CLASS FORCING

SY D. FRIEDMAN
M.I.T.

The method of forcing has had great success in demonstrating the relative consistency and independence of set-theoretic problems with respect to the traditional ZFC axioms, or to extensions of these axioms asserting the existence of large cardinals. One begins with a model M, selects a partial-ordering $P \in M$ and shows that statements of interest hold in extensions of M of the form $M[G]$, when G is P-generic over M.

However, forcing can play another rôle in set theory. Not only is it a tool for establishing relative consistency and independence results, it is also a tool for *proving theorems.* This theorem-proving rôle of forcing in set theory did not become fully apparent until the development of *class forcing.*

In class forcing, the partial-ordering P is no longer assumed to be an element of M, but instead a *class* in M. Section 2 below introduces the necessary definitions. We can nevertheless in this introduction explain the special rôle of class forcing in set theory by posing the basic question:

Question. Do P-generic classes exist?

This question never arises in traditional applications of forcing, for the simple reason that, thanks to the Löwenheim-Skolem Theorem, one can assume that the model M is countable. This assumption assures an easy construction of a P-generic class. Without the countability assumption, our question becomes a serious one, in light of the following:

Fact 1. There exist L-definable class forcings P_0, P_1 such that if G_0, G_1 are P_0, P_1-generic over L, respectively, then:

(a) ZFC holds in $\langle L[G_0], G_0\rangle$ and in $\langle L[G_1], G_1\rangle$.
(b) ZFC (indeed Replacement) fails in $\langle L[G_0, G_1], G_0, G_1\rangle$.

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This Fact forces us to make a choice: we cannot preserve ZFC and have generics for all ZFC preserving class forcings.

The Silver-Solovay theory of $0^\#$ provides a useful criterion for selecting the L-definable forcings which "should" have generics. We say that L is **rigid** if there is no elementary embedding from $\langle L, \in \rangle$ to itself, other than the identity.

Fact 2. L is rigid in class-generic extensions of L. If L is not rigid then there is a smallest inner model in which L is not rigid, and this inner model is $L[0^\#]$, where $0^\#$ is a real.

Now we say that an L-definable forcing P is **relevant** if there is a class which is P-generic over L and which is definable in the inner model $L[0^\#]$. If P_0 and P_1 are relevant forcings then clearly generics for P_0 and for P_1 can coexist, as they both exist definably over $L[0^\#]$. Moreover, by adopting the base theory ZFC $+0^\#$ exists, we can hope to use the theory of relevant forcing to prove new theorems, by constructing objects which actually exist (in the inner model $L[0^\#]$) rather than which may exist in a generic extension of the universe.

In this article we discuss the basic theory and applications of class forcing, with an emphasis on three problems posed by Solovay which can be resolved using it. As class forcing, unlike traditional set forcing, does not in general preserve ZFC, we first isolate the first-order property of *tameness*, necessary and sufficient for this preservation. After mentioning four basic examples, we discuss the question of relevance of class forcing, before turning to the most important technique in the subject, the technique of *Jensen coding*. Armed with these ideas we then proceed to describe the solutions to the Solovay problems. We next discuss *Generic Saturation*, a concept which helps to explain the special rôle of $0^\#$ in this theory. We end by briefly describing some other applications.

For the deeper study of class forcing, including the many proofs omitted here, we refer the reader to Friedman [99].

1. Three Problems of Solovay

Solovay's three problems each demand the existence of a real that neither constructs $0^\#$, nor is attainable by forcing over L.

DEFINITION. If x, y are sets of ordinals then we write $x \leq_L y$ for $x \in L[y]$ and $x <_L y$ for $x \leq_L y, y \not\leq_L x$.

Genericity Problem. Does there exist a real $R <_L 0^{\#}$ such that R does not belong to a generic extension of L?

It was to affirmatively answer this question (when "generic" is interpreted to mean "set-generic") that Jensen proved his Coding Theorem. Roughly speaking he showed that if G is generic for Easton forcing at Successors, the L-definable class forcing that adds a κ-Cohen subset to κ for each L-successor cardinal κ, then there is a real $R <_L 0^{\#}$, obtained by class forcing over $\langle L[G], G\rangle$, such that $L[G] \subseteq L[R]$ and G is definable over $L[R]$. Then R does not belong to a set-generic extension of L as $L[G]$ is not included in any such extension.

Solovay's second problem concerns definability of reals.

DEFINITION. R is an **Absolute Singleton** if for some formula φ, R is the unique solution to φ in every inner model containing R.

Shoenfield's Absoluteness Theorem states that if φ is Π^1_2 (i.e., of the form $\forall\, R \exists\, S\psi$, ψ arithmetical) then $\varphi(R) \longleftrightarrow M \vDash \varphi(R)$ where M is any inner model containing R. Thus any Π^1_2-Singleton (i.e., unique solution to a Π^1_2 formula) is an Absolute Singleton; $0^{\#}$ is an example. Also 0 is trivially an example. Solovay asked if there are in a sense any other examples.

Π^1_2-Singleton Problem. Does there exists a real R, $0 <_L R <_L 0^{\#}$ such that R is a Π^1_2-Singleton?

Suppose that R is set generic over L. Then it can be shown that R belongs to a P-generic extension of L, where there are only countably-many constructible subsets of P, and therefore we can build a P-generic containing any condition in P. So we conclude that if R is nonconstructible and set-generic over L then R cannot be a Π^1_2-Singleton, as there must be other P-generic extensions with reals $R' \neq R$ satisfying any given Π^1_2 formula satisfied by R. This is why the Π^1_2-Singleton Problem requires Jensen's method: an affirmative answer to the Π^1_2-Singleton Problem implies an affirmative answer to the Genericity Problem (for set-genericity).

Solovay's third problem concerns Admissibility Spectra. Let T be a subtheory of ZFC and R a real. The **T-spectrum** of R, $\Lambda_T(R)$, is the class of all ordinals α such that $L_\alpha[R] \vDash T$. A general problem is to characterize the possible T-spectra of reals for various theories T. An important special case is where $T = T_0 =$ (ZFC without the Power Set Axiom and with Replacement restricted to Σ_1 formulas). We may refer to this as "admissibility theory," as an ordinal α is R-admissible if and only if it is either ω or belongs to the T_0-spectrum, or **Admissibility Spectrum**, of R. We denote the latter by $\Lambda(R)$.

There are some basic facts which limit the possibilities for $\Lambda(R)$: First, if R belongs to a set-generic extension of L then $\Lambda(R)$ contains $\Lambda - \beta$ for some ordinal β, where $\Lambda = \Lambda(0)$. This is because if $\alpha \in \Lambda$, $P \in L_\alpha$ then $L_\alpha[G] \vDash T_0$ for P-generic G. Second, if $0^\# \leq_L R$ then $\Lambda(R) - \beta \subseteq L$-inaccessibles for some β. This is because if $0^\# \in L_\beta[R]$ then every α in $\Lambda(R) - \beta$ is in $\Lambda(0^\#)$ and hence is a "Silver indiscernible," an ordinal which is very large (and in particular inaccessible) in L.

Thus to get a nontrivial admissibility spectrum for R without $0^\#$ we need Jensen's methods. An ordinal is **recursively inaccessible** if it is admissible and also the limit of admissibles.

Admissibility Spectrum Problem. Does there exist a real $R \leq_L 0^\#$ such that $\Lambda(R) =$ the recursively inaccessible ordinals?

Of course we must in fact have $R <_L 0^\#$ as otherwise $\Lambda(R)$ is too thin.

Before we can say more about the solutions to the Solovay problems, we must first develop the basic theory of class forcing, to which we turn next.

2. Tameness

We want our class forcings to preserve ZFC. First we isolate a first-order condition that guarantees this.

DEFINITION. A **ground model** is a structure $\langle M, A\rangle$ where:

(a) $\langle M, A\rangle$ is a transitive model of ZFC; i.e., M is a transitive model of ZFC and Replacement holds in M for formulas mentioning A as a unary predicate.

(b) $M \vDash V = L(A) = \cup\{L(A \cap V_\alpha) \mid \alpha \in \mathrm{ORD}\}$.

(b) guarantees that if $M \subseteq N \vDash \mathrm{ZFC}$ then M is definable over $\langle N, A\rangle$.

Suppose $G \subseteq P$ where P is an $\langle M, A\rangle$-**forcing**, i.e., a pre-ordering (reflexive, transitive relation) with greatest element 1^P, definable over $\langle M, A\rangle$. G is **P-generic over** $\langle M, A\rangle$ if G is compatible, upward-closed and $G \cap D \neq \emptyset$ whenever $D \subseteq P$ is dense and $\langle M, A\rangle$-definable.

For any $G \subseteq M$ we define $M[G]$ as follows: A **name** is a set $\sigma \in M$ whose elements are of the form $\langle \tau, a\rangle$, τ a name and $a \in M$ (defined inductively). Interpret names by: $\sigma^G = \{\tau^G \mid \langle \tau, a\rangle \in \sigma \text{ for some } a \in G\}$. Then $M[G] = \{\sigma^G \mid \sigma \text{ a name}\}$. A **$P$-generic extension of** $\langle M, A\rangle$ is a model $\langle M[G], A, G\rangle$ where G is P-generic over $\langle M, A\rangle$. P is an **M-forcing** if it is an $\langle M, A\rangle$-forcing for some A. A **generic extension of** M is a model $\langle M[G], A, G\rangle$ for some choice of A, P and of G P-generic over $\langle M, A\rangle$. $X \subseteq M$ is **generic over** M if X is definable in a generic extension of M.

Set forcings always preserve ZFC but class forcings in general do not. Fix a ground model $\langle M, A\rangle$ and $\langle M, A\rangle$-forcing P. P is ZFC **preserving** if $\langle M[G], A, G\rangle$ is a model of ZFC for all G which are P-generic over $\langle M, A\rangle$. For countable M there is a useful first-order property equivalent to ZFC preservation, called *tameness*, that we now describe. First we consider ZFC − Power:

Definition. $D \subseteq P$ is **predense** $\leq p \in P$ if every $q \leq p$ is compatible with an element of D. $q \in P$ **meets** D if q extends an element of D. P is **pretame** if whenever $p \in P$ and $\langle D_i \mid i \in a\rangle$, $a \in M$ is an $\langle M, A\rangle$-definable sequence of classes predense $\leq p$ there exists $q \leq p$ and $\langle d_i \mid i \in a\rangle \in M$ such that for each $i \in a$, $d_i \subseteq D_i$ and d_i is predense $\leq q$.

Proposition 2.1. *Suppose that M is countable and P is* ZFC − Power *preserving. Then P is pretame.*

Proof. Given $\langle D_i \mid i \in a\rangle$ and p as in the statement of pretameness choose G such that $p \in G$, G P-generic over $\langle M, A\rangle$ and consider $f(i) =$ least rank of an element of $G \cap D_i$. If pretameness failed for $p, \langle D_i \mid i \in a\rangle$ then for every $q \leq p$ and $\alpha \in \mathrm{ORD}(M)$ there would be $r \leq q$ and $i \in a$ with r incompatible with each element of $D_i \cap V_\alpha$. But then by genericity, no ordinal of M can bound the range of f, so replacement fails in $\langle M[G], A, G, M\rangle$. As $\langle M, A\rangle$ is a ground model, replacement fails in $\langle M[G], A, G\rangle$. ⊣

Proposition 2.2. *Suppose that P is pretame, P-forcing is definable (for each formula φ, the relation $p \Vdash \varphi(\sigma_1 \ldots \sigma_n)$ of $p, \sigma_1, \ldots, \sigma_n$ is $\langle M, A\rangle$-definable) and* *the Truth Lemma holds for P-forcing (for G P-generic over $\langle M, A\rangle$, $\langle M[G], A, G\rangle \vDash \varphi(\sigma_1^G \dots \sigma_n^G)$ iff $\exists p \in G, p \Vdash \varphi(\sigma_1 \dots \sigma_n)$). Then P is* ZFC − Power *preserving.*

Proof. Suppose that G is P-generic over M. As $M[G]$ is transitive and contains ω, it is a model of all axioms of ZFC − Power with the possible exception of pairing, union and replacement.

For pairing, given σ_1^G, σ_2^G consider $\sigma = \{\langle\sigma_1, 1^P\rangle, \langle\sigma_2, 1^P\rangle\}$. Then $\sigma^G = \{\sigma_1^G, \sigma_2^G\}$.

For replacement, suppose $f : \sigma^G \longrightarrow M[G]$, f definable (with parameters) in $\langle M[G], A, G\rangle$ and by the Truth Lemma choose $p \in G$, $p \Vdash f$ is a total function on σ. Then for each σ_0 of rank $<$ rank σ, $D(\sigma_0) = \{q \mid$ For some τ, $q \Vdash \sigma_0 \in \sigma \longrightarrow f(\sigma_0) = \tau\}$ is dense $\leq p$. Thus by the Definability of P-forcing and pretameness we get that for each $q \leq p$ there is $r \leq q$ and $\alpha \in \mathrm{ORD}(M)$ such that $D_\alpha(\sigma_0) = \{s \mid s \in V_\alpha$ and for some τ of rank $< \alpha$, $s \Vdash \sigma_0 \in \sigma \longrightarrow f(\sigma_0) = \tau\}$ is predense $\leq r$ for each σ_0 of rank $<$ rank σ. By genericity there is $q \in G$ and $\alpha \in \mathrm{ORD}(M)$ such that $q \leq p$ and $D_\alpha(\sigma_0)$ is predense $\leq q$ for each σ_0 of rank $<$ rank σ. Thus $\mathrm{Range}(f) = \pi^G$ where $\pi = \{\langle\tau, r\rangle \mid \mathrm{rank}\,\tau < \alpha, r \in V_\alpha, r \Vdash \tau \in \mathrm{Range}(f)\}$. So $\mathrm{Range}(f) \in M[G]$.

For union, given σ^G consider $\pi = \{\langle\tau, p\rangle \mid p \Vdash \tau \in \cup\sigma\}$. This is not a set, but for each α we may consider $\pi_\alpha = \pi \cap V_\alpha^M$. By Replacement in $\langle M[G], A, G\rangle$, π_α^G is constant for sufficiently large $\alpha \in \mathrm{ORD}(M)$. For such α we have $\pi_\alpha^G = \cup\sigma^G$. ⊣

Thus the work in establishing the equivalence (for countable M) of ZFC − Power preservation with pretameness resides in:

Lemma 2.3. *(Main Lemma) If P is pretame and M is countable then P-forcing is definable and the Truth Lemma holds for P-forcing.*

Proof. We define a relation $\Vdash^*$, prove the lemma for $\Vdash^*$ and finally show $\Vdash = \Vdash^*$.

Definition **(of $\Vdash^*$).** We say that $D \subseteq P$ is **dense** $\leq p$ if $\forall q \leq p \exists r (r \leq q, r \in D)$.

(a) $p \Vdash^* \sigma \in \tau$ iff $\{q \mid \exists\langle\pi, r\rangle \in \tau$ such that $q \leq r, q \Vdash^* \sigma = \pi\}$ is dense $\leq p$.
(b) $p \Vdash^* \sigma = \tau$ iff for all $\langle\pi, r\rangle \in \sigma \cup \tau$, $p \Vdash^* (\pi \in \sigma \longleftrightarrow \pi \in \tau)$.
(c) $p \Vdash^* \varphi \wedge \psi$ iff $p \Vdash^* \varphi$ and $p \Vdash^* \psi$.
(d) $p \Vdash^* \sim \varphi$ iff $\forall q \leq p (\sim q \Vdash^* \varphi)$.
(e) $p \Vdash^* \forall x \varphi$ iff for all names $\sigma, p \Vdash^* \varphi(\sigma)$.

Note that circularity is avoided in (a), (b) as $\max(\mathrm{rank}\,\sigma, \mathrm{rank}\,\tau)$ goes down (in at most three steps) when these definitions are applied.

SUBLEMMA 2.4. (a) $p \Vdash^* \varphi, q \leq p \longrightarrow q \Vdash^* \varphi$.
(b) *If* $\{q \mid q \Vdash^* \varphi\}$ *is dense* $\leq p$ *then* $p \Vdash^* \varphi$.
(c) *If* $\sim p \Vdash^* \varphi$ *then* $\exists q \leq p\ (q \Vdash^* \sim \varphi)$.

PROOF OF SUBLEMMA 2.4. (a) Clear, by induction on φ, as dense $\leq p \longrightarrow$ dense $\leq q$.
(b) Again by induction on φ. The proof uses the following facts: if $\{q \mid D$ is dense $\leq q\}$ is dense $\leq p$ then D is dense $\leq p$; if $\{q \mid q \Vdash^* \sim \varphi\}$ is dense $\leq p$ then $\forall q \leq p(\sim q \Vdash^* \varphi)$, using (a).
(c) Immediate by (b).

$\dashv$

SUBLEMMA 2.5. *(Definability of* $\Vdash^*$*) For each formula* φ*, the relation* $p \Vdash^* \varphi(\sigma_1 \cdots \sigma_n)$ *of* $p, \sigma_1, \ldots, \sigma_n$ *is* $\langle M, A\rangle$*-definable.*

PROOF OF SUBLEMMA 2.5. It suffices to show that the relations $p \Vdash^* \sigma \in \tau$ and $p \Vdash^* \sigma = \tau$ are $\langle M, A\rangle$-definable. Note that by modifying A if necessary, we may assume that the relations "$x = V_\alpha^M$," "p, q are compatible," "d is predense below p," as well as $(P, \leq)$, are Δ_1-definable over $\langle M, A\rangle$.

Using pretameness we shall define a function F from pairs $(p, \sigma \in \tau)$, $(p, \sigma = \tau)$ into M such that:

(a) $F(p, \sigma \in \tau) = (i, d)$ where $\emptyset \neq d \in M$, $d \subseteq P$, $q \in d \longrightarrow q \leq p$ and either ($i = 1$ and $q \Vdash^* \sigma \in \tau$ for all $q \in d$) or ($i = 0$ and $q \Vdash^* \sigma \notin \tau$ for all $q \in d$).
(b) The same holds for $\sigma = \tau$, $\sigma \neq \tau$ instead of $\sigma \in \tau, \sigma \notin \tau$.
(c) F is Σ_1-definable over $\langle M, A\rangle$.

Given this we can define $p \Vdash^* \sigma \in \tau$ by: $p \Vdash^* \sigma \in \tau$ iff for all $q \leq p$, $F(q, \sigma \in \tau) = (1, d)$ for some d. This definition is correct because Lemma 2.4 gives us that $p \Vdash^* \sigma \in \tau \longleftrightarrow \{q \mid q \Vdash^* \sigma \in \tau\}$ is dense $\leq p$. Similarly for $p \Vdash^* \sigma = \tau$.

Now define F by induction on $\sigma \in \tau$, $\sigma = \tau$. We consider the cases separately.

$\underline{\sigma \in \tau}$: Given p, search for $\langle \pi, r\rangle \in \tau$ and $q \leq p$, $q \leq r$ such that $F(q, \sigma = \pi) = (1, d)$ for some d. If such exist, let $F(p, \sigma \in \tau) = (1, e)$ where e is the union of all such d which appear by the least possible stage α (i.e., this Σ_1 property is true in $\langle V_\alpha^M, A \cap V_\alpha^M\rangle$, α least). If not then $\cup\{d \mid$ For some $q \leq r, F(q, \sigma = \pi) = (0, d)\} \cup \{q \mid q$ is incompatible with $r\} = D(\pi, r)$ is dense below p for each $\langle \pi, r\rangle \in \tau$. So also search for $\langle d(\pi, r) \mid \langle \pi, r\rangle \in \tau\rangle \in M$ and $q \leq p$ such that $d(\pi, r) \subseteq D(\pi, r)$ for each $\langle \pi, r\rangle$ and each $d(\pi, r)$ is predense $\leq q$; if this latter search terminates then set $F(p, \sigma \in \tau) = (0, e)$, where e

consists of all such q witnessed by the least possible stage α. One of these searches must terminate (by pretameness) and hence $F(p, \sigma \in \tau)$ is defined and either of the form $(1, e)$ where $q \in e \longrightarrow q \leq p$, $q \Vdash^* \sigma \in \tau$, or of the form $(0, e)$ where $q \in e \longrightarrow q \leq p, q \Vdash^* \sigma \notin \tau$.

$\underline{\sigma = \tau}$: Given p, search for $\langle \pi, r \rangle \in \sigma \cup \tau$ and $q \leq p, r$ such that $F(q, \pi \in \sigma) = (i, d), q' \in d, F(q', \pi \in \tau) = (1 - i, e)$ and if this search terminates then set $F(p, \sigma = \tau) = (0, f)$ where f is the union of all such e which appear by the least possible stage α. If this search fails then for each $\langle \pi, r \rangle \in \sigma \cup \tau$, $D(\pi, r) = \cup \{e \mid \text{For some } q \leq p, \text{ some } q', d, i,\ F(q, \pi \in \sigma) = (i, d), q' \in d, F(q', \pi \in \tau) = (i, e)\} \cup \{q \mid q \text{ is incompatible with } r\}$ is dense $\leq p$. So also search for $\langle d(\pi, r) \mid \langle \pi, r \rangle \in \sigma \cup \tau \rangle \in M$ and $q \leq p$ such that for each $\langle \pi, r \rangle \in \sigma \cup \tau$, $d(\pi, r) \subseteq D(\pi, r)$ and $d(\pi, r)$ is predense $\leq q$. If this latter search terminates then $q \Vdash^* \sigma = \tau$ for all such q and let $F(p, \sigma = \tau) = (1, f)$, where f consists of all such q witnessed to obey the above by the least stage α. One of these searches must terminate (by pretameness) and hence $F(p, \sigma = \tau)$ is defined and either of the form $(0, f)$ where $q \in f \rightarrow q \leq p, q \Vdash^* \sigma \neq \tau$, or of the form $(1, f)$ where $q \in f \rightarrow q \leq p, q \Vdash^* \sigma = \tau$.

$\dashv$

Now that we have the definability of $\Vdash^*$ we can prove:

SUBLEMMA 2.6. *For G P-generic over M:*
$M[G] \vDash \varphi(\sigma_1^G \ldots \sigma_n^G) \longleftrightarrow \exists p \in G (p \Vdash^* \varphi(\sigma_1 \ldots \sigma_n))$.

PROOF OF SUBLEMMA 2.6. By induction on φ.

$\underline{\sigma \in \tau}(\longrightarrow)$: If $\sigma^G \in \tau^G$ then choose $\langle \pi, r \rangle \in \tau$ such that $\sigma^G = \pi^G$ and $r \in G$. By induction we can choose $p \in G$, $p \leq r$, $p \Vdash^* \sigma = \pi$. Then $p \Vdash^* \sigma \in \tau$. $(\longleftarrow)$ If $p \in G, \{q \mid \exists \langle \pi, r \rangle \in \tau \text{ such that } q \leq r,\ q \Vdash^* \sigma = \pi\} = D$ is dense $\leq p$ then by genericity we can choose $q \in G$, $\langle \pi, r \rangle \in \tau$ such that $q \leq r$, $q \Vdash^* \sigma = \pi$; then by induction $\sigma^G = \pi^G$ and as $r \geq q \in G$ we get $r \in G$ and hence by definition of $\tau^G, \pi^G \in \tau^G$. So $\sigma^G \in \tau^G$.

$\underline{\sigma = \tau}(\longrightarrow)$: Suppose $\sigma^G = \tau^G$. Consider $D = \{p \mid \text{Either } p \Vdash^* \sigma = \tau \text{ or for some } \langle \pi, r \rangle \in \sigma \cup \tau,\ p \Vdash^* \sim (\pi \in \sigma \longleftrightarrow \pi \in \tau)\}$. Then D is dense, using the definition of $p \Vdash^* \sigma = \tau$ and Lemma 2.4(c). By genericity there is $p \in G \cap D$ and by induction it must be that $p \Vdash^* \sigma = \tau$. $(\longleftarrow)$ Suppose $p \in G$, $p \Vdash^* \sigma = \tau$. Then by induction, $\pi^G \in \sigma^G \longleftrightarrow \pi^G \in \tau^G$ for all $\langle \pi, r \rangle \in \sigma \cup \tau$. So $\sigma^G = \tau^G$.

$\underline{\varphi \wedge \psi}$: Clear by induction, using the fact that $p, q \in G \longrightarrow \exists r \in G(r \leq p$ and $r \leq q)$.
$\underline{\sim \varphi}$: Clear by induction, using the density of $\{p \mid p \Vdash^* \varphi$ or $p \Vdash^* \sim \varphi\}$.
$\underline{\forall x \varphi}(\longrightarrow)$: Suppose $M[G] \vDash \forall x\varphi$. As in the proof of $(\longrightarrow)$ for $\sigma = \tau$, there is $p \in G$ such that either $p \Vdash^* \forall x \varphi$ or for some $\sigma, p \Vdash^* \sim \varphi(\sigma)$. By induction the latter is impossible so $p \Vdash^* \forall x \varphi$. $(\longleftarrow)$ Clear by induction.

$\dashv$

SUBLEMMA 2.7. $\Vdash^* = \Vdash$.

PROOF OF SUBLEMMA 2.7. By Sublemma 2.6, $p \Vdash^* \varphi(\sigma_1 \dots \sigma_n) \longrightarrow p \Vdash \varphi(\sigma_1 \dots \sigma_n)$. And $\sim p \Vdash^* \varphi(\sigma_1 \dots \sigma_n) \longrightarrow q \Vdash^* \sim \varphi(\sigma_1 \dots \sigma_n)$ for some $q \leq p$ (by Sublemma 2.4(c)) $\longrightarrow \sim p \Vdash \varphi(\sigma_1 \dots \sigma_n)$ using the countability of M to obtain a generic G, $p \in G$. $\dashv$

This completes the proof of Lemma 2.3. $\dashv$

P is **tame** if P is pretame and in addition $1^P \Vdash$ Power. The latter is first-order for pretame P as pretameness yields the definability of P-forcing. By the Truth Lemma for P-forcing we get:

THEOREM 2.8 (Stanley, M. [97], Friedman [99]). *(Tameness Theorem) Suppose that M is countable. Then P is* ZFC *preserving iff P is tame.*

3. Examples

We next discuss the four basic examples of tame class forcings, which serve as prototypes for more complex examples, such as Jensen coding. In each of these basic examples we take the ground model to be $\langle L, \emptyset \rangle$.

Easton Forcing

A condition in P is a function $p : \alpha(p) \to L$ where $\alpha(p) \in \mathrm{ORD}$ and $p(\alpha) = \emptyset$ unless α is infinite and regular, in which case $p(\alpha) \in 2^{<\alpha} = \{f \mid f : \beta \to 2$ for some $\beta < \alpha\}$. We also require **Easton Support** which means that $\{\beta < \alpha \mid p(\beta) \neq \emptyset\}$ is bounded in α for inaccessible α. For any α, $p(\leq \alpha)$ denotes $p \restriction [0, \alpha]$ and $p(> \alpha)$ denotes $p \restriction (\alpha, \alpha(p))$.

Proposition 3.1. *P is tame and preserves both cofinalities and the* GCH.

Proof. First we verify pretameness. Suppose $p \in P$, $\langle D_i \mid i < \kappa \rangle$ is an L-definable sequence of classes predense $\leq p$ and κ is regular. Let $\langle q_i \mid i < \kappa \rangle$ list all elements of $P(\leq \kappa) = \{q(\leq \kappa) \mid q \in P\}$, using the Easton support requirement. View each $i < \kappa$ as a pair $\langle i_0, i_1 \rangle$ and define $p_0 = p$; $p_{i+1} =$ least $r \leq p_i$ such that $r(\leq \kappa) = p_i(\leq \kappa)$ and $q_{i_0} \cup r(> \kappa)$ is a condition meeting some $r_i \in D_{i_1}$, if possible ($p_{i+1} = p_i$ otherwise); $p_\lambda = \cup\{p_i \mid i < \lambda\}$ for limit $\lambda \leq \kappa$. Then $p^* = p_\kappa \leq p$ has the property: if $r \leq p^*$ meets D_i then r extends r_j for some $j < \kappa$. Thus $d_i = \{r_j \mid r_j \in D_i\}$ is predense $\leq p^*$ for each i, proving pretameness.

To verify the remaining properties we may use:

Lemma 3.2. *(Product Lemma) Suppose that $P = P_0 \times P_1$ where P_0, P_1 are $\langle M, A \rangle$-definable.*

(a) *G_0 P_0-generic over $\langle M, A \rangle$, G_1 P_1-generic over $\langle M[G_0], A, G_0 \rangle \to G_0 \times G_1$ is P-generic over $\langle M, A \rangle$.*
(b) *G P-generic over $\langle M, A \rangle \to G = G_0 \times G_1$ where G_0 is P_0-generic over $\langle M, A \rangle$.* If in addition P_0-forcing is definable *then G_1 is P_1-generic over $\langle M[G_0], A, G_0 \rangle$.*

Proof. (a) Suppose that $D \subseteq P$ is dense and $\langle M, A \rangle$-definable. Then $D_1 = \{p_1 \mid \exists p_0 \in G_0\ (p_0, p_1) \text{ meets } D\}$ is $\langle M[G_0], A, G_0 \rangle$-definable; we claim that it is dense on P_1: given $p_1 \in P_1$ form $D_0(p_1) = \{p_0 \mid (p_0, p_1') \text{ meets } D$ for some $p_1' \leq p_1\}$. Then $D_0(p_1)$ is dense since D is, so $G_0 \cap D_0(p_1) \neq \emptyset$. Thus (p_0, p_1') meets D for some $p_0 \in G_0$, some $p_1' \leq p_1$ and therefore p_1' is an extension of p_1 in D_1.

As D_1 is dense we can choose $p_1 \in G_1 \cap D_1$ and so we get $(p_0, p_1) \in G_0 \times G_1$, (p_0, p_1) meets D. As $G_0 \times G_1$ is compatible and closed upwards (since G_0, G_1 are) we have shown that $G_0 \times G_1$ is P-generic over $\langle M, A \rangle$.

(b) Let $G_0 = \{p_0 \in P_0 \mid (p_0, p_1) \in G \text{ for some } p_1\}$, $G_1 = \{p_1 \mid (p_0, p_1) \in G$ for some $p_0\}$. Clearly $G \subseteq G_0 \times G_1$ and conversely if $(p_0, p_1) \in G_0 \times G_1$ then (p_0, p_1) is compatible with every element of G and hence by genericity of G, $(p_0, p_1) \in G$. If $D_0 \subseteq P_0$ is dense and $\langle M, A \rangle$-definable then $D = \{(p_0, p_1) \mid p_0 \in D_0\} \subseteq P$ is dense and $\langle M, A \rangle$-definable and since G meets D, we get that G_0 meets D_0. So G_0 is P_0-generic over $\langle M, A \rangle$, as compatibility and upward closure for G_0 follow from these properties for G.

Suppose that $D_1 \subseteq P_1$ is $\langle M[G_0], A, G_0\rangle$-definable and dense. Then $D = \{(p_0, p_1) \mid p_0 \Vdash \hat{p}_1 \in D_1\}$ is $\langle M, A\rangle$-definable by the definability of P_0-forcing (where "$\hat{p}_1 \in D_1$" is expressed using a defining formula for D_1). Also D is dense $\leq (p_0, p_1)$ provided $p_0 \Vdash D_1$ is dense. As G_0 is P_0-generic over $\langle M, A\rangle$ we can choose $p_0 \in G_0$, $p_0 \Vdash D_1$ is dense and then the genericity of G over $\langle M, A\rangle$ produces $(p_0', p_1) \in G$, $p_0' \Vdash \hat{p}_1 \in D_1$; then $p_1 \in G_1 \cap D_1$ and as compatibility, upward closure for G_1 are clear, we have shown that G_1 is P_1-generic over $\langle M[G_0], A, G_0\rangle$.

⊣

In the case of Easton forcing, $P \simeq P(> \kappa) \times P(\leq \kappa)$ where $P(> \kappa) = \{p(> \kappa) \mid p \in P\}, P(\leq \kappa) = \{p(\leq \kappa) \mid p \in P\}$ and if G is P-generic then $L[G] = L[G(> \kappa)][G(\leq \kappa)]$; (b) applies as $P(> \kappa)$ is pretame and hence $P(> \kappa)$-forcing is definable. As $P(> \kappa)$ is $\leq \kappa$-closed and for regular κ, $P(\leq \kappa)$ has cardinality κ (by Easton support) we get the preservation of "cof $> \kappa$" for regular κ and hence all cofinalities are preserved. And we have that for regular κ any subset of κ in $L[G]$ belongs to $L[G(\leq \kappa)]$. As $G(\leq \kappa)$ is equivalent to a subset of κ, the GCH follows at regular κ. For singular κ we get $\mathcal{P}(\kappa) = \mathcal{P}(\kappa)$ in $L[G(\leq \kappa^+)]$ and hence $2^\kappa = 2^\kappa$ in $L[G(\leq \kappa^+)] = \kappa^+$. ⊣

Long Easton Forcing

We drop the Easton support requirement. For *successor* cardinals κ we still have that $P(\leq \kappa)$ has cardinality κ, $P(> \kappa)$ is $\leq \kappa$-closed, so the previous arguments show us that P is tame, "cof $> \kappa$" is preserved for *successor* cardinals κ and the GCH is preserved. But not all cardinals need be preserved. A cardinal κ is **Mahlo** if it is inaccessible and in addition $\{\alpha < \kappa \mid \alpha \text{ inaccessible}\}$ is stationary in κ.

THEOREM 3.3. *If κ is Mahlo then κ^+ is collapsed by P; otherwise κ^+ is preserved.*

PROOF. Let $G = \langle G_\alpha \mid \alpha \text{ infinite, regular}\rangle$ be P-generic. For each $\alpha < \kappa$ consider $A_\alpha \subseteq \kappa$ defined by: $\beta \in A_\alpha \longleftrightarrow \alpha \in G_\beta$.

CLAIM. Suppose κ is Mahlo. Then $\{A_\alpha \mid \alpha < \kappa\} \subseteq L$ but for no $\gamma < (\kappa^+)^L$ do we have $\{A_\alpha \mid \alpha < \kappa\} \subseteq L_\gamma$.

Proof of Claim. For any $\alpha < \kappa$ and condition p, we can extend p to q so that $\alpha < \bar{\kappa} < \kappa, \bar{\kappa}$ regular $\longrightarrow p(\bar{\kappa})$ has length greater than α. Thus A_α is forced to belong to L.

Given $\gamma < (\kappa^+)^L$ and a condition p, define $f(\bar{\kappa}) = \text{length}(p(\bar{\kappa}))$ for regular $\bar{\kappa} < \kappa$. As κ is Mahlo, f has stationary domain and hence by Fodor's Theorem we may choose $\alpha < \kappa$ such that length$(p(\bar{\kappa}))$ is less than α for stationary many regular $\bar{\kappa} < \kappa$. Then p can be extended so that A_α is guaranteed to be distinct from the κ-many subsets of κ in L_γ. ⊣ (Claim)

Thus κ^+ is collapsed if κ is Mahlo. Conversely, if κ is not Mahlo, then choose a CUB $C \subseteq \kappa$ consisting of cardinals which are not inaccessible (we may assume that κ is a limit cardinal). Suppose that $\langle D_\alpha \mid \alpha \in C\rangle$ is a definable sequence of dense classes. Given p we can successively extend $p(\geq \alpha^+), \alpha \in C$ so that $\{q \leq p \mid q, p \text{ agree} \geq \alpha^+, q \in D_\alpha\}$ is predense $\leq p$. There is no difficulty in obtaining a condition at a limit stage *less than* κ precisely because conditions are trivial at limit points of C. Thus we have shown that $P(<\kappa) \times P(>\kappa)$ preserves κ^+ as κ-many dense classes can be simultaneously reduced to predense subsets of size $< \kappa$. Finally $P \simeq P(<\kappa) \times P(>\kappa) \times P(\kappa)$ and $P(\kappa)$ preserves κ^+ as it has size κ.

⊣

The previous proof shows that full cofinality preservation is obtained if we consider *Long Easton forcing at Successors*, where κ-Cohen sets are added only for infinite *successor* cardinals κ. We shall consider this and other variants of Long Easton forcing in the next section, on Relevant Forcing.

Reverse Easton Forcing

We consider the iteration defined by: $P(0) = \{\emptyset\}$, the trivial forcing; $P(\leq \alpha) = P(<\alpha) * P(\alpha)$ where $P(\alpha)$ is the trivial forcing unless α is infinite, regular in which case $P(\alpha) = 2^{<\alpha} = \alpha$-Cohen forcing; for limit λ, $P(<\lambda) =$ Direct Limit $\langle P(<\alpha) \mid \alpha < \lambda\rangle$ for λ regular and $P(<\lambda) =$ Inverse Limit $\langle P(<\alpha) \mid \alpha < \lambda\rangle$ for λ singular. (Thus Easton supports are being used.) Let $P =$ Direct Limit $\langle P(<\alpha) \mid \alpha \in \text{ORD}\rangle$.

Proposition 3.4 (Section 2.3 of Friedman [99]). (a) *κ regular $\to P(\leq \kappa)$ has a dense suborder of size κ.*

(b) *For* $\alpha < \beta \leq \infty$, $P(< \beta) \simeq P(\leq \alpha) * P(\alpha, \beta)$ *where* $P(\alpha, \beta)$ *is the natural Reverse Easton iteration of* γ*-Cohen forcings,* $\alpha < \gamma < \beta$*, defined in* $L[G(\leq \alpha)]$.

(c) κ *regular* $\rightarrow P(\leq \kappa) \Vdash P(\kappa, \infty)$ *is* $\leq \kappa$*-closed.*

It follows that $P =$ Direct Limit $\langle P(< \alpha) \mid \alpha \in \text{ORD} \rangle$ is tame and preserves cofinalities and the GCH.

Amenable Forcing

P consists of all $p : \alpha \rightarrow 2$, ordered by extension. P is $\leq \kappa$-closed for all κ and hence tame. Cofinality and GCH preservation are trivial as P adds no new sets.

4. Relevance

Which L-forcings have generics?

PROPOSITION 4.1. *There exist tame* L*-definable forcings* P_0*,* P_1 *such that* not *both* P_0 *and* P_1 *have generics.*

PROOF. For any ordinal α, let $n(\alpha)$ be the least n such that L_α is not a model of Σ_n-replacement, if such an n exists. Let $S_0 = \{\alpha \mid n(\alpha)$ exists and is even$\}$. P_0 consists of all closed p such that $p \subseteq S_0$, ordered by $p \leq q$ iff q is an initial segment of p.

Note that S_0 is unbounded in ORD: Given α, let β be least such that $\beta > \alpha$, $L_\beta \vDash \Sigma_1$-Replacement. Then $n(\beta) = 2$ so $\beta \in S_0$. If $G_0 \subseteq P_0$ is P_0-generic over L then $\cup G_0$ is therefore a closed unbounded subclass of ORD contained in S_0. To show that P_0 is tame, it suffices to show that it is κ^+-distributive for every L-regular κ : If $\langle D_i \mid i < \kappa \rangle$ is an L-definable sequence of classes dense on P_0 and $p \in P_0$ then choose n odd so that $\langle D_i \mid i < \kappa \rangle$ is Σ_n definable and choose $\langle \alpha_i \mid i < \kappa \rangle$ to be first κ-many α such that L_α is Σ_n-elementary in L and $\kappa, p, x \in L_\alpha$ where x is the defining parameter for $\langle D_i \mid i < \kappa \rangle$. We can define $p \geq p_0 \geq p_1 \geq \ldots$ so that p_{i+1} meets D_i and $\cup p_i = \alpha_i$, using the Σ_n-elementarity of L_{α_i} in L. As $n(\alpha_i) = n + 1$ and $n + 1$ is even, we have no problem in defining p_λ to be $\cup\{p_i \mid i < \lambda\} \cup \{\alpha_\lambda\}$ for limit $\lambda \leq \kappa$ and we see that $q = p_\kappa \leq p$ meets each D_i.

Now define P_1 in the same way, but using $S_1 = \{\alpha \mid n(\alpha) \text{ is defined and odd}\}$. Then P_1 is also tame yet if G_0, G_1 are P_0, P_1-generic over L (respectively) then $\cup G_0, \cup G_1$ are disjoint CUB subclasses of ORD. $\dashv$

So we need a criterion for choosing L-definable forcings for which we can have a generic. Our approach is to isolate a "property of transcendence" (#) such that:

(a) In tame class-generic extensions of L, (#) fails.
(b) If (#) is true in V then there is a least inner model $L(\#)$ satisfying (#).

Then our criterion for generic class existence is: P has a generic iff it has one definable over $L(\#)$.

Definition. An amenable $\langle L, A\rangle$ is **rigid** if there is no nontrivial elementary embedding $\langle L, A\rangle \to \langle L, A\rangle$. L is **rigid** if $\langle L, \emptyset\rangle$ is rigid.

We take (#) to be: L is not rigid. First we discuss property (b) above, i.e., that there is a least inner model in which L is not rigid (if there is one at all).

Theorem 4.2 (Kunen, Silver [71], Solovay [67]). *Suppose L is not rigid. Then there is a unique* CUB *class I of L-indiscernibles which* generate *L in the sense that $L = \text{Hull}(I)$, where* Hull *denotes Skolem Hull in L. Moreover I is unbounded in every uncountable cardinal and if $0^\# =$ First-Order theory of $\langle L, \in, i_1, i_2, \dots\rangle$ (where the first ω elements $i_1, i_2, \dots$ of I are introduced as constants) then we have the following:*

(a) *$0^\# \in L[I]$, I is $\Delta_1(L[0^\#])$ in the parameter $0^\#$ and I is unbounded in α whenever $L_\alpha[0^\#] \vDash \Sigma_1$ replacement.*
(b) *$0^\#$, viewed as a real, is the unique solution to a Π^1_2 formula (i.e., a formula of the form $\forall x \exists y \psi$, where x, y vary over reals and ψ is arithmetical).*
(c) *If $f : I \to I$ is increasing, $f \neq$ identity then there is a unique $j : L \to L$ extending f with critical point in I, and every $j : L \to L$ is of this form.*
(d) *If $\langle L, A\rangle$ is amenable then A is $\Delta_1(L[0^\#])$, $\langle L, A\rangle$ is not rigid and a final segment of I is a class of $\langle L, A\rangle$-indiscernibles.*

Remarks. (i) As I is closed and unbounded in every uncountable cardinal it follows that every uncountable cardinal belongs to I and $0^\# =$ First-Order theory of $\langle L, \in, \omega_1, \omega_2, \dots\rangle$.
(ii) The Σ^1_2-absoluteness of L (Shoenfield [61]) implies that the unique solution to a Σ^1_2 formula is constructible; so in a sense (b) is best possible.

(iii) I is a class of *strong* indiscernibles: if $\vec{i}, \vec{j}$ are increasing tuples from I of the same length and $x < \min(\vec{i}), \min(\vec{j})$ then for any φ, $L \vDash \varphi(x, \vec{i}) \longleftrightarrow \varphi(x, \vec{j})$.

In case the conclusion of Theorem 4.2 holds (i.e. in case L is not rigid) we say that "$0^{\#}$ exists" and refer to I as the **Silver Indiscernibles.** Note that Theorem 4.2 implies that if L is not rigid then $L[0^{\#}]$ is the smallest inner model in which L is not rigid, verifying that "L is not rigid" obeys condition (b) of our property of transcendence (#).

Before turning to condition (a) of property (#) we mention Jensen's Covering Theorem and some of its consequences. A set X is **covered in** L if there is a constructible Y such that $X \subseteq Y$, $\operatorname{Card} Y = \operatorname{Card} X$.

THEOREM 4.3 (Jensen, in Devlin-Jensen [75]). *Suppose there exists an uncountable set of ordinals which is not covered in L. Then $0^{\#}$ exists.*

For proofs of Theorems 4.2, 4.3 see Section 3.1 of Friedman [99].

Using the Covering Theorem, we see that the existence of $0^{\#}$ takes many equivalent forms.

THEOREM 4.4. *Each of the following is equivalent to the existence of $0^{\#}$:*

(a) *L is not rigid.*
(b) *$\langle L, A\rangle$ is not rigid for every A such that $\langle L, A\rangle$ is amenable.*
(c) *Some uncountable set of ordinals is not covered in L.*
(d) *Some singular cardinal is regular in L.*
(e) *$\kappa^+ \neq (\kappa^+)^L$ for some singular cardinal κ.*
(f) *Every constructible subset of ω_1 either contains or is disjoint from a closed, unbounded subset of ω_1.*
(g) *$\{\alpha \mid \alpha$ is an L-cardinal$\}$ is Δ_1-definable with parameters.*
(h) *There exists $j : L_\alpha \to L_\beta$, $\operatorname{crit}(j) = \kappa, \kappa^+ \leq \alpha$.*
(i) *There exists $j : L_\alpha \to L_\beta$, $\operatorname{crit}(j) = \kappa, (\kappa^+)^L \leq \alpha, \alpha \geq \omega_2$.*

PROOF. It is straightforward to show that these all follow from the existence of $0^{\#}$; using Theorem 4.2. Also (a), (b) imply the existence of $0^{\#}$ by Theorem 4.2. Conditions (d), (e) each easily imply (c), and we get $0^{\#}$ from (c) via Theorem 4.3. Condition (f) implies (a), since we get an elementary embedding $L \to L \simeq \operatorname{Ult}(L, U) =$ Ultrapower of L by U, where U consists of all constructible subsets of ω_1 containing a closed unbounded subset. (g) implies that $(\kappa^+)^L < \kappa^+$ for κ a sufficiently large cardinal; by taking κ singular we get $0^\#$ via condition (e). To see that (h) implies the existence of $0^\#$, define an ultrafilter U on constructible subsets of κ by: $X \in U$ iff $\kappa \in j(X)$. Then $\mathrm{Ult}(L, U)$ is well-founded, for if not then by Löwenheim-Skolem there would be an infinite descending chain in $\mathrm{Ult}(L_{\kappa^+}, U)$ which contradicts $\kappa^+ \leq \alpha$.

Finally we show that (i) implies the existence of $0^\#$. Define U as before by: $X \in U$ iff $\kappa \in j(X)$. First suppose that κ is at least ω_2. We shall argue that U is *countably complete*, i.e. that if $\langle X_n \mid n \in \omega\rangle$ belong to U then $\cap\{X_n \mid n \in \omega\}$ is nonempty. (This gives $0^\#$ as it implies that $\mathrm{Ult}(L, U)$ is well-founded.) By the Covering Theorem 4.3, there is $F \in L$ of cardinality ω_1 such that $X_n \in F$ for each n. Then as we have assumed that $\kappa \geq \omega_2$, F has L-cardinality less than κ. We may assume that F is a subset of $\mathcal{P}(\kappa) \cap L$, and hence as α is an L-cardinal, F belongs to L_α and there is a bijection $h : F \longleftrightarrow \gamma$ for some $\gamma < \kappa, h \in L_\alpha$. But then $F^* = \{X \in F \mid \kappa \in j(X)\}$ belongs to L_α as $X \in F^* \longleftrightarrow \kappa \in j(h^{-1})(h(X))$ and F^* has nonempty intersection as $j(F^*) = \mathrm{Range}(j \restriction F^*)$ and $\kappa \in \cap j(F^*)$. Thus $\{X_n \mid n \in \omega\}$ has nonempty intersection since it is a subset of F^*. If κ is less than ω_2 then we have $\alpha \geq \omega_2 \geq \kappa^+$ so we have a special case of (h). ⊣

The next theorem verifies (a) of transcendence property (#).

Theorem 4.5 (Beller (in Beller-Jensen-Welch [82]), Friedman [99]). *Suppose that G is P-generic over $\langle L, A\rangle$ and P is tame. Then $L[G] \models 0^\#$ does not exist.*

Proof. Suppose $p_0 \in P, p_0 \Vdash I =$ Silver indiscernibles is unbounded, $i < j$ in $I \to L_i \prec L_j$. Suppose that $p \leq p_0, p \Vdash \hat{\alpha} \in I$. Then $L_\alpha \prec L$ as this is true in any P-generic extension $\langle L[G], A, G\rangle, p \in G$. (By Löwenheim-Skolem we can assume that such a G exists for the sake of this argument.) Thus an L-Satisfaction predicate is definable over $\langle L, A\rangle$ as $L \models \varphi(x)$ iff for some $p \in P$ below p_0, some α with $x \in L_\alpha, p \Vdash \varphi(\hat{x})$ is true in L_α. This is a contradiction if $A = \emptyset$, for then L-satisfaction would be L-definable. But note that for any A such that $\langle L, A\rangle$ is amenable we can apply the same argument, using the fact that by Theorem 4.2(d), $\langle L_\alpha, A \cap L_\alpha\rangle \prec \langle L, A\rangle$ for α in a final segment of I. ⊣

Definition. A forcing P defined over a ground model $\langle L, A\rangle$ is **relevant** if there is a G P-generic over $\langle L, A\rangle$ which is definable (with parameters) over $L[0^\#]$.

Examples of Relevance

Assume that $0^\#$ exists. Then any $L[0^\#]$-*countable* $P \in L$ is relevant, as there are only countably many constructible subsets of P (using the fact that ω_1 is inaccessible in L). Note that this includes the case of any forcing $P \in L$ definable in L.

The situation is far less clear for uncountable $P \in L$. The next result treats the case of κ-Cohen forcing.

PROPOSITION 4.6. *Suppose κ is L-regular and let $P(\kappa)$ denote κ-Cohen forcing in L: conditions are constructible $p : \alpha \to 2$, $\alpha < \kappa$ and $p \le q$ iff p extends q.*

(a) *If κ has cofinality ω in $L[0^\#]$ then $P(\kappa)$ is relevant.*
(b) *If κ has uncountable cofinality in $L[0^\#]$ then $P(\kappa)$ is not relevant.*

PROOF. Let j_n denote the first n Silver indiscernibles $\ge \kappa$.

(a) We use the fact that $P(\kappa)$ is κ-distributive in L. Let $\kappa_0 < \kappa_1 < \dots$ be an ω-sequence in $L[0^\#]$ cofinal in κ. Then any $D \subseteq P(\kappa)$ in L belongs to $\mathrm{Hull}(\kappa_n \cup j_n)$ for some n, where Hull denotes Skolem hull in L. As $\mathrm{Hull}(\kappa_n \cup j_n)$ is constructible of L-cardinality $< \kappa$ we can use the κ-distributivity of $P(\kappa)$ to choose $p_0 \ge p_1 \ge \dots$ successively below any $p \in P(\kappa)$ to meet all dense $D \subseteq P(\kappa)$ in L.

(b) Note that in this case $\kappa \in \mathrm{Lim}\, I$, as otherwise $\kappa = \cup\{\kappa_n \mid n \in \omega\}$ where $\kappa_n = \cup(\kappa \cap \mathrm{Hull}(\bar\kappa + 1 \cup j_n)) < \kappa$, $\bar\kappa = \max(I \cap \kappa)$, and hence κ has $L[0^\#]$-cofinality ω. Suppose $G \subseteq P(\kappa)$ were $P(\kappa)$-generic over L. For any $p \in P(\kappa)$ let $\alpha(p)$ denote the domain of p. Define $p_0 \ge p_1 \ge \dots$ in G so that $\alpha(p_{n+1}) \in I$ and p_{n+1} meets all dense $D \subseteq P(\kappa)$ in $\mathrm{Hull}(\alpha(p_n) \cup j_n)$. Then $p = \cup\{p_n \mid n \in \omega\}$ meets all dense $D \subseteq P(\kappa)$ in $\mathrm{Hull}(\alpha \cup j)$ where $\alpha = \cup\{\alpha(p_n) \mid n \in \omega\} \in I$, $j = \cup\{j_n \mid n \in \omega\}$. But then p is $P(\alpha)$-generic over L, as every constructible dense $\bar D \subseteq P(\alpha)$ is of the form $D \cap P(\alpha)$ for some D as above. So p is not constructible, contradicting $p \in G$.

$\dashv$

As a consequence of Proposition 4.6(b) we see that the basic class forcing examples of Easton and Long Easton forcing are not relevant. However, we can rescue these forcings by restricting to successor cardinals, thereby not adding κ-Cohen sets for κ of uncountable $L[0^\#]$-cofinality.

Theorem 4.7. *Let P be* ***Easton forcing at Successors****: conditions are constructible $p : \alpha(p) \to L$ where $p(\alpha) = \emptyset$ unless α is a successor cardinal of L, in which case $p(\alpha) \in \alpha$-Cohen forcing; we also require that if α is L-inaccessible then $\{\beta < \alpha \mid p(\beta) \neq \emptyset\}$ is bounded in α and define $p \leq q$ iff $p(\alpha)$ extends $q(\alpha)$ for each $\alpha < \alpha(q)$. Then P is relevant.*

Proof. By induction on $i \in I =$ Silver indiscernibles we define $G(< i)$ to be $P(< i)$-generic over L, where $P(< i) =$ Easton forcing at Successors restricted to L_i. For $i = \min I$ take $G(< i)$ to be any $P(< i)$-generic (note that $P(< i)$ is countable in $L[0^\#]$). If $G(< i)$ has been defined we now define $G(< i^*)$ as follows (where $i < i^*$ are adjacent in I) : $P(< i^*)$ factors as $P(< i) \times P(i, i^*)$ where $P(i, i^*)$ is i^+-closed in L, so it suffices to define a $P(i, i^*)$-generic $G(i, i^*)$ and then $G(< i^*) = G(< i) \times G(i, i^*)$; is $P(< i^*)$-generic. To obtain $G(i, i^*)$, successively choose $p_0 \geq p_1 \geq \ldots$ in $G(i, i^*)$ so that p_{n+1} meets all dense $D \subseteq P(i, i^*)$ in $\mathrm{Hull}(i \cup j_n)$ where $j_n =$ first n Siver indiscernibles $\geq i$. Then $\{p \mid p \geq p_n$ for some $n\} = G(i, i^*)$.

Finally if $i \in \mathrm{Lim}\, I$, let $G(< i) = \cup\{G(< j) \mid j \in I \cap i\}$. Note that if $D \subseteq P(< i)$ is dense and constructible then for some $j \in I \cap i$, $D \cap P(< j)$ is dense and constructible and hence is met by $G(< j) \subseteq G(< i)$. So $G(< i)$ is $P(< i)$-generic. Similarly, $G = \cup\{G(< i) \mid i \in I\}$ is P-generic over L (and in fact meets all L-amenable dense $D \subseteq P$). ⊣

Reverse Easton Forcing is relevant, without restriction.

Theorem 4.8. *Let P be the basic example of Reverse Easton forcing. Then P is relevant.*

Proof. Recall that $P(< \alpha)$ has a dense subset of L-cardinality $\leq (\alpha^+)^L$ for each α. By induction on $i \in I$ we define $G(\leq i) = G(< i) * G(i)$ to be $P(\leq i)$-generic over L, where $P(\leq i) = P(< i) * P(i)$, the first $i + 1$ stages in the iteration defining P. We will have: $i \leq j$ in $I \to G(j)$ extends $G(i)$; this will enable us to get through limit stages. For $i = \min I$, take $G(\leq i)$ to be any $P(\leq i)$-generic in $L[0^\#]$. If $G(\leq i)$ has been defined and $i^* = I$-successor to i, then write $P(< i^*)$ as $P(\leq i) * P[i+1, i^*)$ and as $P(\leq i) \Vdash P[i+1, i^*)$ is i^+-closed we can select $G[i+1, i^*)$ to be $P[i+1, i^*)^{G(\leq i)}$-generic over $L[G(\leq i)]$ (the collection of dense sets that must be met is the countable union of subcollections of size i in $L[G(\leq i)]$, using the $\mathrm{Hull}(i \cup j_n)$'s as in the previous proof). Then $G(< i^*) = G(\leq i) * G[i + 1, i^*)$

is $P(< i^*)$-generic over L. We also choose $G(i^*)$ to be $P(i^*)^{G(<i^*)}$-generic over $L[G(< i^*)]$, extending the condition $G(i)$ in this forcing.

For $i \in \operatorname{Lim} I$ take $G(< i)$ to be $\cup\{G(< j) \mid j \in I \cap i\}$, as in the previous proof $G(< i)$ is $P(< i)$-generic over L. And we take $G(i) = \cup\{G(j) \mid j \in I \cap i\}$, which by our construction extends each $G(j)$, $j \in I \cap i$. Again we get genericity for $G(\leq i)$ from that of $G(\leq j)$, $j \in I \cap i$, as $G(< i)$, $G(i)$ extend $G(< j)$, $G(j)$ respectively for each $j \in I \cap i$. ⊣

Before turning to **Long Easton forcing at Successors** (obtained from Easton forcing at Successors by dropping the support condition that $\{\beta < \alpha \mid p(\beta) \neq \emptyset\}$ be bounded in α for L-inaccessible α), we establish the relevance of **Thin Easton forcing at Successors**. The latter is obtained by weakening the support condition in Easton forcing at Successors to: $\{\beta < \alpha \mid p(\beta^+) \neq \emptyset\}$ is nonstationary in α for L-inaccessible α.

THEOREM 4.9. *Let P be Thin Easton forcing at Successors. Then P is relevant.*

PROOF. Factor P as $P(\leq \gamma) \times P(> \gamma)$ for each L-cardinal γ; if γ is a limit L-cardinal then $P(\leq \gamma)$ can be identified with $P(< \gamma)$. Let i be any indiscernible and for any n let j_n be the first n indiscernibles $\geq i$. We can define $p^i_0 \geq p^i_1 \geq \dots$ in $P(\leq i^+)$ such that if $D \subseteq P(\leq i^+)$ is dense and belongs to $\operatorname{Hull}(\gamma^+ \cup j_n)$ then p^i_{n+1} reduces D below γ^+ for any L-cardinal $\gamma < i$. This is possible by successively extending on $[\gamma^{++}, i^+]$ (without violating the nonstationary support requirement). Let $G^i_0 = \{p \in P(\leq i^+) \mid p \geq p^i_n \text{ for some } n\}$.

G^i_0 is *not* $P(\leq i^+)$-generic over L as $p \in G^i_0 \to p(j^+) = \emptyset$ for all $j \in I \cap i$. Notice that for $i_0 < i_1 < \dots < i_n \leq i$ in I, $G^{i_0}_0 \cup \dots \cup G^{i_n}_0$ is a compatible set of conditions. We take $G(\leq i^+) = \{p \in P(\leq i^+) \mid p \geq q_0 \wedge \dots \wedge q_n$ for some $q_l \in G^{i_l}_0, i_0 < \dots < i_n \leq i$ in $I\}$. Now we claim that $G(\leq i^+)$ is $P(\leq i^+)$- generic over L. Indeed if $D \subseteq P(\leq i^+)$ is dense and belongs to $\operatorname{Hull}(\{k_0, \dots, k_m\} \cup j_n)$ with $k_0 < \dots < k_m < i$ in I then p^i_{n+1} reduces D below k^+_m, $p^i_{n+1} \wedge p^{k_m}_{n+2}$ reduces D below $k^+_{m-1}, \dots$ and eventually we get $p^i_{n+1} \wedge p^{k_m}_{n+2} \wedge \dots \wedge p^{k_0}_{n+m+2}$ in $G(\leq i^+)$ meeting D.

Now note that in the above we could have chosen our initial $p^i_0 \in P(\leq i)$ to reduce every dense $D \subseteq P(\ll i) = P \cap L_i$ in $\operatorname{Hull}(\gamma^+ \cup \{i\})$ below γ^+, for any $\gamma < i$. Thus the resulting generic $G(\leq i^+)$ meets every dense $D \subseteq P(\ll i)$ definable over L_i. Now let $G = \cup\{G(\leq i^+) \mid i \in I\}$ and we see that G is P-generic over L. ⊣

In the above proof we use thin supports to guarantee that for $i < j$ in I, the "pre-generics" G_0^i, G_0^j agree at i^+ (indeed they equal $\emptyset$ at i^+). A less severe restriction is to require *coherence on a* CUB:

DEFINITION. Let P denote Long Easton forcing at Successors and suppose that p belongs to $P(\leq \kappa^+)$, where κ is L-regular. For any $\xi \in [\kappa, \kappa^+)$ let f_ξ be the L-least 1-1 function from κ onto ξ. For $s \in P(\kappa^+) = \kappa^+$-Cohen forcing and $\alpha < \kappa$ define s_α as follows: If $\xi = \text{length}(s) \leq \kappa$ or $\alpha \neq \kappa \cap f_\xi[\alpha]$ then $s_\alpha = \emptyset$. Otherwise s_α has domain $[\alpha, \bar{\xi})$ where $\bar{\xi} = \text{ordertype}\, f_\xi[\alpha]$ and $s_\alpha(\delta) = s(f_\xi(\delta))$. We say that p is **coherent at** κ if $p(\kappa^+)_\alpha, p(\alpha^+)$ are compatible for CUB-many $\alpha < \kappa$. A condition p in P is **coherent** if for each L-inaccessible κ in the domain of p, p is coherent at κ. **Coherent Easton forcing at Successors** is the forcing whose conditions are the coherent conditions in Long Easton forcing at Successors.

THEOREM 4.10. *Let P be Coherent Easton forcing at Successors. Then P is relevant.*

PROOF. Follow the proof of the previous Theorem. The only new observation is that by virtue of strong coherence at indiscernibles, we again have the compatibility of G_0^i, G_0^j for $i < j$ in I. ⊣

Remark. Thin Easton forcing at Successors and Coherent Easton forcing at Successors serve as prototypes for Jensen coding, introduced in the next section. In Jensen coding, conditions are sequences of pairs (p_α, p_α^*) where strong coherence is used on the "coding strings" p_α and thinness is used on the "restraints" p_α^*.

Finally we turn to Long Easton forcing at Successors.

THEOREM 4.11. *Let P be Long Easton forcing at Successors. Then P is relevant.*

PROOF. Suppose that p belongs to P and i is an indiscernible. We say that p is *coherent at* i if $p(i^+), \pi(p)(i^+)$ are compatible, where $\pi : L \to L$ is an elementary embedding with critical point i. Equivalently: $p(i^+)_\alpha, p(\alpha^+)$ are compatible for all α in a set X belonging to the L-ultrafilter derived from the embedding π. It suffices to show that if p belongs to $P(\leq i^+)$, is coherent at indiscernibles $\leq i$ and $D \subseteq P(\leq i^+), D \in L$ is L-definable from indiscernibles $\geq i$ then p has an extension meeting D which is coherent at indiscernibles $\leq i$. For then, we can repeat the proof of Theorem 4.9, using conditions which are coherent at indiscernibles $\leq i$ to construct G_0^i, and therefore again obtain the compatibility of G_0^i, G_0^j for $i < j$ in I.

Given p, D as above, inductively extend $p(\alpha^+), \alpha < i, \alpha$ an L-limit cardinal to $q(\alpha^+)$ as follows: if $q \restriction \alpha$ has been defined then let $q(\alpha^+)$ be least so that for some least $r_\alpha \in P(< \alpha), r_\alpha \cup \{q(\alpha^+)\}$ extends $q \restriction \alpha \cup \{p(\alpha^+)\}$ and meets D. Now choose X in the ultrafilter derived from π (X containing all indiscernibles $< i$) such that the r_α cohere for α in X to a condition r in $P(< i)$. Also define $r(i^+)$ to be $\pi(r)(i^+)$. Then r extends p, is coherent at indiscernibles $\leq i$ and meets D. ⊣

Indiscernible Preservation

Though we have shown a number of variants of Easton forcing to be relevant, we can ask for more: that the generic classes preserve indiscernibles. This will be important in the next section, where Jensen coding is introduced, as we can only code a class by a real (in $L[0^\#]$) if the class preserves (a periodic subclass of) the Silver indiscernibles.

Definition. A class $A \subseteq L$ **preserves indiscernibles** if I is a class of indiscernibles for the structure $\langle L[A], A\rangle$.

Theorem 4.12. *For each of Easton at Successors, Reverse Easton, Thin Easton at Successors, Coherent Easton at Successors and Long Easton at Successors there is a generic class G that preserves indiscernibles.*

Proof. The generic classes built earlier for Thin Easton at Successors, Coherent Easton at Successors and Long Easton at Successors preserve indiscernibles. We now treat the case of Reverse Easton forcing. It suffices to build $H \subseteq L_{i_\omega}$ which is $P(< i_\omega)$-generic over L_{i_ω} and such that $t(j_1 \dots j_n) \in H$ iff $t(j'_1 \dots j'_n) \in H$ whenever $j_1 < \cdots < j_n$, $j'_1 < \cdots < j'_n$ belong to $I \cap i_\omega$, $i_\omega = \omega^{\text{th}}$ indiscernible. For then define $t(k_1 \dots k_n) \in G$ iff $t(i_1 \dots i_n) \in H$, $i_1 < \cdots < i_n$ the first n indiscernibles. This is well-defined using the above property of H. And G is P-generic over L: it suffices to consider predense $D \in L$ as P has the ∞-chain condition. Now write $D \in L$ as $s(l_1 \dots l_m)$, $l_1 < \cdots < l_m$ in I, and then $\overline{D} = s(i_1 \dots i_m)$ is predense on $P(< i_\omega)$. If $\bar{p} = t(i_1 \dots i_n) \in H$ meets $\overline{D}$ then $p = t(l_1 \dots l_m, l_{m+1} \dots l_n)$ meets D, where $l_m < l_{m+1} < \cdots < l_n$ belong to I. Also $p \in G$ by definition of G. Finally, note that if $k_1 < \cdots < k_m < l_1 < \cdots < l_m$ and $l_1, \dots, l_m$ are in $\operatorname{Lim} I$, $k_1, \dots, k_m$ in I then for any φ, $\langle L[G], G\rangle \vDash \varphi(k_1 \dots k_m) \longleftrightarrow \varphi(l_1 \dots l_m)$ by the Truth Lemma and the fact that G obeys the same invariance property that characterized H. So I is a class of indiscernibles for $\langle L[G], G\rangle$.

Now we build H. Let $H_2 \subseteq P(< i_2)$ be a $P(< i_2)$-generic in $L[0^\#]$ and $H_1 = H_2 \cap P(< i_1)$. We must now define $H_3 \subseteq P(< i_3)$ to be $P(< i_3)$-generic so that $t(i_1, \vec{j}) \in H_2$ iff $t(i_2, \vec{j}) \in H_3$, where $\vec{j}$ is an increasing sequence from $I - i_\omega$. Note that $H_2(i_1)$, a subset of i_1 generic over $L[H_1]$, is a condition in the i_2-Cohen forcing defined over $L[H_2]$; choose $H_3(i_2)$ to be a generic for this forcing extending $H_2(i_1)$. Now note that for each n there is $t_n(i_1, \vec{j}_n) = p_n \in H_2$ which reduces all predense $D \subseteq P(< i_2)$ in $\mathrm{Hull}(i_1 \cup \{i_1, k_1 \dots k_n\})$ below i_1, where $i_\omega \leq k_1 < \cdots < k_n$ belong to I, using the i_1^+-distributivity of $P(> i_1)^{H_2(\leq i_1)}$ in $L[H_2(\leq i)]$. So if we define $H_3' = \{t_n(i_2, \vec{j}_n) \mid n \in \omega\}$ we have that H_3' reduces all predense $D \subseteq P(< i_3)$, $D \in L$ below i_2. So the desired H_3 can be defined by $H_3 = \{p \in P(< i_3) \mid p(\leq i_2) \in H_3(\leq i_2),\ p \text{ compatible with } H_3'\}$. By construction, $t(i_1, \vec{j}) \in H_2$ iff $t(i_2, \vec{j}) \in H_3$. Note that H_3 was uniquely determined by this last condition, once a choice of $H_3(i_2)$ was made.

H_4 is uniquely determined by $P(< i_4)$-genericity and the condition $t(i_1, i_2, \vec{j}) \in H_3$ iff $t(i_2, i_3, \vec{j}) \in H_4$, as the forcing to add $H_3(i_2)$ is i_1^+-distributive (and the forcing to add $H_3(> i_2)$ is i_2^+-distributive). We must check that $t(i_1, i_3, \vec{j}) \in H_4$ iff $t(i_2, i_3, \vec{j}) \in H_4$. Now any condition in H_4 is extended by one of the form $p = (p_0, p_1)$ where $p_0 \in H_4(\leq i_3)$ and $p_1 = t(i_3, \vec{j})$, as such p reduce all dense $D \subseteq P(< i_4)$, $D \in L$ below i_3. So it suffices to show that $t(i_1, i_3, \vec{j}) \in H_4(\leq i_3)$ iff $t(i_2, i_3, \vec{j}) \in H_4(\leq i_3)$. By definition of H_4 we have $t(i_2, i_3, \vec{j}) \in H_4(\leq i_3)$ iff $t(i_1, i_2, \vec{j}) \in H_3(\leq i_2)$. But the latter implies that $t(i_1, i_2, \vec{j}) = t(i_1, i_3, \vec{j})$ and as $H_3(\leq i_2)$ extends $H_2(\leq i_1)$ we have that $H_4(\leq i_3)$ extends $H_3(\leq i_2)$. So $t(i_1, i_2, \vec{j}) \in H_3(\leq i_2)$ iff $t(i_1, i_2, \vec{j}) \in H_4(\leq i_3)$ iff $t(i_1, i_3, \vec{j}) \in H_4(\leq i_3)$.

In general define H_{m+3} by the condition $t(i_m, i_{m+1}, \vec{j}) \in H_{m+2}$ iff $t(i_{m+1}, i_{m+2}, \vec{j}) \in H_{m+3}$. As above we get that H_{m+3} is $P(< i_{m+3})$-generic and $t(i_1 \dots i_{m+1}, \vec{j}) \in H_{m+2}$ iff $t(i_1 \dots i_m, i_{m+2}, \vec{j}) \in H_{m+3}$. Finally let $H = \cup\{H_m \mid m \in \omega\}$. Then H is $P(< i_\omega)$-generic over L and for any $k_1 < \cdots < k_{l+2} < \vec{j}$ in $I, k_{l+2} < i_\omega \leq \vec{j}$ we have $t(k_1 \dots k_{l+1}, \vec{j}) \in H$ iff $t(k_1 \dots k_l, k_{l+2}, \vec{j}) \in H$. This is enough to imply that $t(\vec{k}_0) \in H$ iff $t(\vec{k}_1) \in H$ whenever $\vec{k}_0, \vec{k}_1$ are increasing sequences from $I \cap i_\omega$. This completes the proof in the case of Reverse Easton forcing.

Easton forcing at Successors can be handled in the same way without need to consider $H(i)$ for $i \in I$, as $H(\alpha)$ is nontrivial only when α is a successor L-cardinal. (Indeed, without the latter restriction the construction fails as there is no available choice for $H(i_2)$.) $\dashv$

5. The Coding Theorem

Class forcing became an important tool in set theory as a result of the following theorem of Jensen (see Beller-Jensen-Welch [82]):

THEOREM 5.1. *(Coding Theorem) Suppose $\langle M, A\rangle$ is a ground model. Then there is an $\langle M, A\rangle$-definable class forcing P such that if $G \subseteq P$ is P-generic over $\langle M, A\rangle$ then:*

(a) $\langle M[G], A, G\rangle \models \mathrm{ZFC}$.

(b) *$M[G] \models V = L[R]$, $R \subseteq \omega$ and $\langle M[G], A, G\rangle \models A, G$ are definable from the parameter R.*

Before discussing the proof of this Theorem, we mention the following corollary, which constitutes a partial positive solution to Solovay's Genericity Problem (for set-genericity):

COROLLARY 5.2. *There is an L-definable class forcing for producing a real R which is not set-generic over L.*

PROOF. Let P_0 be Easton Forcing and let $P_0 * P_1 = P$ be the 2-step iteration where P_1 adds a real R as in Theorem 5.1 such that G_0 is definable over $L[R]$, G_0 denoting the P_0-generic. Then in $L[R]$ there are κ-Cohen sets for every L-regular κ. Thus R is not set-generic over L as no forcing of size κ can add a κ^+-Cohen set. ⊣

In fact R as in Corollary 5.2 can be chosen to satisfy $R <_L 0^\#$, but this property makes use of the relevance of Jensen coding, a topic to be discussed later.

The proof of the Coding Theorem is far easier if one makes the further assumption that $0^\# \notin M$. Indeed, with this extra hypothesis there is a proof, which we provide below, making no use of Jensen's fine structure theory; instead one uses the following consequence of Jensen's Covering Theorem (Theorem 4.3), expressed by Theorem 4.4(i):

PROPOSITION 5.3. *Suppose $0^\#$ does not exist, $j : L_\alpha \to L_\beta$ is Σ_1-elementary, $\alpha \geq \omega_2$. If $\kappa = \mathrm{crit}(j)$ then $\alpha < (\kappa^+)^L$.*

We now give a brief introduction to the coding proof, assuming $0^\# \notin M$. We may assume that $M \models \mathrm{GCH}$, as this can be easily arranged by a preliminary class forcing. Moreover we need not code into a real R; it suffices to code into a *reshaped* subset of ω_1:

DEFINITION. $b \subseteq \omega_1$ is **reshaped** if $\xi < \omega_1 \to \xi$ is countable in $L[b \cap \xi]$.

The following result of Jensen-Solovay [70] provides one of the key ideas in the proof.

PROPOSITION 5.4. *Suppose $V = L[b]$, b a reshaped subset of ω_1. Then there is a* ccc *forcing R^b for adding a real R such that $b \in L[R]$.*

PROOF. Using the fact that b is reshaped we may define $\langle R_\xi \mid \xi < \omega_1\rangle$ by: $R_\xi = L[b\cap\xi]$-least real distinct from the $R_{\xi'}$, $\xi' < \xi$. Separate the R_ξ's by setting $R^*_\xi = \{n \mid n \text{ codes a finite initial segment of } R_\xi\}$.

A condition in R^b is $p = (s(p), s^*(p))$ where $s(p)$ is a finite subset of ω, $s^*(p)$ is a finite subset of b. Extension is defined by: $p \leq q$ iff $s(p) \supseteq s(q)$, $s^*(p) \supseteq s^*(q)$ and $\xi \in s^*(q) \to s(p) - s(q)$ is disjoint from R^*_ξ. This is ccc as $s(p) = s(q) \to p, q$ are compatible. If G is R^b-generic then let $R = \cup\{s(p) \mid p \in G\}$. We get:

$$(*)\ \xi \in b \longleftrightarrow R \cap R^*_\xi \text{ finite.}$$

Thus given R we can test "$\xi \in b$" if we know R_ξ; as R_ξ is computable in $L[b\cap\xi]$ this gives an inductive calculation of $B \cap \xi$ from R. ⊣

There is a perfectly analogous notion of *reshaped subset of* κ^+ for any infinite cardinal κ and if κ is an infinite *successor* cardinal, an analogous forcing R^b for b a reshaped subset of κ^+.

Now we do not necessarily have reshaped sets in our ground model; instead we must force them. A *reshaped string at* κ is a function $s : \alpha \to 2$, $\alpha < \kappa^+$ such that $\xi \leq \alpha \to L[s \restriction \xi] \models \text{Card}\,\xi \leq \kappa$. Reshaped strings at κ of arbitrary length $\alpha < \kappa^+$ do exist and serve to approximate the desired reshaped subsets of κ^+.

We now give a rough description of the forcing conditions. P consists of sequences $p = \langle (p_\alpha, p^*_\alpha) \mid \alpha \in \text{Card}, \alpha \leq \alpha(p)\rangle$ where $\alpha(p) \in \text{Card}$ and:

(a) $p_{\alpha(p)}$ is a reshaped string at $\alpha(p)$, $p^*_{\alpha(p)} = \emptyset$.
(b) For $\alpha \in \text{Card} \cap \alpha(p)$, $(p_\alpha, p^*_\alpha) \in R^{p_{\alpha^+}}$, the forcing for coding p_{α^+}, $A \cap \alpha^+$ by a subset of α^+ using reshaped strings at α.
(c) For α a limit cardinal, $\alpha \leq \alpha(p)$, $p \restriction \alpha$ "exactly codes" p_α.
(d) For α inaccessible, $\alpha \leq \alpha(p)$ there is a CUB $C \subseteq \alpha$ such that $\beta \in C \to p^*_\beta = \emptyset$.

Clause (d) is oversimplified in that "inaccessible" should really be (something like) "$L[p_\alpha] \vDash \alpha$ is inaccessible" and C should be required to belong to (something like) $L[p_\alpha]$. Clause (c) refers to the limit coding, as yet undefined. The key idea that enables one to carry out a fine structure-free proof of the Coding Theorem is the use of *coding delays* in the limit coding. The details are supplied in the proof below.

The 2 main properties of P that must be demonstrated are:

(Extendibility) Suppose $p \in P$ and $f : \alpha \to \alpha$, $f(\beta) < \beta^+$ for successor cardinals $\beta < \alpha$. Then there exists $q \leq p$, $\operatorname{length} q_\beta \geq f(\beta)$ for each successor cardinal $\beta < \alpha$.

(Distributivity) Suppose that D_i is i^+**-dense on** P for each $i < \alpha$: For all p there is $q \leq p$, $q \in D_i$ and $(q_\beta, q^*_\beta) = (p_\beta, p^*_\beta)$ for all $\beta \leq i$. Then for all p there is $q \leq p$, q meets each D_i.

Proposition 5.3 is used to facilitate the proof of Distributivity. Extendibility is not difficult, taking advantage of the coding delays.

Proof of Theorem 5.1 (assuming $0^\# \notin M$). We make the following assumption about the predicate A: If H_α, α an infinite $L[A]$-cardinal, denotes $\{x \in L[A] \mid$ transitive closure (x) has $L[A]$-cardinality $< \alpha\}$ then $H_\alpha = L_\alpha[A]$. This is easily arranged using the fact that the GCH holds in $L[A]$.

Let Card denote all infinite $L[A]$-cardinals. Also $\text{Card}^+ = \{\alpha^+ \mid \alpha \in \text{Card}\}$ and $\text{Card}' =$ all uncountable limit cardinals.

Let α belong to Card.

Definition **(Strings)**. S_α consists of all $s : [\alpha, |s|) \longrightarrow 2$, $\alpha \leq |s| < \alpha^+$ such that $|s|$ is a multiple of α and for all $\eta \leq |s|$, $L_\delta[A \cap \alpha, s \restriction \eta] \vDash \text{Card}(\eta) \leq \alpha$ for some $\delta < (\eta^+)^L \cup \omega_2$.

Thus for $\alpha = \omega, \omega_1$ elements of S_α are "reshaped" in the natural sense mentioned above, but for $\alpha \geq \omega_2$ we insist that $s \in S_\alpha$ be "quickly reshaped" in that $\eta \leq |s|$ is collapsed relative to $A \cap \alpha$, $s \restriction \eta$ before the next L-cardinal. This will be important when we use $\sim 0^\#$ to establish cardinal-preservation, via Proposition 5.3. Elements of S_α are called "strings". Note that we allow the empty string $\emptyset_\alpha \in S_\alpha$, where $|\emptyset_\alpha| = \alpha$. For $s, t \in S_\alpha$ write $s \leq t$ for $s \subseteq t$ and $s < t$ for $s \leq t, s \neq t$.

Definition **(Coding Structures).** For $s \in S_\alpha$ define $\mu^{<s}$, μ^s inductively by: $\mu^{<\emptyset_\alpha} = \alpha, \mu^{<s} = \cup\{\mu^t \mid t < s\}$ for $s \neq \emptyset_\alpha$ and μ^s = least limit of limit ordinals $\mu > \mu^{<s}$ such that $L_\mu[A \cap \alpha, s] \vDash s \in S_\alpha$. And $\mathcal{A}^s = L_{\mu^s}[A \cap \alpha, s]$.

Thus by definition there is $\delta < \mu^s$ such that $L_\delta[A \cap \alpha, s] \vDash \text{Card}(|s|) \leq \alpha$ and $L_{\mu^s} \vDash \text{Card}(\delta) \leq |s|$, when $\alpha \geq \omega_2$.

Definition **(Coding Apparatus).** For $\alpha > \omega$, $s \in S_\alpha$, $i < \alpha$ let $H^s(i) = \Sigma_1$ Skolem hull of $i \cup \{A \cap \alpha, s\}$ in $\mathcal{A}^s$ and $f^s(i)$ = ordertype $(H^s(i) \cap \text{ORD})$. For $\alpha \in \text{Card}^+, b^s = \text{Range}(f^s \restriction B^s)$ where B^s = the successor elements of $\{i < \alpha \mid i = H^s(i) \cap \alpha\}$.

Using the above we will construct a tame, cofinality-preserving forcing P for coding $\langle L[A], A\rangle$ by a subset G_ω of ω_1 which is reshaped in the sense that proper initial segments of (the characteristic function of) G_ω belong to S_ω.

Definition **(A Partition of the Ordinals).** Let B, C, D, E denote the classes of ordinals congruent to $0, 1, 2, 3$ mod 4, respectively. Also for any ordinal α and $X = B, C, D$ or E, we write α^X for the α^{th} element of X (when X is listed in increasing order).

Definition **(The Successor Coding).** Suppose $\alpha \in \text{Card}$ $s \in S_{\alpha^+}$. A **condition in** R^s is a pair (t, t^*) where $t \in S_\alpha, t^* \subseteq \{b^{s\restriction\eta} \mid \eta \in [\alpha^+, |s|)\} \cup |t|$, $\text{Card}(t^*) \leq \alpha$. Extension of conditions is defined by: $(t_0, t_0^*) \leq (t_1, t_1^*)$ iff $t_1 \leq t_0, t_1^* \subseteq t_0^*$ and:

(a) $|t_1| \leq \gamma^B < |t_0|, \gamma \in b^{s\restriction\eta} \in t_1^* \longrightarrow t_0(\gamma^B) = 0$ or $s(\eta)$.
(b) $|t_1| \leq \gamma^C < |t_0|, \gamma = \langle\gamma_0, \gamma_1\rangle, \gamma_0 \in A \cap t_1^* \longrightarrow t_0(\gamma^C) = 0$.

In (b) above, $\langle\cdot,\cdot\rangle$ is an L-definable pairing function on ORD so that $\text{Card}(\langle\gamma_0,\gamma_1\rangle) = \text{Card}\,\gamma_0 + \text{Card}\,\gamma_1$ in L for infinite γ_0, γ_1. An R^s-generic over $\mathcal{A}^s$ is determined by a function $T : \alpha^+ \longrightarrow 2$ such that $s(\eta) = 0$ iff $T(\gamma^B) = 0$ for sufficiently large $\gamma \in b^{s\restriction\eta}$ and such that for $\gamma_0 < \alpha^+ : \gamma_0 \in A$ iff $T(\langle\gamma_0,\gamma_1\rangle^C) = 0$ for sufficiently large $\gamma_1 < \alpha^+$.

Now we come to the definition of the Limit Coding, which incorporates the idea of "coding delays." Suppose $s \in S_\alpha$, $\alpha \in \text{Card}'$ and $p = \langle(p_\beta, p_\beta^*) \mid \beta \in \text{Card} \cap \alpha\rangle$ where $p_\beta \in S_\beta$ for each $\beta \in \text{Card} \cap \alpha$. A natural definition of "p codes s" would be: for $\eta < |s|$, $p_\beta(f^{s\restriction\eta}(\beta)) = s(\eta)$ for sufficiently large $\beta \in \text{Card} \cap \alpha$. There are a number of problems with this definition however. First, to avoid conflict with the Successor Coding we should use $f^{s\restriction\eta}(\beta)^D$ instead of $f^{s\restriction\eta}(\beta)$. Second, to lessen conflict with codings at $\beta \in \mathrm{Card}' \cap \alpha$ we only require the above for $\beta \in \mathrm{Card}^+ \cap \alpha$. However there are still serious problems in making sure that the coding of s is consistent with the coding of p_β by $p \restriction \beta$ for $\beta \in \mathrm{Card}' \cap \alpha$.

We introduce coding delays to facilitate extendibility of conditions. The rough idea is to code not using $f^{s\restriction\eta}(\beta)^D$, but instead just after the least ordinal $\geq f^{s\restriction\eta}(\beta)^D$ where p_β takes the value 1. In addition, we "precode" s by a subset of α, which is then coded with delays by $\langle p_\beta \mid \beta \in \mathrm{Card} \cap \alpha\rangle$; this "indirect" coding further facilitates extendibility of conditions.

Definition. Suppose $\alpha \in \mathrm{Card}$, $X \subseteq \alpha$, $s \in S_\alpha$. Let $\tilde{\mu}^s$ be defined just as we defined μ^s but with the requirement "limit of limit ordinals" replaced by the weaker condition "limit ordinal". Then note that $\tilde{\mathcal{A}}^s = L_{\tilde{\mu}^s}[A \cap \alpha, s]$ belongs to $\mathcal{A}^s$, contains s and $\Sigma_1\,\mathrm{Hull}(\alpha \cup \{A \cap \alpha, s\})$ in $\tilde{\mathcal{A}}^s = \tilde{\mathcal{A}}^s$. Now X **precodes** s if X is the Σ_1 theory of $\tilde{\mathcal{A}}_s$ with parameters from $\alpha \cup \{A \cap \alpha, s\}$ (viewed as a subset of α).

Definition **(Limit Coding)**. Suppose $s \in S_\alpha$, $\alpha \in \mathrm{Card}'$ and $p = \langle (p_\beta, p^*_\beta) \mid \beta \in \mathrm{Card} \cap \alpha\rangle$ where $p_\beta \in S_\beta$ for each $\beta \in \mathrm{Card} \cap \alpha$. We wish to define "$p$ codes s". First we define a sequence $\langle s_\gamma \mid \gamma \leq \gamma_0\rangle$ of elements of S_α as follows. Let $s_0 = \emptyset_\alpha$. For limit $\gamma \leq \gamma_0$, $s_\gamma = \cup\{s_\delta \mid \delta < \gamma\}$. Now suppose s_γ is defined and let $f_p^{s_\gamma}(\beta) =$ least $\delta \geq f^{s_\gamma}(\beta)$ such that $p_\beta(\delta^D) = 1$, if such a δ exists. If for cofinally many $\beta \in \mathrm{Card}^+ \cap \alpha, f_p^{s_\gamma}(\beta)$ is undefined, then set $\gamma_0 = \gamma$. Otherwise define $X \subseteq \alpha$ by: $\delta \in X$ iff $p_\beta((f_p^{s_\gamma}(\beta) + 1 + \delta)^D) = 1$ for sufficiently large $\beta \in \mathrm{Card}^+ \cap \alpha$. If Even (X) precodes an element t of S_α extending s_γ such that $f_p^{s_\gamma}, X \in \mathcal{A}^t$ then set $s_{\gamma+1} = t$. Otherwise let $s_{\gamma+1}$ be $s_\gamma * X^E$, if this results in $f_p^{s_\gamma} \in \mathcal{A}^{s_{\gamma+1}}$; if not, then $\gamma_0 = \gamma$. Now p **exactly codes** s if $s = s_\gamma$ for some $\gamma \leq \gamma_0$ and p **codes** s if $s \leq s_\gamma$ for some $\gamma \leq \gamma_0$.

Note that the Successor Coding only restrains p_β from taking certain nonzero values, so there is no conflict between the Successor Coding and these delays. The advantage of delays is that they give us more control over *where* the Limit Coding takes place, thereby enabling us to avoid conflict between the Limit Codings at different cardinals.

Definition **(The Conditions)**. A **condition in** P is a sequence $p = \langle (p_\alpha, p^*_\alpha) \mid \alpha \in \mathrm{Card}, \alpha \leq \alpha(p)\rangle$ where $\alpha(p) \in \mathrm{Card}$ and:

(a) $p_{\alpha(p)} \in S_{\alpha(p)}, p^*_{\alpha(p)} = \emptyset$.
(b) For $\alpha \in \mathrm{Card} \cap \alpha(p)$, $(p_\alpha, p^*_\alpha) \in R^{p_{\alpha^+}}$.

(c) For $\alpha \in \mathrm{Card}', \alpha \leq \alpha(p)$, $p \restriction \alpha \in \mathcal{A}^{p_\alpha}$, $p \restriction \alpha$ exactly codes p_α.
(d) For $\alpha \in \mathrm{Card}', \alpha \leq \alpha(p), \alpha$ inaccessible in $\mathcal{A}^{p_\alpha} \longrightarrow$ there exists a CUB $C \subseteq \alpha$, $C \in \mathcal{A}^{p_\alpha}$ such that $\beta \in C \longrightarrow p^*_\beta = \emptyset$.

For $\alpha \in \mathrm{Card}, P^{<\alpha}$ denotes the set of all conditions p such that $\alpha(p) < \alpha$. Conditions are ordered by: $p \leq q$ iff $\alpha(p) \geq \alpha(q)$, $p(\alpha) \leq q(\alpha)$ in $R^{p_{\alpha^+}}$ for $\alpha \in \mathrm{Card} \cap \alpha(p) \cap (\alpha(q)+1)$ and $p_{\alpha(p)}$ extends $q_{\alpha(p)}$ if $\alpha(q) = \alpha(p)$. Also for $s \in S_\alpha$, $\omega < \alpha \in \mathrm{Card}$, P^s denotes $P^{<\alpha}$ together with all $p \restriction \alpha$ for conditions p such that $\alpha(p) = \alpha$, $p_{\alpha(p)} \leq s$. To order conditions in P^s, define $p^+ = p$ for $p \in P^{<\alpha}$ and for $p \in P^s - P^{<\alpha}$, $p^+ \restriction \alpha = p$ and $p^+(\alpha) = (s \restriction \eta, \emptyset)$ where η is least such that $p \in P^{s \restriction \eta}$; then $p \leq q$ iff $p^+ \leq q^+$ as conditions in P. Finally, $P^{<s} = \cup\{P^{s\restriction\eta} \mid \eta < |s|\} \cup P^{<\alpha}$.

It is worth noting that (c) above implies that f^{p_α} dominates the coding of p_α by $p \restriction \alpha$, in the sense that f^{p_α} strictly dominates each $f^{p_\alpha \restriction \eta}_{p \restriction \alpha}$, $\eta < |p_\alpha|$ on a tail of $\mathrm{Card}^+ \cap \alpha$. The purpose of (d) is to guarantee that extendibility of conditions at (local) inaccessibles is not hindered by the Successor Coding (see the proof of Extendibility below).

We now embark on a series of lemmas which together show that P preserves cofinalities and if G is P-generic over $\langle L[A], A\rangle$ then for some reshaped $X \subseteq \omega_1$, $L[A,G] = L[X]$ and A is $L[X]$-definable from the parameter X. Then X can be coded by a real via a ccc forcing using the Solovay method described earlier.

Lemma 5.5 (Distributivity for R^s). *Suppose $\alpha \in \mathrm{Card}$, $s \in S_{\alpha^+}$. Then R^s is α^+-distributive in $\mathcal{A}^s$: if $\langle D_i \mid i < \alpha\rangle \in \mathcal{A}^s$ is a sequence of dense subsets of R^s and $p \in R^s$ then there is $q \leq p$ such that q meets each D_i.*

Proof. Choose $\mu < \mu^s$ to be a large enough limit ordinal such that $p, \langle D_i \mid i < \alpha\rangle$, $\mathcal{A}^{<s} \in \mathcal{A} = L_\mu[A \cap \alpha^+, s]$. Let $\langle \alpha_i \mid i < \alpha\rangle$ enumerate the first α elements of $\{\beta < \alpha^+ \mid \beta = \alpha^+ \cap \Sigma_1 \text{ Hull of } (\beta \cup \{p, \langle D_i \mid i < \alpha\rangle, \mathcal{A}^{<s}\}) \text{ in } \mathcal{A}\}$.

Now write p as (t_0, t^*_0) and successively extend to (t_i, t^*_i) for $i \leq \alpha$ as follows: (t_{i+1}, t^*_{i+1}) is the least extension of (t_i, t^*_i) meeting D_i such that t^*_{i+1} contains $\{b^{s\restriction\eta} \mid \eta \in H_i \cap [\alpha^+, |s|)\}$ where $H_i = \Sigma_1$ Hull of $\alpha_i \cup \{p, \langle D_i \mid i < \alpha\rangle, \mathcal{A}^{<s}\}$ in $\mathcal{A}$ and: (a) If $b^{s\restriction\eta} \in t^*_i, s(\eta) = 1$ then $t_{i+1}(\gamma^\beta) = 1$ for some $\gamma \in b^{s\restriction\eta}, \gamma > |t_i|$. (b) If $\gamma_0 \notin A, \gamma_0 < |t_i|$ then $t_{i+1}(\langle\gamma_0, \gamma_1\rangle^C) = 1$ for some $\gamma_1 > |t_i|$.

The lemma reduces to:

Claim. $(t_\lambda, t^*_\lambda) =$ greatest lower bound to $\langle (t_i, t^*_i) \mid i < \lambda\rangle$ exists for limit $\lambda \leq \alpha$.

PROOF OF CLAIM. We must show that $t_\lambda = \cup\{t_i \mid i < \lambda\}$ belongs to S_α. Note that $\langle t_i \mid i < \lambda\rangle$ is definable over $\overline{H}_\lambda$ = transitive collapse of H_λ and by construction, t_λ codes $\overline{H}_\lambda$ definably over $L_{\bar{\mu}_\lambda}[t_\lambda]$, where $\bar{\mu}_\lambda$ = height of $\overline{H}_\lambda$. So t_λ is reshaped, as $|t_\lambda|$ is singular, definably over $L_{\bar{\mu}_\lambda}[t_\lambda]$. By Proposition 5.3, $\bar{\mu}_\lambda < (|t_\lambda|^+)^L$ if $\alpha \geq \omega_2$. So t_λ belongs to S_α. ⊣ (Claim)

⊣

The next lemma illustrates the use of coding delays.

LEMMA 5.6 (Extendibility for P^s). *Suppose that α is a limit cardinal, $p \in P^s$, $s \in S_\alpha$, $X \subseteq \alpha, X \in \mathcal{A}^s$. Then there exists $q \leq p$ such that $X \cap \beta \in \mathcal{A}^{q_\beta}$ for each $\beta \in \mathrm{Card} \cap \alpha$.*

PROOF. Let $Y \subseteq \alpha$ be chosen so that $\mathrm{Even}(Y)$ precodes s and $\mathrm{Odd}(Y)$ is the Σ_1 theory of $\mathcal{A}$ with parameters from $\alpha \cup \{A \cap \alpha, s\}$, where $\mathcal{A}$ is an initial segment of $\mathcal{A}^s$ of limit height large enough to extend $\widetilde{\mathcal{A}}^s$ and contain X, p. For $\beta \in \mathrm{Card} \cap \alpha$ let $\overline{\mathcal{A}}_\beta$ =transitive collapse of Σ_1 Hull$(\beta \cup \{A \cap \alpha, s\})$ in $\mathcal{A}$. Then for sufficiently large $\beta \in \mathrm{Card}' \cap \alpha$, either Even $(Y \cap \beta)$ precodes $s_\beta \in S_\beta$ where s_β = pre-image of s under the natural embedding $\overline{\mathcal{A}}_\beta \longrightarrow \mathcal{A}$, or $|p_\beta| < (\beta^+)^{\overline{\mathcal{A}}_\beta}$ in which case f^{p_β} is dominated by the function $g(\gamma) = (\gamma^+)^{\overline{\mathcal{A}}_\gamma}$ on a final segment of $\mathrm{Card}^+ \cap \beta$.

Define q as follows: $q_\beta = s_\beta$ if Even $(Y \cap \beta)$ precodes $s_\beta \in S_\beta$, $q_\beta = p_\beta * (Y \cap \beta)^E$ for other $\beta \in \mathrm{Card}' \cap \alpha$, $q_\beta = p_\beta * \vec{0} * 1 * (Y \cap \beta)^D$ where $\vec{0}$ has length $g(\beta)$, for $\beta \in \mathrm{Card}^+ \cap \alpha$.

As $g \restriction \beta, Y \cap \beta$ are definable over $\overline{\mathcal{A}}_\beta$ for $\beta \in \mathrm{Card}' \cap \alpha$ we get $g \restriction \beta, Y \cap \beta \in \mathcal{A}^{s_\beta}$ when Even $(Y \cap \beta)$ precodes $s_\beta \in S_\beta$. Also $g \restriction \beta, Y \cap \beta \in \mathcal{A}^{q_\beta}$ for other $\beta \in \mathrm{Card}' \cap \alpha$ as Odd $(Y \cap \beta)$ codes $\overline{\mathcal{A}}_\beta$. And note that for all $\beta \in \mathrm{Card}' \cap \alpha, g \restriction \beta$ dominates f^{p_β} on a final segment of $\mathrm{Card}^+ \cap \alpha$ (and hence $q \restriction \beta$ exactly codes q_β), unless Even $(Y \cap \beta)$ precodes s_β and $s_\beta = p_\beta$, in which case $q \restriction \beta$ exactly codes $q_\beta = s_\beta$ because $p \restriction \beta$ does.

So we conclude that for sufficiently large $\beta \in \mathrm{Card}' \cap \alpha, q \restriction \beta$ exactly codes q_β and $X \cap \beta \in \mathcal{A}^{q_\beta}$. Apply induction on α to obtain this for all $\beta \in \mathrm{Card}' \cap \alpha$. Finally, note that the only problem in verifying $q \leq p$ is that the restraint p^*_β may prevent us from making the extension q_β of p_β when $q_\beta = s_\beta$, Even $(Y \cap \beta)$ precodes s_β. But property (d) in the definition of condition guarantees that $p^*_\beta = \emptyset$ for β in a CUB $C \subseteq \alpha$, $C \in \mathcal{A}^s$. We may assume that $C \in \mathcal{A}$ and hence for sufficiently large β as above we get $\beta \in C$ and hence $p^*_\beta = \emptyset$. So $q \leq p$

on a final segment of $\text{Card} \cap \alpha$, and we may again apply induction to get $q \leq p$ everywhere. ⊣

The key idea of Jensen's proof lies in the verification of distributivity for P^s. Before we can state and prove distributivity we need some definitions.

DEFINITION. Suppose $i < \beta \in \text{Card}$ and $D \subseteq P^s$, $s \in S_{\beta^+}$. D is i^+-**predense** on P^s if $\forall p \in P^s \exists q \in P^s (q \leq p, q$ meets D and $q \restriction i^+ = p \restriction i^+)$. $X \subseteq \text{Card} \cap \beta^+$ is **thin** if for each inaccessible $\gamma \leq \beta$, $X \cap \gamma$ is not stationary in γ. A function $f :\ \text{Card} \cap \beta^+ \longrightarrow V$ is **small** if for each $\gamma \in \text{Card} \cap \beta^+$, $\text{Card}(f(\gamma)) \leq \gamma$ and Support $(f) = \{\gamma \in \text{Card} \cap \beta^+ \mid f(\gamma) \neq \emptyset\}$ is thin. If $D \subseteq P^s$ is predense and $p \in P^s$, $\gamma \in \text{Card} \cap \beta^+$ we say that p **reduces** D **below** γ if for some $\delta \in \text{Card}^+$, $\delta \leq \gamma$, $q \leq p \longrightarrow$ there exists $r \leq q (r$ meets D and $r \restriction [\delta, \beta] = q \restriction [\delta, \beta])$. Finally, for $p \in P^s$, f small, $f \in \mathcal{A}^s$ we define $\Sigma^p_f =$ all $q \leq p$ in P^s such that whenever $\gamma \in \text{Card} \cap \beta^+$, $D \in f(\gamma), D$ predense on $P^{p_{\gamma^+}}$, we have that q reduces D below γ.

LEMMA 5.7 (Distributivity for P^s). *Suppose $s \in S_{\beta^+}$, $\beta \in \text{Card}$.*

(a) *If $\langle D_i \mid i < \beta \rangle \in \mathcal{A}^s$, D_i i^+-dense on P^s for each $i < \beta$ and $p \in P^s$ then there is $q \leq p$, q meets each D_i.*
(b) *If $p \in P^s$, f small in $\mathcal{A}^s$ then there exists $q \leq p$, $q \in \Sigma^p_f$.*

PROOF. We demonstrate (a) and (b) by a simultaneous induction on β. If $\beta = \omega$ or belongs to Card^+ then by induction, (a) and (b) reduce to the following: If S is a collection of β-many predense subsets of P^s, $S \in \mathcal{A}^s$ then $\{q \in P^s \mid q$ reduces each $D \in S$ below $\beta\}$ is dense on P^s. The latter follows from Lemma 5.5, since P^s factors as $R^s * Q$ where $1^{R^s} \Vdash Q$ is β^+-cc, and hence any $p \in P^s$ can be extended to $q \in P^s$ such that $D^q = \{r \mid r \cup q(\beta)$ meets $D\}$ is predense $\leq q \restriction \beta$ for each $D \in S$.

Now suppose that β is inaccessible. We first show that (b) holds for f, provided $f(\beta) = \emptyset$. First select a CUB $C \subseteq \beta$ in $\mathcal{A}^s$ such that $\gamma \in C \rightarrow f(\gamma) = \emptyset$ and extend p so that $f \restriction \gamma$, $C \cap \gamma$ belong to $\mathcal{A}^{p_\gamma}$ for each $\gamma \in \text{Card} \cap \beta^+$. Then we can successively extend p on $[\beta_i^+, \beta_{i+1}]$ in the least way so as to meet Σ^p_f on $[\beta_i^+, \beta_{i+1}]$, where $\langle \beta_i \mid i < \beta \rangle$ is the increasing enumeration of C. At limit stages λ, we still have a condition, as the sequence of first λ extensions belongs to $\mathcal{A}^{p_{\beta_\lambda}}$. The final condition, after β steps, is an extension of p in Σ^p_f.

Now we prove (a) in this case. Suppose $p \in P^s$ and $\langle D_i \mid i < \beta \rangle \in \mathcal{A}^s$, D_i is i^+-dense on P^s for each $i < \beta$. Let $\mu_0 < \mu^s$ be a large enough limit ordinal so that $\langle D_i \mid i < \beta\rangle$, p, $\tilde{\mu}^s \in L_{\mu_0}[A \cap \beta^+, s]$ and for $i < \beta$ let $\mu_i = \mu_0 + \omega \cdot i < \mu^s$. For any X we let $H_i(X)$ denote $\Sigma_1\,\mathrm{Hull}(X \cup \{\langle D_i \mid i < \beta\rangle, p, \tilde{\mu}^s, s, A \cap \beta^+\})$ in $L_{\mu_i}[A \cap \beta^+, s]$.

Let f_i : $\mathrm{Card} \cap \beta \to V$ be defined by: $f_i(\gamma) = H_i(\gamma)$ if $i < \gamma \in H_i(\gamma)$ and $f_i(\gamma) = \emptyset$ otherwise. Then each f_i is small in $\mathcal{A}^s$ and we inductively define $p = p^0 \geq p^1 \geq \cdots$ in P^s as follows: $p^{i+1} =$ least $q \leq p^i$ such that:

(a) $q(\beta)$ meets all predense $D \subseteq R^s$, $D \in H_i(\beta)$,

(b) q meets $\Sigma^{p^i}_{f_i}$ and D_i,

(c) $q \restriction i^+ = p^i \restriction i^+$.

For limit $\lambda \leq \beta$ we take p^λ to be the greatest lower bound to $\langle p^i \mid i < \lambda\rangle$, if it exists.

Claim. p^λ is a condition in P^s, where $p^\lambda(\gamma) = (\cup\{p^i_\gamma \mid i < \lambda\}, \cup\{p^{i\,*}_\gamma \mid i < \lambda\})$ for each $\gamma \in \mathrm{Card} \cap \beta^+$.

Suppose that γ belongs to $H_\lambda(\gamma) \cap \beta$. First we verify that $p^\lambda_\gamma = \cup\{p^i_\gamma \mid i < \lambda\}$ belongs to S_γ. Let $\bar{H}_\lambda(\gamma)$ be the transitive collapse of $H_\lambda(\gamma)$ and write $\bar{H}_\lambda(\gamma)$ as $L_{\bar{\mu}}[\bar{A}, \bar{s}]$, $\bar{P} =$ image of $P^s \cap H_\lambda(\gamma)$ under transitive collapse, $\bar{\beta} =$ image of β under collapse. Also write $\bar{P}$ as $\bar{R}^{\bar{s}} * P^{\bar{G}_{\bar{\beta}}}$ where $\bar{G}$ denotes an $\bar{R}^{\bar{s}}$-generic (just as P^s factors as $R^s * P^{G_\beta}, G_\beta$ denoting an R^s-generic).

Now the construction of the p^i's (see conditions (a), (b)) was designed to guarantee: (i) $\bar{G}_{\bar{\beta}} = \{\bar{p} \in R^{\bar{s}} \mid \bar{p} \text{ is extended by some } \bar{p}^i(\bar{\beta}), i < \lambda\}$ is $R^{\bar{s}}$-generic over $\bar{H}_\lambda(\gamma)$, where $\bar{p}^i =$ image of p^i under collapse, and (ii) for each $\bar{\delta}$ in (Card^+ of $\bar{H}_\lambda(\gamma)$), $\gamma < \bar{\delta} < \bar{\beta}$, $\{\bar{p} \mid \bar{p} \text{ is extended by some } \bar{p}^i \restriction [\gamma, \bar{\delta}) \text{ in } \bar{P}^{\bar{p}^i_{\bar{\delta}}}_\gamma\}$ is $\bar{P}^{\bar{G}_{\bar{\delta}}}_\gamma$-generic over $\mathcal{A}^{\bar{G}_{\bar{\delta}}} = \cup\{\mathcal{A}^{\bar{p}^i_{\bar{\delta}}} \mid i < \lambda\}$, where $\bar{P}^{\bar{G}_{\bar{\delta}}}_\gamma = \cup\{\bar{P}^{\bar{p}^i_{\bar{\delta}}}_\gamma \mid i < \lambda\}$ and $\bar{P}^{\bar{p}^i_{\bar{\delta}}}_\gamma$ denotes the image under collapse of $P^{p^i_\delta}_\gamma = \{q \restriction [\gamma, \delta) \mid q \in P^{p^i_\delta}\}$, $\bar{\delta} =$ image of δ under collapse.

Note. We do *not* necessarily have property (ii) above for $\bar{\delta} = \bar{\beta}$, and this is the source of our need for $\sim 0^\#$ in this proof.

By induction, we have the distributivity of P^t for $t \in S_\delta$, $\delta \in \mathrm{Card}^+ \cap \beta$, and hence that of $\bar{P}^{\bar{t}}$ for $\bar{t} \in \bar{S}_{\bar{\delta}}$, $\bar{\delta} \in$ (Card^+ of $\bar{H}_\lambda(\gamma)$), $\bar{\delta} < \bar{\beta}$. So the "weak" genericity of the preceding paragraph implies that:

(d) $L_{\bar{\beta}}[A \cap \gamma, p^\lambda_\gamma] \vDash |p^\lambda_\gamma|$ is a cardinal.

Also:

(e) $L_{\bar{\mu}}[A \cap \gamma, p^\lambda_\gamma] \vDash |p^\lambda_\gamma|$ is Σ_1-singular.

Thus $p^\lambda_\gamma \in S_\gamma$ (by (e)) provided we can show that when $\gamma \geq \omega_2$, $\bar\mu < (|p^\lambda_\gamma|^+)^L$. But $\bar H_\lambda(\gamma) \tilde\to H_\lambda(\gamma)$ gives a Σ_1-elementary embedding with critical point $|p^\lambda_\gamma|$, so by Proposition 5.3, this is true. Also note that we now get $p^\lambda \restriction \gamma \in \mathcal{A}^{p^\lambda_\gamma}$ as well, since $p^\lambda \restriction \gamma$ is definable over $\bar H_\lambda(\gamma)$ and we defined $\mathcal{A}^{p^\lambda_\gamma}$ to be large enough to contain $\bar H_\lambda(\gamma)$, since $L_{\bar\beta} \vDash |p^\lambda_\gamma|$ is a cardinal by (d) and $\bar\beta$ is a cardinal of $L_{\bar\mu}$.

The previous argument applies also if $\gamma = \beta$, using the distributivity of R^s, or if $\gamma = \beta \cap H_\lambda(\gamma)$, using the fact that p^λ_β collapses to p^λ_γ. If $\gamma < \gamma^* = \min(H_\lambda(\gamma) \cap [\gamma, \beta))$ then we can apply the first argument to get the result for γ^*, and then the second argument to get the result for γ.

Finally, to prove the Claim we must verify the restraint condition (d) in the definition of P. Suppose γ is inaccessible and for $i < \lambda$ let C^i be the least CUB subset of γ in $\mathcal{A}^{p^i_\gamma}$ disjoint from $\{\bar\gamma < \gamma \mid p^{i\,*}_{\bar\gamma} \neq \emptyset\}$. If $\lambda < \gamma$ then $\cap\{C^i \mid i < \lambda\}$ witnesses the restraint condition for p^λ at γ, if $\gamma < \lambda$ then the restraint condition for p^λ at γ follows by induction on λ and if $\gamma = \lambda$ then $\Delta\{C^i \mid i < \lambda\}$ witnesses the restraint condition for p^λ at γ, where Δ denotes diagonal intersection.

Thus the Claim and therefore (a) is proved in case β is inaccessible. To verify (b) in this case, note that as we have already proved (b) when $f(\beta) = \emptyset$ it suffices to show: if $\langle D_i \mid i < \beta\rangle \in \mathcal{A}^s$ is a sequence of dense subsets of P^s then $\forall p \exists q \leq p$ (q reduces each D_i below β). But using distributivity we see that $D^*_i = \{q \mid q$ reduces D_i below $i^+\}$ is i^+-dense for each $i < \beta$, so again by distributivity there is $q \leq p$ reducing D_i below i^+ for each i.

We are now left with the case where β is singular. The proof of (a) can be handled using the ideas from the inaccessible case as follows. Choose $\langle \beta_i \mid i < \lambda_0\rangle$ to be a continuous and cofinal sequence of cardinals $< \beta$, $\lambda_0 < \beta_0$. First we argue that $p \in P^s$ can be extended to meet Σ^p_f for any f small in $\mathcal{A}^s$ provided $f(\beta) = \emptyset$: extend p if necessary so that for each $\gamma \in \mathrm{Card} \cap \beta^+$, $f \restriction \gamma$ and $\{\beta_i \mid \beta_i < \gamma\}$ belong to $\mathcal{A}^{p_\gamma}$. Now perform a construction like the one used to prove distributivity in the inaccessible case, extending p successively on $[\beta_0, {\beta_i}^+]$ so as to meet Σ^p_f on $[\beta_0, {\beta_i}^+]$ as well as appropriate $\Sigma^{p^i}_{f_i}$'s defined on $[\beta_0, {\beta_i}^+]$ to guarantee that p^λ is a condition for limit $\lambda \leq \lambda_0$. Note that each extension is made on a bounded initial segment of $[\beta_0, \beta)$ and therefore by induction $\Sigma^p_f, \Sigma^{p^i}_{f_i}$ can be met on these intervals. The result is that p can be extended to meet Σ^p_f on a final segment of $\mathrm{Card} \cap \beta$ and therefore by induction can be extended to meet Σ^p_f. Second, use the density of Σ^p_f when $f(\beta) = \emptyset$ to carry out the distributivity proof as we did in the inaccessible case. And again, (b) follows from (a). This complete the proof of Lemma 5.7. $\dashv$

Theorem 5.1 now follows, as the argument of the previous lemma also shows:

LEMMA 5.8 (Distributivity for P). *If $\langle D_i \mid i < \kappa\rangle$ is $\langle M, A\rangle$-definable where D_i is i^+-dense for each $i < \kappa$ and $p \in P$ then there exists $q \leq p, q$ meets each D_i.*

Thus P is tame and preserves cofinalities. ⊣

The proof of Theorem 5.1 in the general case is far more difficult; we refer the reader to Section 4.3 of Friedman [99].

Large Cardinal Preservation

The forcing used to prove the Coding Theorem preserves a number of large cardinal properties consistent with $V = L[R]$, $R \subseteq \omega$, such as the Mahlo and α-Erdös properties. In addition for any m, n a predicate A^* can be adjoined to $\langle M, A\rangle$ so that if κ is Σ^n_m-indescribable then κ is Σ^n_m-indescribable relative to A^*, and then A^* can be coded by a real, via a modification of the forcing described above, so as to preserve Σ^n_m-indescribability. Preservation of Π^n_m-indescribability for $n > 1$ is an open problem.

Relevance

It is at this point that we see the importance of indiscernible preservation:

PROPOSITION 5.9. *Suppose that $A \subseteq L$ preserves indiscernibles. Then there is a real $R \in L[A, 0^\#]$ generic over $\langle L[A], A\rangle$ such that A is definable in $L[R]$. Moreover R preserves indiscernibles.*

The following proof of Proposition 5.9 is reminiscent of the proof of relevance for Coherent Easton forcing at Successors.

PROOF. First assume that $A = \emptyset$. For any indiscernible i let j_n be the first n indiscernibles $\geq i$. Then define $s_n \in S^{i^+}$ and $p^n \in P^{s_n}$ inductively, meeting the following conditions: $s_0 = \emptyset, p^n =$ the trivial condition. $s_{n+1} = \pi_i(p^n)_{i^+}$ where $\pi_i : L \to L$ is an elementary embedding with critical point i, $p^{n+1} =$ least $q \leq p^n$ in P^{s_n} meeting $\Sigma^{p^n}_{f_n}$ where $f_n(\beta) = \mathrm{Hull}(\beta \cup j_n)$ if $\beta \in \mathrm{Hull}(\beta \cup j_n)$, $f_n(\beta) = \emptyset$ otherwise. (β ranges over $\mathrm{Card} \cap i^+$ and when $\beta = i$ we take $p^n_{\beta^+}$ to be s_n.) Let $G^i_0 = \{p \mid p \text{ is extended by some } p^n\}$.

G^i_0 is *not* P^{s_n}-generic over $\mathcal{A}^{s_n}$ in general as all conditions in G^i_0 have empty restraint at indiscernibles $< i$. But notice that for $i_0 < i_1 < \cdots < i_n \leq i$ in

I, $G_0^{i_0} \cup \cdots \cup G_0^{i_n}$ is a compatible set of conditions. We take G^i to be $\{p \mid p$ is extended by $q_0 \wedge \cdots \wedge q_n$ for some $q_l \in G_0^{i_l}$, $i_0 < \cdots < i_n \leq i$ in $I\}$. Now we claim that G^i is P^{s_n}-generic over $\mathcal{A}^{s_n}$ for each n. Indeed, if D is predense on P^{s_n} and belongs to $\mathcal{A}^{s_n}$, $D \in \text{Hull}(\{k_0, \cdots, k_m\} \cup j_n)$ with $k_0 < \cdots < k_m < i$ in I then p^{n+1} reduces D below k_m^+, p^{n+2} reduces D below $k_{m-1}^+, \cdots$ and eventually we get p^{n+m+2} in G^i meeting D.

It follows that $G^i(< i) = G^i \cap P^i$ is generic over L_i (for L_i- definable dense sets) and hence G is P-generic over L where $G = \cup\{G^i(< i) \mid i \in I\}$. Clearly G preserves indiscernibles.

If $A \neq \emptyset$ then first force to obtain the GCH, preserving indiscernibles, and then apply the above argument. ⊣

COROLLARY 5.10 (Jensen). *There is $R <_L 0^\#$, R not set-generic over L. Hence the Genericity Problem has an affirmative solution when "generic" is interpreted to mean"set-generic".*

Not every $A \subseteq L$ can be coded generically by a real, in the presence of $0^\#$, as a result of Paris' work on "patterns of indiscernibles":

DEFINITION. For $\alpha, \beta \in \text{ORD}$, $\beta \neq 0$ let $I_{\alpha,\beta} = \{i_{\alpha+\beta\gamma} \mid \gamma \in \text{ORD}\}$ where $\langle i_\alpha \mid \alpha \in \text{ORD}\rangle$ is the increasing enumeration of I.

THEOREM 5.11 (Paris [74]). *If $R \subseteq \omega$, $0^\# \notin L[R]$ then for some $\alpha, \beta < \omega_1$, $I_{\alpha,\beta} =$ the Silver indiscernibles for $L[R]$.*

There exist classes $A \subseteq L$ which are generic over L, yet relative to which $I_{\alpha,\beta}$ is *not* a class of indiscernibles for any α, β. It follows that A cannot be generically coded by a real R, as any such R satisfies the hypothesis of Paris' Theorem. However this is the only restriction.

THEOREM 5.12. *If $I_{\alpha,\beta}$ is a class of indiscernibles for $\langle L[A], A\rangle$, $\alpha, \beta < \omega_1$ then there is a real $R \in L[A, 0^\#]$ generic over $\langle L[A], A\rangle$ such that A is definable in $L[R]$. Moreover $I_{\alpha,\beta}$ is a class of indiscernibles for $L[R]$.*

In addition:

THEOREM 5.13. *For any $\alpha, \beta < \omega_1$ there exists a real R such that $I_{\alpha,\beta} =$ the Silver indiscernibles for $L[R]$.*

Theorems 5.12, 5.13 are proved by first using Reverse Easton methods to create $A^* \subseteq L$ such that $I_{\alpha,\beta}$ is a *generating* class of indiscernibles for $\langle L[A^*], A^* \rangle$ and then using the method of Proposition 5.9 to code A^* by a real, preserving the indiscernibility of $I_{\alpha,\beta}$.

6. The Solovay Problems

We are now prepared to discuss the solutions to the three problems posed in Section One. For a full treatment of this material, we refer the reader to Chapters 5,6,7 of Friedman [99].

The Genericity Problem

We show that there is a real $R <_L 0^\#$ which is not class-generic over L. First recall the statement of the Truth Lemma, which holds for all tame L-forcings:

Truth Lemma If G is P-generic over $\langle L, A \rangle$ then $\langle L[G], A, G \rangle \vDash \varphi(\sigma_1^G \dots \sigma_n^G)$ iff there exists $p \in G$, $p \Vdash \varphi(\sigma_1 \dots \sigma_n)$.

We also have:

Uniform Definability Lemma The relation "$p \Vdash \varphi(\sigma_1 \dots \sigma_n)$" is definable as a relation of $p, \varphi, \langle \sigma_1 \dots \sigma_n \rangle$ over $\langle L, \mathrm{Sat}\langle L, A \rangle \rangle$ where $\mathrm{Sat}\langle L, A \rangle$ denotes the Satisfaction relation for $\langle L, A \rangle$.

Remark. $\langle L, \mathrm{Sat}\langle L, A \rangle \rangle$ is amenable, as $\langle L, A \rangle$ amenable $\rightarrow \langle L_i, A \cap L_i \rangle \prec \langle L, A \rangle$ for sufficiently large $i \in I$.

A consequence is the following:

Fact If G is P-generic over $\langle L, A \rangle$ then $\mathrm{Sat}\langle L[G], A, G \rangle$ is definable over $\langle L[G], \mathrm{Sat}\langle L, A \rangle, G \rangle$.

Using this Fact we can see a strategy for producing a real R *not* generic over L: If $R \in L[G]$, G P-generic over $\langle L, A \rangle$ then by the Fact and Tarski's Undefinability of Satisfaction, $\mathrm{Sat}\langle L, A \rangle$ *cannot* be definable over $\langle L[G], A, G \rangle$ and hence *cannot* be definable over $\langle L[R], A \rangle$. Thus:

PROPOSITION 6.1. *R generic over $L \rightarrow$ For some amenable $\langle L, A \rangle$,* $\mathrm{Sat}\langle L, A \rangle$ *is* not *definable over $\langle L[R], A \rangle$.*

THEOREM 6.2. *There exists $R <_L 0^\#$ such that* $\mathrm{Sat}\langle L, A\rangle$ *is definable over* $\langle L[R], A\rangle$ *for every amenable* $\langle L, A\rangle$.

To prove Theorem 6.2 we define for each $i \in I$ a forcing $P_i \subseteq L_{i^+}$ for producing $X_i \subseteq i$ such that for each constructible $A \subseteq i$, $\mathrm{Sat}\langle L_i, A\rangle$ is definable over $\langle L_i[X_i], A, X_i\rangle$. This forcing P_i is of the Easton variety and hence preserves cofinalities. The main part of the proof consists in showing that there is a single $X \subseteq \mathrm{ORD}$ definable in $L[0^\#]$ such that $X \cap i$ is P_i-generic for all $i \in I$ simultaneously, and such that X preserves indiscernibles. Then for each amenable $\langle L, A\rangle$, $\mathrm{Sat}\langle L, A\rangle$ is definable over $\langle L[X], A, X\rangle$ and X can be coded by a real $R <_L 0^\#$ with the same property, using the fact that X preserves indiscernibles and Proposition 5.9.

The proof is not special to the Sat operator and can be used to prove:

THEOREM 6.3. *Suppose $F : \mathcal{P}_L(\omega_1) \to \mathcal{P}_L(\omega_1)$ is constructible where $\mathcal{P}_L(\omega_1) =$ all constructible subsets of ω_1. Then there is a real $R <_L 0^\#$ such that $F(A)$ is definable over $\langle L_{\omega_1}[R], A\rangle$ for all $A \in \mathcal{P}_L(\omega_1)$.*

The Π^1_2-Singleton Problem

The following result gives an affirmative solution to this problem:

THEOREM 6.4. *There is a real R generic over L such that $0 <_L R <_L 0^\#$ and R is the unique solution to a Π^1_2 formula.*

The heart of the matter is to build an L-definable forcing with a unique generic, in the form of a real. To guarantee uniqueness we design our forcing so as to make our generic "guess" at which ordinals belong to $I =$ the Silver indiscernibles. Of course no generic can correctly answer this question, but we arrange that only one generic does a reasonable job of guessing, in the sense that other potential generics would in fact produce CUB classes disjoint from I, an impossibility. More precisely, a generic consists of a real R and a class A such that:

(a) R codes A as in Jensen coding.
(b) There is a $\Sigma_1(L)$ procedure $(i_1 \dots i_n) \mapsto p(i_1 \dots i_n)$ such that the generic corresponding to (R, A) is $\{p(i_1 \dots i_n) \mid i_1 < \cdots < i_n \text{ belong to } I\}$.
(c) A adds CUB sets so as to "kill" any $(i_1 \dots i_n)$ such that $p(i_1 \dots i_n)$ disagrees with R.

(d) No $(i_1 \dots i_n) \in I^n$ can be killed.

It follows that $\{p(i_1 \dots i_n) \mid i_1 < \cdots < i_n \text{ in } I\}$ is the *only* generic, as by (c) another generic R' would kill $(i_1 \dots i_n) \in I^n$ such that $p(i_1 \dots i_n)$ disagrees with R', an impossibility by (d).

Of course there is a circularity here, as to design P we need the procedure in (b), which is defined assuming that we know P. This is resolved using the Recursion Theorem.

The killing method above involves forcing of the Reverse Easton variety and the coding of A by R uses Jensen coding, a variety of Coherent Easton forcing at Successors. Thus unlike the solution to the Genericity Problem, here we must mix the relevance arguments for two different types of class forcing together, to obtain a generic in $L[0^\#]$ for P.

The Admissibility Spectrum Problem

We first describe the proof of:

THEOREM 6.5 (David [89], Friedman [99]). *There is a real $R <_L 0^\#$ such that $\Lambda(R) \subseteq$ the recursively inaccessible ordinals.*

We wish to arrange that α R-admissible $\to$ α recursively inaccessible. Suppose that we have $D \subseteq \omega_1$ such that α D-admissible $\to$ α recursively inaccessible. (α is **D-admissible** if $L_\alpha[D]$ obeys ZFC $-$ Power, with replacement restricted to formulas which are Σ_1 and mention D as a predicate.) Then we may hope to code D by a real R with the same property. However if we code D by R in the usual way (with almost disjoint forcing) we only obtain:

$$\alpha \ R\text{-admissible} \to \alpha \ D \cap \omega_1^{L_\alpha}\text{-admissible}$$

The reason is that to decode D from R we need to know the almost disjoint coding reals R_ξ and it is only for $\xi < \omega_1^{L_\alpha}$ that we have $R_\xi \in L_\alpha$. Thus the recovery of D from R is not "fast enough". On the other hand we would be in good shape if D were to have the following stronger properties:

$$(*) \ \alpha \ D \cap \xi\text{-admissible}, \ L_\alpha[D \cap \xi] \vDash \xi = \omega_1 \to \alpha \text{ recursively inaccessible}$$

$$(**) \ \alpha \ D\text{-admissible and } L_\alpha[D] \vDash \omega_1 \text{ does not exist} \to \alpha \text{ recursively inaccessible}$$

For then we need only recover $D \cap \omega_1^{L_\alpha}$ inside $L_\alpha[R]$ to guarantee that α is recursively inaccessible (or inadmissible relative to R), a recovery that can be successfully made.

The question is how to obtain $D \subseteq \omega_1$ obeying $(*), (**)$. The natural thing to do is to force with conditions d which are bounded subsets of ω_1 obeying $(*), (**)$ for $\xi \leq \sup(d)$, ordered by end extension. We now come to the key part of the argument, which is contained in the following two observations:

(a) Extendibility for this forcing is trivial because given d and $\xi > \sup(d)$ we are free to extend d to length ξ by *killing all admissibles* between $\sup(d)$ and ξ. It is important for this argument that we are only concerned with killing admissibility, not with preserving it.

(b) Distributivity for this forcing is easily established assuming the following: There exists $D' \subseteq \omega_2$ such that:

$$(*') \ \alpha \ D' \cap \xi\text{-admissible}, \ L_\alpha[D' \cap \xi] \vDash \xi = \omega_2 \to \alpha \text{ recursively inaccessible}$$

$$(**') \ \alpha \ D'\text{-admissible and } L_\alpha[D'] \vDash \omega_2 \text{ does not exist} \to \alpha \text{ recursively inaccessible}$$

Thus we are faced with the original difficulty, but one cardinal higher! However note that we need not already have all of D' before we can start building D; thus the idea of the proof (as in other Jensen coding constructions) is to build $R, D, D', D'', \ldots$ simultaneously and check distributivity for any final segment of the forcing.

To solve the Admissibility Spectrum Problem we must introduce the requirement of admissibility *preservation* into the above. This requires the method of *Strong Coding.*

THEOREM 6.6. *There is a real $R <_L 0^\#$ such that $\Lambda(R) =$ the recursively inaccessible ordinals.*

We approach the problem as in the previous proof. Of course the Extendibility property is more difficult to establish (Distributivity is approximately the same). Indeed the desired extension of d to d' of length $\geq \xi$ must be made so as to preserve the admissibility of recursively inaccessible ordinals. Thus our conditions must be constructed out of sets which are generic for "local" versions of the full forcing. In fact we construct a strong coding forcing $P^\beta \subseteq L_\beta$ at each admissible β and then inductively build P^β out of sets which are generic for the various $P^{\beta'}, \beta' < \beta$.

The main difficulty is in showing that the desired locally generic sets actually exist; note that we want a P^β-generic over L_β to exist where β may be uncountable. The proof of local generic existence is by a simultaneous induction with the proofs of Extendibility and Distributivity and requires a substantial use of the kind of fine structure theory used in the construction of higher gap morasses.

7. Generic Saturation

Suppose that P is an L-forcing which has a generic; need it have a generic definable in $L[0^\#]$? Not necessarily, as the forcing P could produce a real R that guarantees the countability of $\omega_1^{L[0^\#]}$, and clearly no such real can exist in $L[0^\#]$. However we can weaken this slightly to obtain a positive result:

Definition. Suppose that $M \subseteq N$ are inner models of ZFC. We say that N is **generically saturated over** M if whenever an M-forcing has a generic, then it has one definable in a set-generic extension of N.

With a mild assumption about $\infty =$ the class of all ordinals, it can be shown that $L[0^\#]$ is generically saturated over L. This assumption involves the concept of an *Erdös cardinal.*

Definition. A cardinal κ is **α-Erdös** if whenever $A \subseteq \kappa$ and C is CUB in κ there exists $X \subseteq C$ such that ordertype $X = \alpha$ and $\gamma \in X \to X - \gamma$ is a set of indiscernibles for $\langle L[A], A, \delta\rangle_{\delta<\gamma}$. We say that ∞ is **α-Erdös** if this holds where κ is replaced by ∞ and indiscernibility is only required for Σ_1 formulas.

Theorem 7.1. *Suppose ∞ is $\omega+\omega$-Erdös. Then $L[0^\#]$ is generically saturated over L.*

Theorem 7.1 is proved by starting with G P-generic over $\langle L, A\rangle$ and using $\omega+\omega$ indiscernibles for $\langle L[G, 0^\#], A, G\rangle$ to produce another P-generic G^*, which is "periodic". The latter means that for some $\alpha \in$ ORD and $0 < \beta \in$ ORD, $I_{\alpha,\beta} = \{i_{\alpha+\beta\gamma} \mid \gamma \in \text{ORD}\}$ is a class of indiscernibles for $\langle L[G^*], A, G^*\rangle$, where $I = \langle i_\alpha \mid \alpha \in \text{ORD}\rangle$ is the increasing enumeration of I. Then by an absoluteness argument, such a G^* may be defined in a set-generic extension of $L[0^\#]$ in which α and β are countable.

PROOF OF THEOREM 7.1. Suppose that $G \subseteq P$ is P-generic over $\langle L, A\rangle$. We shall construct another P-generic G^* (in a set-generic extension of V) such that G^* has periodic indiscernibles.

Let X be a set of indiscernibles for $\langle L[0^\#, G], G, A\rangle$ of ordertype $\omega + \omega$ such that $\alpha \in X \to \alpha$ is Σ_1-stable in $0^\#, G, A$. The latter means that $\langle L_\alpha[0^\#, G \cap L_\alpha], G \cap L_\alpha, A \cap L_\alpha\rangle$ is Σ_1-elementary in $\langle L[0^\#, G], G, A\rangle$. We can obtain X as $C = \{\alpha \mid \alpha \text{ is } \Sigma_1\text{-stable in } 0^\#, G, A\}$ is CUB.

Choose $\langle D(\alpha_1 \dots \alpha_n) \mid \alpha_1 < \cdots < \alpha_n \text{ in ORD}\rangle$ such that each $\langle L, A\rangle$-definable open dense $D \subseteq P$ is of the form $D(\alpha_1 \dots \alpha_n)$ for some $\alpha_1 < \cdots < \alpha_n$ in I. Also assume that this sequence is $\Delta_1\langle L, \mathrm{Sat}\langle L, A\rangle\rangle$. Let $D^*(\alpha_1 \dots \alpha_n) = \cap\{D(\vec\beta) \mid \vec\beta$ a subsequence of $\langle \alpha_1 \dots \alpha_n\rangle\}$.

For $j_0 \in X$ choose the least $t_{j_0}(\vec{k_0}(j_0), j_0, \vec{k_1}(j_0))$ in $D(j_0) \cap G$. By the choice of the indiscernibles X, we can write this as $t_0(\vec{k_0}, j_0, \vec{k_1}(j_0))$ and $j_0 < j_1$ in $X \to \vec{k_1}(j_0) < j_1$.

Next for $j_0 < j_1$ in X choose the least $t_{j_0,j_1}(\vec{k^1_0}(j_0, j_1), j_0, \vec{k^1_1}(j_0, j_1), j_1, \vec{k^1_2}(j_0, j_1))$ in $D^*(\vec{k_0}, j_0, \vec{k_1}(j_0), j_1, \vec{k_1}(j_1)) \cap G$. By the choice of X we can write this as $t_1(\vec{k^1_0}, j_0, \vec{k^1_1}(j_0), j_1, \vec{k^1_2}(j_0, j_1))$, and by Σ_1-stability this is less than j_2 whenever $j_1 < j_2$ in X. But we want to argue that in fact $\vec{k^1_2}(j_0, j_1)$ can be chosen *independently of* j_0.

Assuming this, we have $t_1(\vec{k^1_0}, j_0, \vec{k^1_1}(j_0), j_1, \vec{k^1_2}(j_1)) \in D^*(\vec{k_0}, j_0, \vec{k_1}(j_0), j_1, \vec{k_1}(j_1)) \cap G$ for $j_0 < j_1$ in X. By modifying t_1 we can guarantee that $\vec{k^1_1}(j_0) = \vec{k^1_2}(j_0)$ for all $j_0 \in X$, $j_0 \neq \min X$. Also we can arrange that $\vec{k_0} \subseteq \vec{k^1_0}$, $\vec{k_1}(j_0) \subseteq \vec{k^1_1}(j_0)$ for $j_0 \in X$. By indiscernibility, the structure $\langle \vec{k^1_1}(j_0), <\rangle$ with a unary predicate for $\vec{k_1}(j_0)$ has isomorphism type independent of the choice of $j_0 \in X$.

Build $t_2(\vec{k^2_0}, j_0, \vec{k^2_1}(j_0), j_1, \vec{k^2_1}(j_1), j_2, \vec{k^2_1}(j_2)) \in D^*(\vec{k^1_0}, j_0, \vec{k^1_1}(j_0), j_1, \vec{k^1_1}(j_1), j_2, \vec{k^1_1}(j_2)) \cap G$ similarly, so that $\vec{k^1_0} \subseteq \vec{k^2_0}$ and for $j_0 \in X$, $\vec{k^1_1}(j_0) \subseteq \vec{k^2_1}(j_0)$ with the isomorphism type of $\langle \vec{k^2_1}(j_0), <\rangle$ with unary predicates for $\vec{k_1}(j_0), \vec{k^1_1}(j_0)$ independent of j_0. Continue with $t_3, t_4, \dots$.

Let $i_\alpha = \min X$ and $\beta = \mathrm{ordertype}(\cup\{\vec{k^n_1}(j_0) \mid n \in \omega\})$, an ordinal independent of the choice of $j_0 \in X$. In a generic extension where α is countable we may also arrange that $\cup\{\vec{k^n_0} \mid n \in \omega\} = I \cap i_\alpha$.

For any indiscernible i_γ define $\vec{k^n_1}(i_\gamma) \subseteq I \cap (i_\gamma, i_{\gamma+\beta})$ so that $\langle I \cap (i_\gamma, i_{\gamma+\beta}), <\rangle$ with a predicate for $\vec{k^n_1}(i_\gamma)$ is isomorphic to $\langle \cup\{\vec{k^n_1}(j_0) \mid n \in \omega\}, <\rangle$ with a predicate for $\vec{k^n_1}(j_0)$, for $j_0 \in X$. Define: $G^* = \{p \in P \mid p$ is extended by some $t_n(\vec{k^n_0}, i_{\alpha_1}, \vec{k^n_1}(i_{\alpha_1}), \cdots, i_{\alpha_n}, \vec{k^n_1}(i_{\alpha_n}))$ where $\alpha \leq \alpha_1 < \dots < \alpha_n$ are of the form $\alpha + \beta\gamma, \gamma \in \mathrm{ORD}\}$. Using the indiscernibility of $I - i_\alpha$ in $\langle L, A\rangle$, G^* is compatible and meets every $\langle L, A\rangle$-definable open dense subclass of P. Thus G^* is P-generic and $I_{\alpha,\beta}$ is a class of indiscernibles for $\langle L[G^*], A, G^*\rangle$.

To complete the proof we return to the problem of making $\vec{k_2^1}(j_0, j_1)$ independent of j_0. First a lemma:

LEMMA 7.2. *Let $x < y$ by the maximum difference order on finite sets of ordinals: $x < y$ iff $\alpha \in y$ where α is the greatest element of the symmetric difference of x and y. For any $j_0 < j_1$ in X and any open dense D definable in $\langle L, A\rangle$ there exists $t(\vec{\ell_0}, j_0, \vec{\ell_1}, j_1, \vec{\ell_2}, \vec{\ell}) \in L_{\min(\vec{\ell})} \cap D \cap G$ such that $\vec{\ell_0} < j_0 < \vec{\ell_1} < j_1 < \vec{\ell_2} < \vec{\ell}$ belong to I and $\vec{\ell_0} \cup \vec{\ell_1} \cup \vec{\ell_2}$ is the $<$-least finite set of **ordinals** (not necessarily indiscernibles) x such that $t(x \cap j_0, j_0, x \cap (j_0, j_1), j_1, x - j_1, \vec{\ell})$ belongs to $L_{\min(\vec{\ell})} \cap D \cap G$.*

PROOF. Let x be $<$-least such that for some t and indiscernibles $\vec{\ell} > \max(x)$, $t(x \cap j_0, j_0, x \cap (j_0, j_1), j_1, x - j_1, \vec{\ell}) \in L_{\min(\vec{\ell})} \cap D \cap G$. If some $\alpha \in x$ were not in I then there would be a $t^*(x^* \cap j_0, j_0, x^* \cap (j_0, j_1), j_1, x^* - j_1, \vec{\ell^*}) = t(x \cap j_0, j_0, x \cap (j_0, j_1), j_1, x - j_1, \vec{\ell})$ with $\vec{\ell}$ an initial segment of $\vec{\ell^*}$ and $x^* - \alpha = x^* - (\alpha + 1)$, as α is L-definable from indiscernibles $< \alpha$ and indiscernibles $> \vec{\ell}$. So let $\vec{\ell_0}, \vec{\ell_1}, \vec{\ell_2}$ be $x \cap j_0, x \cap (j_0, j_1), x - j_1$. $\dashv$

Now for $j_0 < j_1$ in X choose the least $t_{j_0,j_1}(\vec{k_0^1}(j_0, j_1), j_0, \vec{k_1^1}(j_0, j_1), j_1, \vec{k_{2,0}^1}(j_0, j_1), \vec{k_{2,1}^1}(j_0, j_1))$ to satisfy Lemma 7.2 with $D = D^*(\vec{k_0}, j_0, \vec{k_1}(j_0), j_1, \vec{k_1}(j_1))$, and $\vec{\ell}$ denoted by $\vec{k_{2,1}^1}(j_0, j_1)$. By the choice of X we can write this as $t_1(\vec{k_0^1}, j_0, \vec{k_1^1}(j_0), j_1, \vec{k_{2,0}^1}(j_0, j_1), \vec{\infty})$, where $\vec{\infty}$ denotes an arbitrary sequence of large indiscernibles (of the appropriate length). Note that $\langle \vec{k_0^1}, \vec{k_1^1}(j_0), \vec{k_{2,0}^1}(j_0, j_1)\rangle$ is definable in $\langle L[G], A, G\rangle$ from $\vec{k_0}, j_0, \vec{k_1}(j_0), j_1, \vec{k_1}(j_1), \vec{\infty}$ and therefore $\vec{k_{2,0}^1}(j_0, j_1)$ is definable in $\langle L[G], A, G\rangle$ from $\vec{k_1}(j_1), \vec{\infty}$ and ordinals $\leq j_1$.

CLAIM. $\vec{k_{2,0}^1}(j_0, j_1)$ is independent of j_0.

PROOF. Let $j_0 < j_1 < \ldots < j$ be the first $\omega + 1$ elements of X and for any n, m let $\vec{k}(j_n, j)(m) = m^{\text{th}}$ element of $\vec{k_{2,0}^1}(j_n, j)$. If the Claim fails then for some fixed m, $\vec{k}(j_0, j)(m) < \vec{k}(j_1, j)(m) < \cdots$ is an increasing sequence of indiscernibles with supremum $\ell \in I$ (using the fact that $X - j$ has ordertype $>$ length$(\vec{\infty})$). As these ordinals are definable in $\langle L[G], A, G\rangle$ from ordinals in $(j + 1) \cup \vec{k_1}(j) \cup \vec{\infty}$ we get that ℓ has cofinality $\leq j$ in $L[G]$. But $0^\# \notin L[G]$ (as G is generic over L) so by Jensen's Covering Theorem, ℓ has L-cofinality $<$ (j^+ in $L[G]$). As $\ell \in I, \ell$ is L-regular and hence j^+ in $L < j^+$ in $L[G]$.

But then in $L[G]$ there is a CUB $C \subseteq j$ such that $D \subseteq j, D$ CUB, $D \in L \to C \subseteq D \cup \alpha$ for some $\alpha < j$. Now $I \cap j$ is the intersection of countably many such D's and therefore as j has uncountable cofinality (in $L[G, 0^\#]$) we get $C \subseteq I \cup \alpha$ for some $\alpha < j$. This yields $0^\# \in L[G]$, contradiction.

This proves the Claim. ⊣

With the Claim we see that there is a P-generic G^* (in a set-generic extension of V) such that $\langle L[G^*], A, G^* \rangle$ has a periodic class of indiscernibles $I_{\alpha,\beta}$. It now follows by absoluteness that there is such a G^* definable in a set-generic extension of $L[0^\#]$ in which α and β are countable. This completes the proof of Theorem 7.1. ⊣

It can be shown that there can be no countable bound on the α and β of the previous proof, using the solution to the Π^1_2-Singleton Problem. (See Section 8.2 of Friedman [99].)

8. Further Results

The material below is discussed in Chapter 8 of Friedman [99].

Strict Genericity

In set forcing, one may show that an inner model of a generic extension is itself a generic extension. This can fail for class forcing.

Definition. Let $\langle M, A \rangle$ be a ground model. A real R is **generic over** M if it belongs to a generic extension of M (via a forcing amenable to M). R is **strictly generic over** M if for some amenable structure $\langle M, A \rangle$, some forcing P definable over $\langle M, A \rangle$ and some G P-generic over $\langle M, A \rangle$, R belongs to $M[G]$ and G is definable over $\langle M[R], A \rangle$.

Theorem 8.1. *There is a real $R <_L 0^\#$ such that R is generic over L (for an L-definable forcing) but not strictly generic over L.*

As with the solution to the Genericity Problem, Theorem 8.1 is reduced to the violation of a definability property: If R is strictly generic over L then for some A amenable to L, $\text{Sat}\langle L[R], \emptyset \rangle$ is definable over $\langle L[R], A \rangle$. The latter can be violated using class forcing.

Minimal Universes

The minimal model of $V = L[0^{\#}]$ can be "minimized" by a class which does *not* construct $0^{\#}$:

THEOREM 8.2. *Suppose that for no α is $L_\alpha[0^{\#}]$ a model of* ZFC. *Then there is $A \subseteq$ ORD definable in $L[0^{\#}]$ such that $0^{\#} \notin L[A]$ and for no α is $\langle L_\alpha[A], A\cap\alpha\rangle$ elementary in $\langle L[A], A\rangle$.*

This result is partial evidence for the conjecture that $0^{\#}$ is generic over some proper inner model of $L[0^{\#}]$.

Countable Π^1_2 Sets

Assume that $R^{\#}$ exists for every real R. Kechris-Woodin [83] showed that a nonempty countable Π^1_2 set must have an ordinal-definable element; we show that in a sense their result is optimal. First some definitions.

DEFINITION. A set of reals X is n-**absolute** if for some formula φ, $R \in X \leftrightarrow L[R] \models \varphi(R, \omega_1, \dots, \omega_n)$, where ω_k denotes the ω_k of V. An n-**absolute singleton** is a real R such that $\{R\}$ is n-absolute. When $n = 0$ we say absolute, absolute singleton.

THEOREM 8.3 (Kechris-Woodin [83]). *Assume that $R^{\#}$ exists for every real R. A nonempty countable Π^1_2 set contains an n-absolute singleton for some n.*

Our next result demonstrates the optimality of the previous theorem.

THEOREM 8.4. *For each n there is a countable Π^1_2 set X_n such that $R \in X_n \to R$ is* not *an n-absolute singleton.*

Not all elements of countable Π^1_2 sets are n-absolute singletons for some n:

THEOREM 8.5. *There exists a countable Π^1_2 set X and $R \in X$ such that for all n, R is* not *an n-absolute singleton.*

Not every absolute singleton belongs to a countable Π^1_2 set: If a set is Σ^1_2 (with a constructible parameter) and contains a non-constructible real then it has a constructibly-coded perfect closed subset, and a code for this perfect closed set can be computed as a Σ^1_2 function applied to an index $n \in \omega$ for the given Σ^1_2

set X_n. Moreover $\{n \mid X_n \text{ has a perfect closed subset}\}$ is Σ^1_2. It follows that in L there is a perfect closed set C, with code recursive in the complete Σ^1_2 subset of ω, such that $R \in C \to R$ does not belong to any Π^1_2 set whose complement contains a non-constructible real. In particular $R \in C \to R$ does not belong to a countable Π^1_2 set. The set C contains elements which are Δ^1_3 in L, and hence which are absolute singletons.

An open problem is to provide a revealing characterization of the reals which belong to a countable Π^1_2 set.

In Harrington-Kechris [77] it is proved: If X is a nonempty Π^1_2 set then X has an element R such that either $R \leq_L 0^\#$ or $0^\# \leq_L R$. Our next result implies that $0^\#$ has least nonzero L-degree among reals with this property, even when X is restricted to have a unique element.

THEOREM 8.6. *There exists a sequence $\langle (R^n_0, R^n_1) \mid n \in \omega \rangle$ of pairs of reals such that:*

(a) $R \leq_L R^n_0,\ R \leq_L R^n_1 \to R \in L$.
(b) $\{\langle R, n, i \rangle \mid R = R^n_i\}$ *is* Π^1_2.
(c) $n \in 0^\# \leftrightarrow n \in R^n_0 \leftrightarrow n \in R^n_1$.

COROLLARY 8.7. *Suppose R is a non-constructible real and every Π^1_2-singleton is $\leq_L$-comparable with R. Then $0^\# \leq_L R$.*

Thus $0^\#$ is the least "canonical" Π^1_2-singleton.

New Σ^1_3 Facts

If M is an inner model, $0^\# \notin M$ then of course there is a true Σ^1_3 sentence not holding in M, namely the sentence asserting the existence of $0^\#$; can this effect be achieved by forcing over M?

THEOREM 8.8. *There exists an ω-sequence of Σ^1_3 sentences $\langle \varphi_n \mid n \in \omega \rangle$ such that if M is an inner model, $0^\# \notin M$:*

(a) φ_n *is false in M for some n.*
(b) *For each n, some generic extension of M satisfies φ_n.*

Moreover if $M = L[R]$, R a real then the generic extensions in (b) can be taken as inner models of $L[R, 0^\#]$.

The proof is based on the following, which may be of independent interest.

THEOREM 8.9. *There exists an L-definable function $n : L$-Singulars $\to \omega$ such that if M is an inner model, $0^\# \notin M$:*

(a) *For some n, $M \models \{\alpha \mid n(\alpha) \leq n\}$ is stationary.*
(b) *For each n there is a generic extension of M in which $0^\#$ does not exist and $\{\alpha \mid n(\alpha) \leq n\}$ is non-stationary.*

In (a) of the previous theorem, we intend that whenever $C \subseteq$ ORD is CUB and M-definable then there is $\alpha \in C$, $n(\alpha) \leq n$. In (b) we intend that the generic extension satisfy ZFC and have a definable CUB class $C \subseteq$ ORD such that $\alpha \in C \to n(\alpha) > n$.

Killing Admissibles Revisited

DEFINITION. α is **quasi R-admissible** if every well-ordering in $L_\alpha[R]$ has ordertype less than α.

R-admissibility implies quasi R-admissibility, but not conversely, as the limit of the first ω R-admissibles is quasi R-admissible but not R-admissible. Let $\Lambda^*(R)$ denote $\{\alpha > \omega \mid \alpha$ is quasi R-admissible$\}$, a CUB class of ordinals containing $\Lambda(R)$.

THEOREM 8.10. *Suppose φ is Σ_1 and $L \models \varphi(\kappa)$ whenever κ is an L-cardinal. Then there is a real $R <_L 0^\#$ such that $\Lambda^*(R) \subseteq \{\alpha \mid L \models \varphi(\alpha)\}$.*

COROLLARY 8.11 (Beller (in Beller-Jensen-Welch [82]), David [82]). *Suppose α is countable, $L_\alpha \models$ ZF. Then for some real R, α is the* least *ordinal such that $L_\alpha[R] \models$ ZF.*

COROLLARY 8.12. *There is a real $R <_L 0^\#$ such that $\Lambda^*(R) \subseteq \{\alpha \mid L_\alpha \models$ ZF $-$ Power$\}$.*

Non-Characterizability of Admissibility Spectra

There cannot be a simple characterization of admissibility spectra, by virtue of the following result.

THEOREM 8.13. *Let $X = \{A \subseteq \omega_1^L \mid A \in L$ and for some real $R, \omega_1^{L[R]} = \omega_1^L$ and $\Lambda(R) \cap \omega_1^L = A\}$. Then $X =_L 0^\#$.*

Δ_1-Coding

The results described here (with the exception of Theorem 8.22) are taken from Friedman-Veličković [97]. A real R Δ_1-**codes** a class $A \subseteq$ ORD iff A is Δ_1-definable over $L[R]$. Every L-amenable class A is Δ_1-coded by $0^{\#}$. The next result provides a converse to this result.

PROPOSITION 8.14. *Suppose that L-Card $= \{\alpha \mid \alpha$ is a cardinal of $L\}$ is Σ_1 over $L[R]$, R a real. Then $0^{\#} \leq_L R$.*

PROOF. Suppose that the Σ_1 definition has parameters less than κ, where κ is a singular cardinal. As κ^+ is an L-cardinal, by reflection there must be unboundedly many $\alpha < \kappa^+$, $\alpha \in L$-Card. But then $(\kappa^+)^L < \kappa^+$, which implies that $0^{\#}$ exists. As this argument can be carried out in $L[R]$, in fact $0^{\#} \leq_L R$. ⊣

We introduce a sufficient condition for an L-amenable class to be Δ_1-coded by a real which is class-generic over L. To motivate it we first indicate a necessary condition for Δ_1-codability:

DEFINITION. Suppose that x is an extensional set (i.e., $\langle x, \in\rangle$ satisfies the axiom of extensionality). Let $\bar{x}$ denote the transitive collapse of x. For $A \subseteq$ ORD we say that x **preserves** A if $\langle \bar{x}, \in, A \cap \bar{x}\rangle$ is isomorphic to $\langle x, \in, A \cap x\rangle$.

DEFINITION. For a set x and ordinal δ, $x[\delta]$ denotes $\{f(\gamma) \mid \gamma < \delta,\ f \in x,$ f a function whose domain includes $\gamma\}$. We say that x **strongly preserves** $A \subseteq$ ORD if $x[\delta]$ is extensional and preserves A for each cardinal δ. A sequence of extensional sets $t_0 \subseteq t_1 \subseteq \cdots$ is **tight** if it is continuous (i.e., $t_\lambda = \cup\{t_i \mid i < \lambda\}$ for limit λ) and for each i: $t_i = t_{i+1}$ or $t_i \in t_{i+1}$, $\langle \bar{t}_j \mid j < i\rangle$ belongs to the least ZF$^-$ model containing $\bar{t}_i$ as an element which correctly computes Card$(\bar{t}_i)$.

Condensation Condition Suppose that t is transitive, κ is regular, $\kappa \in t$ and $x \in t$. Then:

(a) There is a tight κ-sequence $t_0 \prec t_1 \prec \cdots \prec t$ such that $x \in t_0$ and for each $i < \kappa$: Card$(t_i) = \kappa$, t_i strongly preserves A.
(b) If κ is inaccessible then there exists $t_0 \prec t_1 \prec \cdots \prec t$ as above, but where Card$(t_i) = \omega_i$.

THEOREM 8.15. *(Δ_1-Coding Theorem) Suppose that A is L-amenable and obeys the Condensation Condition in L. Then A is Δ_1-coded in a tame class-generic extension of $\langle L, A\rangle$ by a real R such that $L, L[R]$ have the same cofinalities.*

Corollary 8.16. *Suppose that A is L-amenable, obeys the Condensation Condition in L and preserves indiscernibles. Then A is Δ_1-definable over $L[R]$ for some indiscernible preserving real R such that $L, L[R]$ have the same cofinalities.*

We can apply the above to show that $L\text{-cof}\,\omega = \{\alpha \mid \alpha$ has L-cofinality $\omega\}$ is Δ_1-definable in $L[R]$, where R is a real not constructing $0^\#$.

Lemma 8.17. *There is a real R_0, class-generic over L, such that $R_0 <_L 0^\#$, R_0 preserves all L-cardinals with the exception of ω_1^L and the Condensation Condition holds for $A = L\text{-cof}\,\omega$ in $L[R_0]$.*

Corollary 8.18. *There exists a real $R <_L 0^\#$ such that R is class-generic over L, R preserves indiscernibles and all L-cardinals greater than ω_1^L, and $L\text{-cof}\,\omega$ is Δ_1 over $L[R]$.*

Corollary 8.19. *There is a real $R <_L 0^\#$ such that every quasi R-admissible has uncountable L-cofinality.*

Corollary 8.20. *There is a real $R <_L 0^\#$ such that the function $f(\alpha) = [\alpha]^\omega \cap L$ is Δ_1 over $L[R]$.*

An **immune partition** is $F : \mathrm{ORD} \to 2$ such that neither $\{\alpha \mid F(\alpha) = 0\}$ nor $\{\alpha \mid F(\alpha) = 1\}$ contains an infinite constructible set.

Corollary 8.21. *There is a real $R <_L 0^\#$ such that some immune partition is $\Delta_1(L[R])$.*

We consider the "characterization problem" for Δ_1-definability in a real: Is there an exact constructible criterion for a subset of an L-cardinal κ to be the intersection with κ of a predicate which is Δ_1-definable in $L[R]$ for some real R that preserves L-cardinals? The answer is "No" when κ is ω_3^L.

Theorem 8.22. *Let $S = \{X \subseteq \omega_3^L \mid X = \omega_3^L \cap A$ for some $A = \mathrm{ORD}$, A Δ_1-definable in $L[R]$ for some real R that preserves L-cardinals$\}$. Then $S =_L 0^\#$.*

Theorem 8.22 rules out any simple characterization of when an L-amenable predicate can be Δ_1-definable in a real not constructing $0^\#$.

Minimal Coding

We have the following strengthening of the Coding Theorem.

THEOREM 8.23. *Suppose that* $A \subseteq \mathrm{ORD}$ *and* $\langle L[A], A\rangle$ *is a model of* $\mathrm{ZFC} + \mathrm{GCH}$. *Then there is an* $\langle L[A], A\rangle$*-definable class forcing* P *such that if* $G \subseteq P$ *is* P*-generic over* $\langle L[A], A\rangle$*:*

(a) $\langle L[A, G], A, G\rangle$ *is a model of* $\mathrm{ZFC} + \mathrm{GCH}$.
(b) $L[A, G] = L[R]$ *for some real* R *and* A, G *are definable over* $L[R]$ *from the parameter* R.
(c) $L[A]$ *and* $L[R]$ *have the same cofinalities.*
(d) R *is* ***minimal over*** $L[A]$*: if* $x \in L[R]$ *then either* $x \in L[A]$ *or* $R \in L[A, x]$.

Thus a universe obeying GCH can be "coded minimally" by a real. Note that in clause (d) of the Theorem, x is any set constructible from R, not necessarily a real.

Further Applications to Descriptive Set Theory

Solovay [70] established the consistency of a number of regularity properties for projective sets of reals, using a natural model in which ω_1 **is inaccessible to reals**, (i.e., ω_1 is an inaccessible cardinal in $L[R]$ for each real R). In this section we construct other models with this property, which can be applied to the study of regularity properties for projective sets and projective prewellorderings.

A set of reals is $\underline{\Sigma}^1_1$ if it is the continuous image of a Borel set and is $\underline{\Pi}^1_1$ if its complement is $\underline{\Sigma}^1_1$. It is $\underline{\Sigma}^1_{n+1}$ if it is the continuous image of a $\underline{\Pi}^1_n$ set and is $\underline{\Pi}^1_{n+1}$ if its complement is $\underline{\Sigma}^1_{n+1}$. A set of reals is $\underline{\Delta}^1_n$ if both it and its complement are $\underline{\Sigma}^1_n$. Similar definitions apply to k-ary relations on the reals. It a set of reals (or k-ary relation in reals) is $\underline{\Sigma}^1_n$ for some n then we say that it is **projective**.

Regularity Properties

DEFINITION. Measure ($\underline{\Sigma}^1_n$) is the assertion that every $\underline{\Sigma}^1_n$ set of reals is Lebesgue Measurable. Category ($\underline{\Sigma}^1_n$) is the assertion that every $\underline{\Sigma}^1_n$ set of reals has the Baire Property, i.e., has meager symmetric difference with some Borel set. Perfect ($\underline{\Sigma}^1_n$) is the assertion that any uncountable $\underline{\Sigma}^1_n$ set of reals contains a perfect closed subset. Similar definitions apply to $\underline{\Pi}^1_n, \underline{\Delta}^1_n$.

In ZFC one may prove Measure ($\underline{\Sigma}^1_1$), Category ($\underline{\Sigma}^1_1$), Perfect ($\underline{\Sigma}^1_1$). In Gödel's model L one has $\sim$Measure ($\underline{\Delta}^1_2$), $\sim$Category ($\underline{\Delta}^1_2$), $\sim$Perfect ($\underline{\Pi}^1_1$) using the fact that in L there is a $\underline{\Delta}^1_2$ wellordering of the reals (and the Kondo-Addison Uniformization Theorem for $\underline{\Pi}^1_1$). By extending ZFC slightly we get:

THEOREM 8.24. (Solovay [69]) *Assume that* ω_1 *is inaccessible to reals. Then the following hold: Measure* ($\underline{\Sigma}^1_2$), *Category* ($\underline{\Sigma}^1_2$), *Perfect* ($\underline{\Sigma}^1_2$).

Our next result implies that the previous Theorem is optimal. The proof is based on David [83].

THEOREM 8.25. *Assume the consistency of an inaccessible cardinal. Then there is a model in which:*

(a) ω_1 *is inaccessible to reals.*
(b) *There is a* Δ^1_3 *wellordering of the reals, and hence* $\sim$*Measure* (Δ^1_3), $\sim$*Category* (Δ^1_3).
(c) $\sim$*Perfect* (Π^1_2).

Remark. We use $\Sigma^1_n, \Pi^1_n, \Delta^1_n$ to denote the "effective" versions of $\underline{\Sigma}^1_n, \underline{\Pi}^1_n, \underline{\Delta}^1_n$; see Moschovakis [80] for details.

Another axiom with consequences for regularity properties of projective sets is Martin's Axiom (MA). (We take MA to include the hypothesis $\sim$ CH.)

THEOREM 8.26. *MA implies Measure* ($\underline{\Sigma}^1_2$), *Category* ($\underline{\Sigma}^1_2$).

Again this is optimal.

THEOREM 8.27. *This is a model of MA in which:*

(a) $\omega_1 = \omega_1^L$.
(b) *There is a* Δ^1_3 *wellordering of the reals.*

Remark. Perfect (Π^1_1) fails in the above model, as this property implies that ω_1^L is countable. It is not known if (a) can be replaced by "ω_1 is inaccessible to reals" in the previous theorem (assuming the consistency of a weakly compact cardinal; this is a necessary assumption for the consistency of MA $+\omega_1$ inaccessible to reals).

Theorem 8.25 generalizes to higher levels of the projective hierarchy. Recall that κ is **Mahlo** if κ is inaccessible and $\{\bar\kappa < \kappa \mid \bar\kappa \text{ regular}\}$ is stationary.

THEOREM 8.28. *Assume the consistency of a Mahlo cardinal. Then there is a model in which:*

(a) *Measure* ($\underline{\Sigma}^1_3$), *Category* ($\underline{\Sigma}^1_3$). *Perfect* ($\underline{\Sigma}^1_3$).

(b) *There is a* Δ^1_4 *wellordering of the reals.*
(c) $\sim$*Perfect* (Π^1_3).

Remark. To go further, one must replace L by a sufficiently $\underset{\sim}{\Sigma}^1_3$ correct model. Thus, assuming the consistency of "every set has a sharp" together with a Mahlo cardinal, one obtains a model of Measure $(\underset{\sim}{\Sigma}^1_4)$, Category $(\underset{\sim}{\Sigma}^1_4)$, Perfect $(\underset{\sim}{\Sigma}^1_4)$, $\sim$Perfect (Π^1_4) with a Δ^1_5 wellordering of the reals. However the author does not know if this use of #'s is necessary.

Prewellorderings

A **prewellordering** is a reflexive, transitive well-founded relation. A wellordering is obtained by identifying two elements a, b when $a \leq b$, $b \leq a$; the **length** of the prewellordering is the ordertype of its associated wellordering.

$\underset{\sim}{\delta}^1_n$ denotes the supremum of the lengths of all $\underset{\sim}{\Delta}^1_n$ prewellorderings of the reals.

THEOREM 8.29. *(Classical)* $\underset{\sim}{\delta}^1_1 = \omega_1$.

Kunen and Martin showed that $\underset{\sim}{\delta}^1_2$ is at most ω_2 (see Martin [77]). The next result shows that this result is the best possible.

THEOREM 8.30. *It is consistent with* ZFC *that* $\underset{\sim}{\delta}^1_2 = \omega_2$.

Using the Condensation Condition, we can simultaneously have ω_1 inaccessible to reals:

THEOREM 8.31. *(*Friedman-Woodin [96]*) Assuming the consistency of an inaccessible, there is a model in which* $\underset{\sim}{\delta}^1_2 = \omega_2$ *and* ω_1 *is inaccessible to reals.*

There is no explicit bound on $\underset{\sim}{\delta}^1_3$ provable in ZFC, even with the added hypothesis that ω_1 is inaccessible to reals.

THEOREM 8.32. *(Section 8.4 of* Friedman [99]*) Assuming the consistency of an inaccessible, there is a model in which* ω_1 *is inaccessible to reals and there is a* Π^1_2 *wellordering of some set of reals of length* κ*, for any pre-chosen* L*-definable cardinal* κ *(and hence* $\underset{\sim}{\delta}^1_3 \geq \kappa$*).*

9. **Some Open Problems**

1. Can one code a class by a real, preserving Π^n_m-indescribability?
2. Define *n-generic over L* as follows: R is *0-generic over L* iff R is generic over L. R is *$n+1$-generic over L* iff R is generic over an inner model of $L[S]$, where S is n-generic over L. Does $n+1$-genericity imply n-genericity for some n? Is there a real $R <_L 0^\#$ which is *not* n-generic over L for any n?
3. Is $0^\#$ generic over some proper inner model of $L[0^\#]$?
4. Can one prove that $L[0^\#]$ is generically saturated over L in the theory ZFC $+0^\#$ exists?
5. Is $L[0^\#]$ the *least* inner model which is generically saturated over L?
6. Is there a reasonable notion of "forcing" with the property that every real either constructs $0^\#$ or can be obtained by "forcing" over L?
7. Is there a real R, $0 <_L R <_L 0^\#$, which is the unique solution to a Π^1_2 formula φ which *provably in* ZFC has at most one solution?
8. Is there a simple characterization of the reals which belong to a countable Π^1_2 set?
9. Assuming only the consistency of an inaccessible cardinal, is it consistent for each n that all $\underset{\sim}{\Sigma}^1_n$ sets of reals be Lebesgue Measurable and have the Baire and Perfect Set properties, while there is a Δ^1_{n+1} wellordering of the reals?
10. Assuming only the consistency of a weakly compact cardinal, is it consistent to have Martin's Axiom, ω_1 inaccessible to reals with a Δ^1_3 wellordering of the reals?
11. Is it consistent for Δ^1_3-reducibility and L-reducibility to coincide?
12. Assuming only the consistency of an inaccessible cardinal, is it consistent for Post's Problem to fail in HC = the hereditarily countable sets?
13. Is there a *remarkable real*; i.e., a real $R <_L 0^\#$ such that R is not generic over L, R is a Π^1_2-singleton, $\Lambda(R) =$ the recursively inaccessible ordinals and R has minimal L-degree? It has not yet been shown that there is a real $R <_L 0^\#$ which has more than one of these properties simultaneously.

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