A characterisation of $0^\#$ in terms of forcing

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Abstract

We show that "saturation" of the universe with respect to forcing over L with partial orders on ω_1 is equivalent to the existence of $0^\#$.

If P is a constructible forcing notion then we say that $G \subseteq P$ is *P-generic* iff G is P-generic over L. The statement that all countable constructible forcings have generics is rather weak, and holds for example in $L[R]$ where R is a Cohen real over L. But it is not possible that all constructible forcings have generics: consider the forcing that collapses ω_1 to ω with finite conditions.

Definition. V is *L-saturated for ω_1-forcings* iff whenever P is a constructible forcing of L-cardinality ω_1 such that for any $p \in P$ there is a P-generic containing p in some ω_1-preserving extension of V, then there is a P-generic in V.

Theorem 1 *The following are equivalent:*
(a) V is L-saturated for ω_1-forcings.
(b) $0^\#$ exists.

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Proof. (a) → (b) The existence of $0^\#$ is equivalent to the statement that every stationary constructible subset of ω_1 contains a CUB subset (see [2]). Now use the following:

Fact. (Baumgartner, see [1]) If X is a stationary constructible subset of ω_1 then there is a forcing $P \in L$ of L-cardinality ω_1 which preserves cardinals over V and adds a CUB subset to X. (P adds a CUB subset of X using "finite conditions".)

(b) → (a) Assume that $0^\#$ exists and suppose that P is a constructible forcing of L-cardinality ω_1 such that every condition in P belongs to a generic in an ω_1-preserving extension of V. We will show that there is a P-generic in V. Assume that the universe of P is exactly ω_1. Let P be of the form $t(\vec{i}, \omega_1, \vec{\infty})$ where $\vec{i} < \omega_1 < \vec{\infty}$ is a finite increasing sequence of indiscernibles and t is an L-term. We claim that if $\vec{i} < k_0 < k_1$ are countable indiscernibles and G_{k_0} is P_{k_0}-generic over L then there is G_{k_1} containing G_{k_0} which is P_{k_1}-generic over L, where $P_k = t(\vec{i}, k, \vec{\infty})$. If not, then player I wins the open game $\mathcal{G}(k_0, k_1, G_{k_0})$ where I chooses constructible dense subsets of P_{k_1} and II responds with increasingly strong conditions meeting these dense sets which are compatible with all conditions in G_{k_0}. The latter is a property of the model $L[G_{k_0}]$. Let $p \in P_{k_0}$ be a condition forcing that I wins $\mathcal{G}(k_0, k_1, G_{k_0})$. Then p forces that I wins $\mathcal{G}(k_2, k_3, G_{k_2})$, where $k_2 < k_3$ are any indiscernibles $\geq k_0$ and G_{k_2} denotes the P_{k_2}-generic. But now let G be a P-generic containing p in an ω_1-preserving extension of V. As G preserves ω_1 over V, there are indiscernibles $k_2 < k_3$ with $k_0 \leq k_2$ such that $G \cap k_2$ is P_{k_2}-generic and $G \cap P_{k_3}$ is P_{k_3}-generic, so clearly player II has a winning strategy in the game $\mathcal{G}(k_2, k_3, G \cap P_{k_2})$, in contradiction to the choice of p.

Now it is easy to build a P-generic: List the countable indiscernibles greater than $\vec{i}$ as $j_0 < j_1 < j_2 < \cdots$ and inductively choose P_{j_α}-generic G_α such that $\alpha < \beta$ implies $G_\alpha \subseteq G_\beta$. At the first step, G_{j_0} is an arbitrary P_{j_0}-generic. By the previous paragraph there is no difficulty at the successor steps, where one extends G_{j_α} to $G_{j_{\alpha+1}}$. At limit stages λ, the P_{j_λ}-genericity of the union G_{j_λ} of the G_{j_α}, $\alpha < \lambda$, follows by indiscernibility. The desired P-generic is the union of the G_{j_α}, $\alpha < \omega_1$. □

Remark. The proof of (a) implies (b) shows that the Theorem still holds if "ω_1-preserving extension" is taken to be "ω_1-preserving set-generic extension" in the definition of L-saturation for ω_1-forcings.

Question. Suppose that $0^{\#}$ exists. Then does part (a) of the Theorem hold (in the obvious sense) for constructible ω_1^{+L}-forcings; i.e. constructible P of L-cardinality ω_1^+ of L?

References

[1] Abraham, U. and Shelah, S. Forcing closed unbounded sets, J. Symbolic Logic 48 (1983) 643-657.

[2] Jech, T. *Set Theory: Millenium Edition*, Springer-Verlag, 2003.