Cantor's Set Theory from a Modern Point of View

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October 3, 2002

Abstract

Georg Ferdinand Ludwig Philipp Cantor, the inventor of set theory, spent virtually his entire mathematical career in Halle and was the founder of the Deutsche Mathematische Vereinigung. At its first meeting, in Halle in 1891, this Society elected Cantor as its first president.

Cantor's work uncovered a structure to infinity which no one had foreseen, and which Cantor himself found difficult to believe. His ideas were very much opposed (especially by Poincaré and Kronecker), but fortunately also defended (by Dedekind, Weierstrass and Mittag-Leffler). His work is fundamental to our current understanding of mathematics.

Although Cantor's version of set theory led to paradox, this has been adequately circumvented through the introduction of axioms for set theory. The emerging theory ZFC (Zermelo-Fraenkel set theory with the axiom of Choice) has served as an excellent foundation for mathematics: its language is adequate to express almost all mathematical assertions and its axioms are adequate to prove almost all theorems of mathematics, once they are expressed in set-theoretic terms.

Set theory today exhibits two interconnected aspects, the Pure and the Applied.

Applied set theory refers to the many successes of set theory either in solving mathematical problems or in showing that they are unsolvable, by virtue of their being neither provable nor refutable in ZFC. For example, Cantor himself applied set theory to answer an important question in the study of Fourier series. But he was unable to resolve the Continuum Problem (concerning the cardinality of the set of real numbers), and indeed this problem was later shown to be unsolvable in ZFC by Gödel and Cohen.

Pure set theory refers to the deeper investigation of the structure of infinity. Although Cantor left us with an excellent theory of counting into the transfinite and a powerful theory of cardinality, he provided no clear picture of the universe of sets as a whole. There is now evidence that such a picture is starting to emerge through the detailed analysis of subuniverses of the universe of all sets called *inner models*.

This talk emphasizes the exciting recent developments in set theory's pure side, with some reference to the striking applications which have served as important motivation for the subject's development.

Cantor earned his doctorate in Berlin in 1867 and was appointed in Halle in 1869, where he soon after habilitated. Under the influence of Heine, his work moved to the study of trigonometric series, which led him necessarily into his investigations in set theory. In addition to developing his profound theory of transfinite numbers and cardinality, he showed

*AMS-Klassifikation: 03E15, 03E35, 03E45, 04E55. Key Words: projective sets, constructible sets, forcing, inner models, Woodin cardinals.

that the algebraic numbers are countable (and therefore almost all reals are transcendental), the reals are not countable and n-dimensional Euclidean space can be put into 1-1 correspondence with the real line. He was appointed to a chair in Halle in 1879.

Cantor founded the DMV in 1890 and was elected its first president the year after.

Cantor's ideas were revolutionary, and met significant opposition, especially from Kronecker. Mittag-Leffler persuaded Cantor to withdraw a paper that he had submitted to Acta Mathematica, on the grounds that he had submitted it "about 100 years too soon"! Fortunately Cantor also had the support of some important mathematicians, such as Dedekind.

Cantor's two key contributions were to provide us with theories of *transfinite counting* and of *infinite cardinality* .

Transfinite counting arises naturally when one considers the *Cantor derivative*: If C is a closed set of real numbers, then C' denotes the set of all limit points of C. Thus $C \supseteq C' \supseteq C'' \supseteq \cdots$. Write $C^\infty = C \cap C' \cap C' \cap \cdots$. Then $C^\infty \supseteq (C^\infty)'$, and this inclusion may be strict. Thus one must keep counting past ∞: $C^\infty \supseteq C^{\infty+1} \supseteq C^{\infty+2} \supseteq \cdots$!

The key feature of this new sequence of counting stages is that it forms a *wellordering*, a linear ordering with no infinite descending sequence of elements. Cantor showed that any two wellorderings are comparable in the sense that one is isomorphic to an initial segment of the other, and that the wellorderings can be canonically represented by special wellorderings where the ordering relation is given by the membership relation $\in$. Sets which carry such a special wellordering are called *ordinals*.

Cantor's view was that each well-defined set carries a wellordering. Thus every set can be placed in 1-1 correspondence with an ordinal. But this ordinal is *not* unique. A *cardinal* is an ordinal which cannot be put into 1-1 correspondence with a smaller ordinal. Then every set can be put into 1-1 correspondence with a *unique* cardinal, called the *cardinality* of that set.

Zermelo later justified Cantor's view by proving that every set can indeed be wellordered, using the Axiom of Choice. Thus, Cantor's theory does provide an excellent theory of cardinality for arbitrary sets. This theory left however one major gap:

The Continuum Problem: What is the cardinality of the continuum?
The Continuum Hypothesis (CH): Every uncountable set of reals has the same cardinality as the set of all reals.

Cantor failed to solve this problem, and in fact it was later discovered that this problem cannot be solved using the currently accepted axioms for set theory.

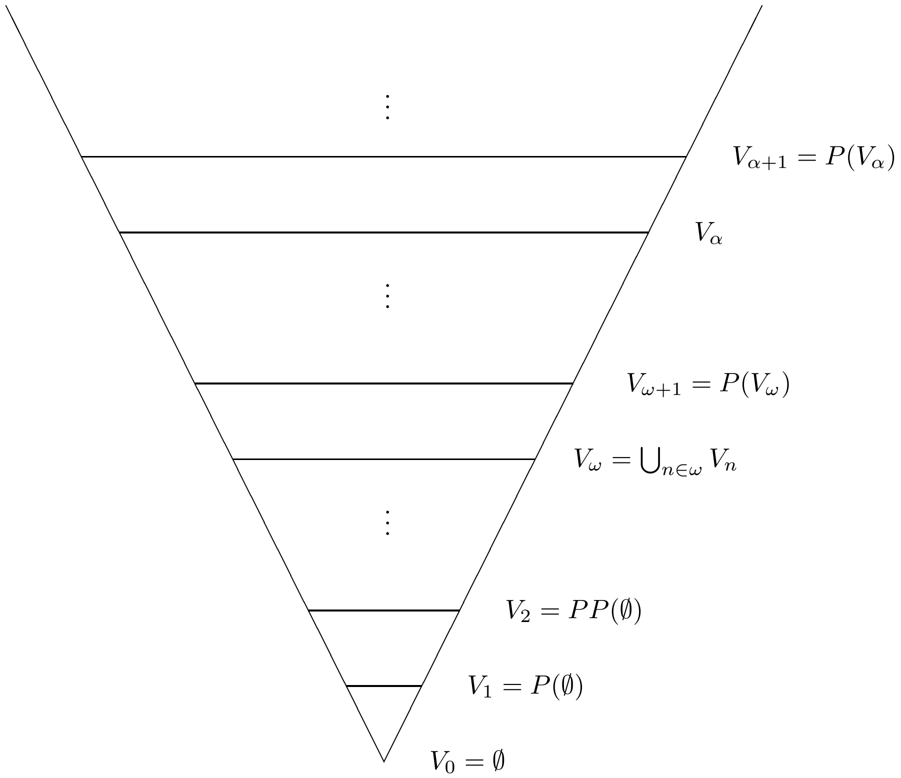
Paradoxes

Cantor, and independently Burali-Forti, found paradoxes in Cantor's set theory. The most famous version of the paradox is due to Russell: If x consists of all sets y such that $y \notin y$, then $x \in x \leftrightarrow x \notin x$!

Zermelo's proposal was to avoid the paradoxes by confining oneself to established principles of set-formation. He developed an axiomatic theory of sets, Zermelo set theory, which captures these principles and to the present day does appear to be free of contradiction. Fraenkel added a missing axiom scheme to Zermelo's system, resulting in ZFC, Zermelo-Fraenkel set theory with the axiom of choice. This is the standard axiom system for set theory today.

The Universe of Sets V

As a consequence of the ZFC axioms one has the following attractive picture of the set-theoretic universe V:



This provides the first picture of the universe of sets, based on the ordinals and power set operation. But this is *not* a canonical description, in light of the vagueness of the power set operation. A consequence of this vagueness is that the continuum problem remains unsolved.

The Vagueness of Power Set

One possible way to avoid the vagueness of power set is to focus on *definable sets*, as did the descriptive set-theorists of the 1930s. A set of reals is *Borel* iff it belongs to the smallest σ-algebra containing all open sets. It is Σ^1_1 (or analytic) iff it is the continuous image of a Borel set and is Π^1_1 iff its complement is Σ^1_1. Inductively, a Σ^1_{n+1} set is the continuous image of a Π^1_n set and a Π^1_{n+1} set is the complement of a Σ^1_{n+1} set. A set is *projective* iff it is Σ^1_n or Π^1_n for some n.

The descriptive set-theorists of the 1930s were able to show that Σ^1_1 sets do not violate the continuum hypothesis, in the sense that each uncountable Σ^1_1 set of reals does have the same cardinality as the reals. But they were unable to extend this to Π^1_1 sets. Later work of Gödel and Cohen implies that this indeed cannot be done in ZFC.

Another approach to circumventing the vagueness of the power set operation is due to Gödel. Recall that in our picture of V, at a successor stage $V_{\alpha+1}$ one takes the all subsets of the previous stage V_α. Gödel modified this hierarchy by instead taking only those subsets of the previous stage which can be produced using simple set-theoretic operations. He showed that iteration of this "weak" power set operation gives rise to a hierarchy of L_α's whose union L is a model of ZFC. Moreover his hierarchy grows so slowly that sets are easily labelled by ordinal numbers, yielding the striking consequence that CH holds in L. This provided the first consistent interpretation of ZFC in which CH is demonstrably true.

Gödel himself did not consider L to be the correct interpretation of V, but only as a tool for showing that statements are consistent with the axioms of ZFC. This point of view is almost universal among set-theorists today. Indeed there are other consistent interpretations of the ZFC axioms. Cohen developed a technique for adding new sets to L while preserving the ZFC axioms. In fact, he was able to obtain such an interpretation where CH is false: A *Cohen real over* L is a real that belongs to every open dense set of reals which L can "describe". By adding many Cohen reals to L, one obtains a consistent failure of CH.

Cohen's method, called *forcing*, is very general. Another of its applications is the following: A real in $[0, 1]$ is *random over* L iff it belongs to every measure one subset of $[0, 1]$ which L can "describe". Solovay[70] used this notion of forcing to obtain an interpretation of ZFC in which no projective set of reals violates the continuum hypothesis.

Canonical Universes

Thus we are faced with a dilemma: Must we accept different universes with different kinds of mathematics; universes where CH holds and universes where it does not? This kind of undecidability is certainly very troubling, and has led set-theorists to search for a canonical, acceptable interpretation, or *standard model*, of ZFC which provides the "correct" answers to undecidable problems. Gödel's L is surely canonical, but rejected as being too restrictive, given the ease with which it can be modified by forcing. Unfortunately the universes constructed using forcing are not canonical: If there is one Cohen (random) real over L, then there are many. How does one obtain canonical universes which are larger than L?

An answer came from measure theory. From work of Scott[61] and Solovay[71], we have: If there is a countably additive extension of Lebesgue measure to all sets of reals, then V is not L. And any model of ZFC with such a measure can be converted into a model of ZFC with a certain type of large cardinal number called a *measurable cardinal* (and vice-versa). Now Silver[71] showed that if there is a measurable cardinal then there is a canonical subuniverse of V, or *inner model*, with a measurable cardinal. This provided the first canonical inner model larger than Gödel's L.

Is Silver's model the desired standard model of ZFC?

To answer this question we consider the role of *large cardinal hypotheses*, of which the assumption of a measurable cardinal is an example. We say that two set-theoretic assertions φ, ψ are *consistency-equivalent* iff one can show that ZFC $+\varphi$ has a model iff ZFC $+\psi$ has a model. Experience shows that any natural set-theoretic assertion φ is consistency-equivalent to an assertion ψ, where ψ is either a large cardinal hypothesis or simply one of the statements $0 = 0$ or $0 = 1$. Thus large cardinal hypotheses calibrate the strength of natural set-theoretic assertions.

Hypotheses beyond the existence of a measurable cardinal are needed for this calibration. A nice example arises from descriptive set theory:

A is *Wadge reducible* to B iff for some continuous f, $x \in A$ iff $f(x) \in B$.
WP_n: If A, B are Σ^1_n but not Π^1_n then A is Wadge reducible to B and vice-versa.

Now we have

WP_1 is consistency equivalent to the *existence of #'s*, a large cardinal axiom below a measurable cardinal (Martin[70] and Harrington[78]).
WP_2 is consistency equivalent to (a little more than) the existence of a Woodin cardinal, a cardinal much larger than a measurable cardinal (Hjorth[96]).
WP_n seems to require $n-1$ Woodin cardinals.

Thus our desired standard model for ZFC should be big enough to permit the existence of Woodin cardinals. Silver's model is therefore too small.

The construction of canonical inner models for large cardinals at and above the level of Woodin cardinals is a central, ongoing project in pure set theory. These models however cannot be built using only the axioms of ZFC. Instead, one shows that if there is a large cardinal of a certain kind, then there is a canonical inner model with this kind of large cardinal. But isn't this circular? Why should we allow these large cardinals at all? It is reasonable to assert that L is too restrictive, but need we assume that Woodin cardinals exist? Perhaps our position should be that WP_n is simply false for $n > 1$!

The justification of large cardinal hypotheses remains an important challenge for pure set theory. One approach to this problem makes use of endomorphisms. An inner model M is *rigid* iff there is no 1-1 embedding $M \to M$ which preserves the basic set-theoretic operations union, product, difference, etc. The smallest large cardinal axiom, written "$0^\#$ exists", is equivalent to the statement that L is not rigid. If L is not rigid, there is a "canonical" inner model $L^\#$ bigger than L which satisfies "L is not rigid". Now do this again: If $L^\#$ is not rigid then there is a canonical inner model $L^{\#\#}$ where this is true, etc. There is a canonical # operation which iterated in this way leads to models with Woodin cardinals. This is analogous to Gödel's construction of L, obtained through iteration of a weak power set operation. In Gödel's context, the axioms of ZFC justify the use of this operation. In the current context, one must argue for the non-rigidity of the models that arise in a canonical #-iteration. Such an argument can provide a justification for the existence of inner models with Woodin cardinals.

However no canonical # iteration is known which goes very far *beyond* Woodin cardinals. Luckily, most current applications of large cardinals do not use stronger hypotheses. But finding such an operation remains an important problem if we wish to achieve a satisfying picture of the set-theoretic universe, a picture which would not only give rise to numerous further applications of set theory and justify the use of large cardinal hypotheses, but also lend credence to the claim that the paradoxes that threatened Cantor in the infancy of set theory have been definitively resolved.

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