal to the given condition q_γ and $\dot{q}_0$ equal to $\dot{q}$. Given q_{γ_n} and $\dot{q}_n$ we define $q_{\gamma_{n+1}}$ and $\dot{q}_{n+1}$ as follows: Let H_{γ_n} be a Q_{γ_n}-generic containing q_{γ_n} and let q_n be $\dot{q}_n^{H_{\gamma_n}}$. Then q_n belongs to $Q_\alpha \cap M[G(<\alpha^*)]$ and the restriction of q_n to γ_n belongs to H_{γ_n}. As q_{γ_n} is $(M[G(<\gamma_n^*)], Q_{\gamma_n})$-generic and D_n belongs to $M[G(<\alpha^*)]$ we can find q_{n+1} below q_n in $D_n \cap M[G(<\alpha^*)]$ whose restriction to γ_n belongs to H_{γ_n}. This describes a Q_{γ_n}-name $\dot{q}_n'$. Now apply induction to q_{γ_n} and ($\dot{q}_n'$ restricted to γ_{n+1}) to obtain $q_{\gamma_{n+1}}$ whose restriction to γ_n is

q_{γ_n} and which forces $\dot{q}'_n$ restricted to γ_{n+1} to belong to the $Q_{\gamma_{n+1}}$-generic. Finally, set $\dot{q}_{n+1}$ to be the $Q_{\gamma_{n+1}}$-name which is forced by $q_{\gamma_{n+1}}$ to equal the Q_{γ_n}-name $\dot{q}'_n$.

Let q be the limit of the q_{γ_n}'s. Then q belongs to Q_α as the sequence of q_{γ_n}'s belongs to $L[G(< \beta^*)]$. And the restriction of q to γ is q_γ. We claim that for each n, q forces that $\dot{q}_n$ belongs to the Q_α-generic. For suppose that H_α is Q_α-generic and contains q. Let q_n be $\dot{q}_n^{H_\alpha}$. Then q_n belongs to $M[G(< \alpha^*)]$ and q_n restricted to γ_k belongs to the Q_{γ_k}-generic for all $k \geq n$. Thus q_n restricted to β belongs to the Q_β-generic, again using the fact that $M[G(< \alpha^*)]$ belongs to $L[G(< \beta^*)]$. As the support of q_n is contained in β it follows that q_n belongs to H_α.

Thus q forces that $\dot{q}_n$ belongs to the Q_α-generic for each n. This implies that q forces $\dot{q}$ to belong to the Q_α-generic and that q is $(M[G(< \alpha^*)], Q_\alpha)$-generic, because q forces that $\dot{q}_n$ belongs to $D_{n-1} \cap M[G(< \alpha^*)]$ for $n > 0$. □

The above argument shows that $Q[\alpha, \kappa)$ is L-proper in $L[G][H_\alpha]$ for any α in $\bar{D}$ and Q_α-generic H_α.

Lemma 5 *The forcing Q preserves ω_2.*

Proof. It suffices to show that Q is ω_2-cc in $L[G]$. We can think of Q as a subset of $L_{\omega_2}[G]$ (it has a dense subset contained in that model). Now suppose that X is a maximal antichain in Q. By reflection, $X \cap L_\alpha[G]$ is a maximal antichain in $Q \cap L_\alpha[G]$ for a club of α's in ω_2. Choose such an α in $\bar{D}$ of uncountable cofinality; then by virtue of countable support, $Q \cap L_\alpha[G]$ equals Q_α and therefore $X \cap Q_\alpha$ is a maximal antichain in Q_α. But again by countable support, if q is any condition in Q, $q \restriction \alpha$ = (q restricted to α) is trivial on a final segment of α and therefore equivalent to a condition in Q_α. It follows that $q \restriction \alpha$ is compatible with a condition in $X \cap Q_\alpha$ and therefore so is q. We have shown that $X \cap Q_\alpha$ is a maximal antichain in Q and therefore equals X. □

Lemma 6 *Q forces BPFA. In fact, Q forces the bounded forcing axiom for L-proper forcings.*

Proof. Let H be Q-generic over $L[G]$. We want to show that the bounded forcing axiom for L-proper forcings holds in $L[G][H]$. Suppose that φ is a Σ_1 fact with parameter A, where A is a subset of ω_1 and φ holds in an L-proper forcing extension of $L[G][H]$. As $P*Q$ has (a dense subset of) size ω_2 and ω_2 is preserved, we can choose α in $\bar{D}$ of uncountable cofinality such that A belongs to $L[G(<\alpha), H_\alpha]$. Now $L[G][H]$ factors as $L[G(<\alpha), H_\alpha][G[\alpha, \omega_2), H[\alpha, \omega_2)]$, and by Lemmas 3, 4, the second factor is L-proper over the first. It follows that φ holds in an L-proper forcing extension of $L[G(<\alpha), H_\alpha]$. Set $\beta = \alpha^+_{\bar{D}}$. As the forcing $P(<\alpha) * Q_\alpha$ is an element of L_β and L_β is Σ_2 elementary in L, it follows that $L_\beta[G(<\alpha), H_\alpha]$ is Σ_2 elementary in $L[G(<\alpha), H_\alpha]$ and therefore φ holds in an extension of $L[G(<\alpha), H_\alpha]$ via a proper forcing in the $H(\beta)$ of $L[G(<\alpha), H_\alpha]$, one of the forcings included in the sum $\dot{Q}(\alpha)^{H_\alpha}$. As this holds for all sufficiently large α in $\bar{D}$ of uncountable cofinality, it follows that the set of conditions which for some such α force $\dot{Q}(\alpha)^{H_\alpha}$ to be an L-proper forcing guaranteeing that φ holds in $L[G][H_\beta]$, and therefore in $L[G][H]$, is dense. By genericity there is such a condition in the generic, and therefore φ holds in $L[G][H]$, as desired. □

Proof of Theorem 1. Con (b) implies Con (a) is proved in [4]. Conversely, start with a reflecting cardinal in L and consider the model $L[G, H]$ above. Then BPFA holds there and the inner model $L[G]$ where G is P-generic over L has the correct ω_2. Clearly the generic H adds reals which are generic over the model $L[G]$ (for example, it adds Cohen reals cofinally often in the iteration) and therefore it adds reals that do not belong to $L[G]$. □

We now establish a similar result for the full PFA.

Theorem 7 *Assume the consistency of a supercompact. Then it is consistent that PFA holds and there is an inner model M not containing all reals with the correct ω_2.*

Proof. Suppose that κ is supercompact and let f be a fast function, i.e., a function $f : \kappa \to \kappa$ such that for any x and cardinal λ there is a λ-supercompactness embedding $j : V \to M$ such that x belongs to $H(j(f)(\kappa))^M$. Such a fast function can be obtained as follows: Recall that Laver [5] produced a function $g : \kappa \to H(\kappa)$ such that for any x and cardinal λ there is a λ-supercompactness embedding $j : V \to M$ such that $j(g)(\kappa) = x$; now set $f(\alpha) =$ the least cardinal β such that $g(\alpha)$ belongs to $H(\beta)$.

As before add G which turns κ into ω_2 using finite sets of closed intervals and countable side conditions. Then perform a countable support diagonal iteration Q over $V[G]$ of length κ which at stages $\alpha < \kappa$ of uncountable cofinality forces with the sum of all V-proper forcings in the $H(f(\alpha))$ of $V[G(<\alpha)]^{Q_\alpha}$. As before it follows that the iteration Q is V-proper in $V[G]$ and preserves ω_2.

Let H be Q-generic over $V[G]$. We claim that PFA, and indeed the forcing axiom for all V-proper forcings, holds in $V[G, H]$: Suppose that R in $V[G, H]$ is V-proper and let $\dot{R}$ be a name for R. Choose $j : V \to M$ with a high degree of supercompactness such that $\dot{R}$ belongs to $H(j(f)(\kappa))^M$. Note that the generic G for P can be extended to a generic G^* for the forcing $j(P)$ of M (which turns $j(\kappa)$ into ω_2) by choosing G^* to include a condition (A, S) where κ belongs to L_A. Also choose H^* so that $G * H \subseteq G^* * H^*$ and $H^*(\kappa)$ is R-generic. Then $j : V \to M$ lifts to $j^* : V[G, H] \to M[G^*, H^*]$ in the model $V[G^* * H^*]$.

Now suppose we are given a collection $\mathcal{D}$ of $\aleph_1$-many dense sets on R in $V[G, H]$. Then $H^*(\kappa)$ meets each element of $\mathcal{D}$ as it is R-generic over $M[G, H]$, and all of the dense sets in $\mathcal{D}$ in fact belong to $M[G, H]$ due to the high degree of supercompactness of $j : V \to M$. It follows that $j^*[H^*(\kappa)]$ meets every dense set in $j^*[\mathcal{D}] = j^*(\mathcal{D})$. But again by the high degree of supercompactness of j, $j^*[H^*(\kappa)]$ is an element of $M[G^*, H^*]$ and therefore by elementarity, there is a compatible subset of R in $V[G, H]$ meeting each element of $\mathcal{D}$.

Finally, note that $V[G]$ is an inner model of $V[G, H]$ with the correct ω_2 which does not contain all reals. $\square$

Remark. By a result of Velickovic ([8]), the previous theorem does not hold with PFA replaced by SPFA.

Question. Can Theorem 1 (b) hold for BSPFA or for BMM?

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