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JENSEN'S Σ^* THEORY AND THE COMBINATORIAL CONTENT OF $V = L$

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An awkward feature of the standard fine structure theory of the J_α's (see Jensen [72]) is that special parameters are required to make good sense of the notion of "Σ_n Skolem hull", parameters which may not be preserved in condensation arguments.

The purpose of this article is to indicate how a reformulation of Jensen's Σ^* theory (developed for the study of core models) can be used to provide a more satisfactory treatment of uniformization, hulls, and Skolem functions for the J_α's. Then we use this approach to fine structure to formulate a principle intended to capture the combinatorial content of the axiom $V = L$.

§**1. Fine structure revisited.** We begin with a simplified definition of the J-hierarchy. Inductively we define $\widetilde{J}_\alpha$, $\alpha \in \mathrm{ORD}$ (and then $J_\alpha = \widetilde{J}_{\omega\alpha}$) : $\widetilde{J}_n = V_n$ for $n \leq \omega$. Suppose $\widetilde{J}_\lambda$ is defined for a limit λ and let $W_n^\lambda(e, x)$ be a canonical universal $\Sigma_n(\widetilde{J}_\lambda)$ predicate (also defined inductively). For $e \in \widetilde{J}_\lambda$ let $X_1^\lambda(e) = \{x \mid W_1^\lambda(e, x)\}$, and for $n \geq 1$ let $X_{n+1}^\lambda(e) = \{X_n^\lambda(\overline{e}) \mid W_{n+1}^\lambda(e, \overline{e})\}$. Then $\widetilde{J}_{\lambda+n} = \{X_n^\lambda(e) \mid e \in \widetilde{J}_\lambda\}$. For all limit λ, $\widetilde{J}_\lambda = \bigcup\{\widetilde{J}_\delta \mid \delta < \lambda\}$. It is straightforward to verify that the $\widetilde{J}_\lambda$, λ limit, behave like, and in fact equal, the usual J_α's.

Let M denote some J_α, $\alpha > 0$. (More generally, our theory applies to "acceptable J-models".) We make the following definitions, inductively. We order finite sets of ordinals by the maximum difference order: $x < y$ iff $\alpha \in y$, where α is the largest element of $(y - x) \cup (x - y)$.

1) A Σ_1^* formula is just a Σ_1 formula. A predicate is $\underline{\Sigma_1^*}$ (Σ_1^*, respectively) if it is definable by a Σ_1^* formula with (without, respectively) parameters. ρ_1^M = Σ_1^* projectum of M = least ρ such that there is a $\underline{\Sigma_1^*}$ subset of ρ not in M and p_1^M = least p such that $A \cap \rho_1^M \notin M$ for some A Σ_1^* in parameter p (where p is a finite set of ordinals). $H_1^M = H_{\rho_1^M}^M$ = sets x in M such that

$$M\text{-card}(\text{transitive closure } (x)) < \rho_1^M.$$

For any $x \in M$, $M_1(x)$ = first reduct of M relative to $x = \langle H_1^M, A_1(x)\rangle$, where $A_1(x) \subseteq H_1^M$ codes the Σ_1^* theory of M with parameters from $H_1^M \cup \{x\}$ in the natural way: $A_1(x) = \{\langle y, n\rangle \mid$ the nth Σ_1^* formula is true at $\langle y, x\rangle$, $y \in H_1^M\}$. A

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good Σ_1^* function is just a Σ_1 function, and for any $X \subseteq M$ the Σ_1^* hull (X) is just the Σ_1 hull of X.

2) For $n \geq 1$, a Σ_{n+1}^* formula is one of the form $\varphi(x) \longleftrightarrow M_n(x) \models \psi$, where ψ is Σ_1. A predicate is $\underline{\Sigma_{n+1}^*}$ (Σ_{n+1}^*, respectively) if it is defined by a Σ_{n+1}^* formula with (without, respectively) parameters. $\rho_{n+1}^M = \Sigma_{n+1}^*$ projectum of M = least ρ such that there is a $\underline{\Sigma_{n+1}^*}$ subset of ρ not in M and $p_{n+1}^M = p_n^M \cup p$, where p is least such that $A \cap \rho_{n+1}^M \notin M$ for some A Σ_{n+1}^* in parameter $p_n^M \cup p$. $H_{n+1}^M = H_{\rho_{n+1}^M}^M$ = sets x in M such that M-card(transitive closure $(x)) < \rho_{n+1}^M$. For any $x \in M$, $M_{n+1}(x) =$ $(n+1)$st reduct of M relative to $x = \langle H_{n+1}^M, A_{n+1}(x)\rangle$, where $A_{n+1}(x) \subseteq H_{n+1}^M$ codes the Σ_{n+1}^* theory of M with parameters from $H_{n+1}^M \cup \{x\}$ in the natural way: $A_{n+1}(x) = \{\langle y, m\rangle |$ the mth Σ_{n+1}^* formula is true at $\langle y, x\rangle, y \in H_{n+1}^M\}$. A good Σ_{n+1}^* function f is a function whose graph is Σ_{n+1}^* with the additional property that for $x \in \mathrm{Dom}(f)$, $f(x) \in \Sigma_n^*$ hull $(H_n^M \cup \{x\})$. The Σ_{n+1}^* hull (X) for $X \subseteq M$ is the closure of X under good Σ_{n+1}^* functions.

FACTS. (a) *φ, ψ are Σ_n^* formulas $\longrightarrow \varphi \vee \psi, \varphi \wedge \psi$ are Σ_n^* formulas.*

(b) *φ Σ_n^* or Π_n^* (= negation of Σ_n^*) $\longrightarrow \varphi$ is Σ_{n+1}^*.*

(c) *$Y \subseteq \Sigma_n^*$ hull $(X) \longrightarrow \Sigma_n^*$ hull $(Y) \subseteq \Sigma_n^*$ hull (X).*

(d) *f good Σ_n^* function $\longrightarrow f$ good Σ_{n+1}^* function.*

(e) *Σ_n^* hull $(X) \subseteq \Sigma_{n+1}^*$ hull (X).*

(f) *There is a Σ_n^* relation $W(e, x)$ such that if $S(x)$ is Σ_n^* then for some $e \in \omega$, $S(x) \longleftrightarrow W(e, x)$ for all x.*

(g) *The structure $M_n(x) = \langle H_n^M, A_n(x)\rangle$ is amenable.*

(h) $H_n^M = J_{\rho_n^M}^{A_n}$, *where* $A_n = A_n(0)$.

(i) *Suppose $H \subseteq M$ is closed under good Σ_n^* functions and $\pi : \overline{M} \longrightarrow M, \overline{M}$ transitive, Range$(\pi) = H$ and $p_{n-1}^M \in H$ (if $n > 1$). Then π preserves Σ_n^* formulas: for $\Sigma_n^* \varphi$ and $x \in \overline{M}$, $\overline{M} \models \varphi(x) \longleftrightarrow M \models \varphi(\pi(x))$. And (for $n > 1$), $\pi(p_{n-1}^{\overline{M}}) = p_{n-1}^M$.*

Proof of (i). Note that $H \cap M_{n-1}(\pi(x))$ is Σ_1-elementary in $M_{n-1}(\pi(x))$, and $\pi^{-1}[H \cap M_{n-1}(\pi(x))] = \langle J_\rho^A, A(x)\rangle$ for some $\rho, A, A(x)$. But (by induction on n) $A = A_{n-1}^{\overline{M}} \cap J_\rho^A$, $A(x) = A_{n-1}(x)^{\overline{M}} \cap J_\rho^A$, and $\rho = \rho_{n-1}^{\overline{M}}$ using our assumption about the parameter p_{n-1}^M. Also, $\pi^{-1}(p_{n-1}^M) = \bar{p}$ must be $p_{n-1}^{\overline{M}}$ as $\overline{M} = \Sigma_{n-1}^*$ hull of $H_{n-1}^{\overline{M}} \cup \{p_{n-1}^{\overline{M}}\}$.

THEOREM 1. *By induction on $n > 0$:*

(1) *If $\varphi(x, y)$ is Σ_n^* then $\exists y \in \Sigma_{n-1}^*$ hull $(H_{n-1}^M \cup \{x\})$ $\varphi(x, y)$ is also Σ_n^*.*

(2) *If $\varphi(x_1 \cdots x_k)$ is $\Sigma_m^*, m \geq n$, and $f_1(x), \cdots, f_k(x)$ are good Σ_n^* functions, then $\varphi(f_1(x) \cdots f_k(x))$ is Σ_m^*.*

(3) *The domain of a good Σ_n^* function is Σ_n^*.*

(4) *Good Σ_n^* functions are closed under composition.*

(5) *(Σ_n^* Uniformization) If $R(x, y)$ is Σ_n^* then there is a good Σ_n^* function $f(x)$ such that $x \in \mathrm{Dom}(f) \longleftrightarrow \exists y \in \Sigma_{n-1}^*$ hull $(H_{n-1}^M \cup \{x\})$ $R(x, y) \longleftrightarrow R(x, f(x))$.*

(6) *There is a good Σ_n^* function $h_n(e, x)$ such that for each x, Σ_n^* hull $(\{x\}) = \{h_n(e, x) | e \in \omega\}$.*

PROOF. The base case $n = 1$ is easy (take Σ_0^* hull $(X) = M$ for all X). Now we prove it for $n > 1$, assuming the result for smaller n.

(1) Write $\exists y \in \Sigma_{n-1}^*$ hull $(H_{n-1}^M \cup \{x\})\varphi(x, y)$ as $\exists \bar{y} \in H_{n-1}^M \varphi(x, h_{n-1}(e, \langle x, \bar{y}\rangle))$

using (6) for $n-1$. Since h_{n-1} is good Σ^*_{n-1} we can apply (2) for $n-1$ to conclude that $\varphi(x, h_{n-1}(e, \langle x, \bar{y}\rangle))$ is Σ^*_n. Since the quantifiers $\exists e \exists \bar{y} \in H^M_{n-1}$ range over H^M_{n-1}, they preserve Σ^*_n-ness.

(2) $\varphi(f_1(x) \cdots f_k(x)) \longleftrightarrow \exists x_1 \cdots x_k \in \Sigma^*_{n-1}$ hull $(H^M_{n-1} \cup \{x\})$ $[x_i = f_i(x)$ for $1 \leq i \leq k \wedge \varphi(x_1 \cdots x_k)]$. If $m = n$ then this is Σ^*_n by (1). If $m > n$ then reason as follows: the result for $m = n$ implies that $A_n(\langle f_1(x) \cdots f_k(x)\rangle)$ is Δ_1 over $M_{n+1}(x)$. Thus $A_{m-1}(\langle f_1(x) \cdots f_k(x)\rangle)$ is Δ_1 over $M_{m-1}(x)$. So as φ is Σ^*_m we get that $\varphi(f_1(x) \cdots f_k(x))$ is also Σ_1 over $M_{m-1}(x)$, hence Σ^*_m.

(3) If $f(x)$ is good Σ^*_n, then $\text{dom}(f) = \{x | \exists y \in \Sigma^*_{n-1}$ hull of $H^M_{n-1} \cup \{x\}(y = f(x))\}$ is Σ^*_n by (1).

(4) If f, g are good Σ^*_n, then the graph of $f \circ g$ is Σ^*_n by (2). Also, $f \circ g(x) \in \Sigma^*_{n-1}$ hull$(H^M_{n-1} \cup \{x\})$ since the latter hull contains $g(x)$, f is good Σ^*_n, and Fact (c) holds.

(5) Using (6) for $n-1$, let $\overline{R}(x, \bar{y}) \longleftrightarrow R(x, h_{n-1}(\bar{y})) \wedge \bar{y} \in H^M_{n-1}$. Then $\overline{R}$ is Σ^*_n by (2) for $n-1$ and, using Σ_1 uniformization on $(n-1)$st reducts, we can define a good Σ^*_n function $\overline{f}$ such that $\overline{R}(x, \overline{f}(x)) \longleftrightarrow \exists \bar{y} \in H^M_{n-1} \overline{R}(x, \bar{y})$. Let $f(x) = h_{n-1}(\overline{f}(x))$. Then f is good Σ^*_n by (4).

(6) Let W be universal Σ^*_n as in Fact (f). By (5) there is a good Σ^*_n $g(e, x)$ such that $\exists y \in \Sigma^*_{n-1}$ hull$(H^M_{n-1} \cup \{x\})$ $W(e, \langle x, y\rangle) \longleftrightarrow W(e, \langle x, g(e, x)\rangle)$ (and $g(e, x)$ defined $\longrightarrow W(e, \langle x, g(e, x)\rangle)$). Let $h_n(e, x) = g(e, x)$. If $y \in \Sigma^*_n$ hull $(\{x\})$ then, for some e, $W(e, \langle x, y'\rangle) \longleftrightarrow y' = y$, so $y = h_n(e, x)$. Clearly $h_n(e, x) \in \Sigma^*_n$ hull $(\{x\})$, since h_n is good Σ^*_n. □

REMARK. It follows from Jensen [72] (and is explicitly indicated in Jensen [89]) that for any J_α there is a parameter p such that, over J_α, $\Sigma_n = \Sigma^*_n$ relative to p and all Σ_n functions are good Σ^*_n relative to p. We shall make use of this fact in the next section (see the proof of Proposition 2).

§2. The combinatorial content of $V = L$. In this section we provide an axiomatic treatment of the Σ^* theory introduced in §1. When establishing combinatorial principles in $L[R]$, R a real, one makes use of a *standard Skolem system* for R (defined below), of which the system of canonical Σ^*_n Skolem functions for the J^R_α's (relativized to p_{n-1}) constitutes the canonical example. Our principal goal is to provide combinatorial axioms for a system of functions which guarantee that it is in fact a standard Skolem system for some real. These axioms can then be used to formulate a single combinatorial principle which captures the full power of Jensen's fine structure theory.

Some notation: For $\delta = \lambda + n$, λ limit or 0 and $n \in \omega$, $\text{Seq}(\delta)$ denotes all finite sequences from λ together with all finite sequences from δ of length $\leq n$. Let $x * y$ denote the concatenation of the sequences x, y. For λ limit or $0, \widetilde{J}^R_\lambda$ denotes J^R_δ where $\omega \cdot \delta = \lambda$.

A *standard Skolem system* for a real R is a system $\vec{F} = \langle F^\delta_n | n > 0, \delta \in \text{ORD}, n > 1 \longrightarrow \delta \text{ limit}\rangle$, where F^δ_n is a partial function from $\omega \times \text{Seq}(\delta)$ to δ, obeying (A)–(E) below. For any limit $\lambda, x \in \text{Seq}(\lambda), n \geq 1$ let $H^\lambda_n(x) = \{F^\lambda_n(k, x) | k \in \omega\}$ and if $\bar{\lambda} = \text{ordertype } (H^\lambda_n(x))$ let $\pi^n_{\bar{\lambda}\lambda}(x) : \bar{\lambda} \longrightarrow \lambda$ be the increasing enumeration of $H^\lambda_n(x)$. We say $y \in H^\lambda_n(x)$, for $y \in \text{Seq}(\lambda)$, if each coordinate of y belongs to $H^\lambda_n(x)$.

(A) (Monotonicity). $\delta_1 \le \delta_2 \longrightarrow F_1^{\delta_1} \subseteq F_1^{\delta_2}, x \in H_1^\lambda(x) \subseteq H_2^\lambda(x) \subseteq \cdots \subseteq \lambda$ for limit λ, and $x \in \mathrm{Seq}(\lambda)$.

(B) (Condensation). Let $\pi = \pi^n_{\bar\lambda\lambda}(x)$. Then for $m \le n$, and $\bar x \in \mathrm{Seq}(\bar\lambda)$, $\pi(F_m^{\bar\lambda}(k,\bar x)) \simeq F_m^\lambda(k,\pi(\bar x))$. Also, $\tilde\pi(F_1^{\bar\lambda+m}(k,\bar x)) \simeq F_1^{\lambda+m}(k,\tilde\pi(\bar x))$ for $\bar x \in \mathrm{Seq}(\bar\lambda+m)$, where $\tilde\pi$ is the extension of π to $\bar\lambda+m$ obtained by sending $\bar\lambda+i$ to $\lambda+i$.

(C) (Continuity). For limit $\lambda, F_1^\lambda = \bigcup\{F_1^\delta \mid \delta < \lambda\}$. For all $x \in \mathrm{Seq}(\lambda)$ and $y < \lambda$, $F_{n+1}^\lambda(x) \simeq y$ iff there is some $z \in \mathrm{Seq}(\lambda)$ such that for all $w \in \mathrm{Seq}(\lambda), F_{n+1}^{\bar\lambda}(\bar x) \simeq \bar y$, where $\bar\lambda =$ ordertype $(H_n^\lambda(z*w))$ and $\pi^n_{\bar\lambda\lambda}(z*w)$ sends $\bar x, \bar y$ to x, y.

(D) $\langle F_n^\delta \mid \delta < \lambda, n < \omega\rangle$ is uniformly $\Delta_1(\tilde J_\lambda^R)$ for limit λ, in the parameter R.

(E) For limit $\lambda, H_1^\lambda(x) = \lambda \cap \Sigma_1$ Skolem hull of $\langle x, R\rangle$ in $\tilde J_\lambda^R$ and $\bigcup_n H_n^\lambda(x) = \lambda \cap$ Skolem hull of $\langle x, R\rangle$ in $\tilde J_\lambda^R$ for $x \in \mathrm{Seq}(\lambda)$.

Intuitively, F_n^λ is a Σ_n^* Skolem function for $\tilde J_\lambda^R$ relativized to p_{n-1}, and $F_1^{\lambda+n}$ is the nth approximation to $F_1^{\lambda+\omega}$.

Proposition 2. *For every real R there exists a standard Skolem system for R.*

Proof. Let $\psi \longmapsto \psi_n^*$ be a recursive translation on formulas so that for limit λ, $\tilde J_{\lambda+n}^R \models \psi \longleftrightarrow \tilde J_\lambda^R \models \psi_n^*$ (where $\tilde J_\alpha^R$ is defined just like $\tilde J_\alpha$, but relativized to R). Fix a recursive enumeration $\langle \varphi_k(v) \mid k \in \omega\rangle$ of Δ_0 formulas with a predicate $\underline{R}$ denoting R and sole free variable v. Let $<_R$ denote the ordering of $L[R]$ given by: $x <_R y$ iff $\exists\lambda \in \mathrm{Lim} \cup \{0\} \exists n \in \omega$ [$y \in \tilde J_{\lambda+n+1}^R - \tilde J_{\lambda+n}^R$ and either ($x \in \tilde J_{\lambda+n}^R$) or ($\lambda$ limit, $x \in \tilde J_{\lambda+n+1}^R$, $e <_R f$ where e, f are $<_R$-least such that $X_{n+1}^{\lambda,R}(e) = x$, $X_{n+1}^{\lambda,R}(f) = y$) or ($\lambda = 0$ and $x <_L y$)].

Now define $\vec F = \langle F_n^\delta \mid \delta \in \mathrm{ORD}, n > 0, n > 1 \longrightarrow \delta \text{ limit}\rangle$ as follows:

(a) $F_1^n(k,x) \simeq y$ iff $L_n^R \models \exists w : \langle y, w\rangle$ is $<_R$-least such that $\varphi_k(\langle x, y, w\rangle)$.

(b) For λ limit, $F_1^\lambda = \bigcup\{F_1^\delta \mid \delta < \lambda\}$.

(c) For λ limit, $n > 0$, $F_1^{\lambda+n}(k,x) \simeq y$ iff for some $m \le n$, $\tilde J_{\lambda+m}^R \models$ ($\exists w$: $\langle y, w\rangle$ is $<_R$-least such that $\varphi_k(\langle x, y, w\rangle)$) and if ψ denotes the formula in parentheses then ψ_m^* is Σ_n^*.

(d) For λ limit, $n > 1$, $F_n^\lambda(k,x) = F(k, x * p_{n-1})$, where F is the canonical Σ_n^* Skolem function for $\tilde J_\lambda^R$ (restricted to $\omega \times \mathrm{Seq}(\lambda)$) as in (6) of Theorem 1.

The verification that $\vec F$ is a standard Skolem system for R is straightforward. To prove Continuity, use the fact that for some choice of $z \in \mathrm{Seq}(\lambda)$ we have $\Sigma_n^* = \Sigma_n$ relative to z (over $\tilde J_\lambda^R$). □

An *abstract Skolem system* is a system $\vec F$ obeying properties (A), (B), (C) from the definition of standard Skolem system. We would like to prove that every abstract Skolem system is a standard Skolem system for some real. However, standard systems share one further property which we must also impose:

(Stability). For λ limit, $x \in \mathrm{Seq}(\lambda)$ let $\pi : \bar\lambda \longrightarrow \lambda$ be the increasing enumeration of $H_1^\lambda(x)$. Then π extends uniquely to a Σ_1-elementary embedding of $\langle \tilde J_{\bar\lambda}^{\vec F}, \vec F \restriction \bar\lambda\rangle$ into $\langle \tilde J_\lambda^{\vec F}, \vec F \restriction \lambda\rangle$. Also for λ limit, $x \in \mathrm{Seq}(\lambda)$, if $\pi : \bar\lambda \longrightarrow \lambda$ is the increasing enumeration of $H^\lambda(x) = \bigcup_n H_n^\lambda(x)$ then π extends uniquely to an elementary embedding of $\langle \tilde J_{\bar\lambda}^{\vec F}, \vec F \restriction \bar\lambda\rangle$ into $\langle \tilde J_\lambda^{\vec F}, \vec F \restriction \lambda\rangle$.

Though Stability is not combinatorial, we shall see that any abstract Skolem system can be made stable without changing its "cofinality function". This fact will enable us to formulate combinatorial principles which are universal for principles which depend only on cofinality.

THEOREM 3. *The following are equivalent*:

(a) $\vec{F}$ *is a stable, abstract Skolem system.*

(b) $\vec{F}$ *is a standard Skolem system in a CCC forcing extension of V.*

Note that (b) $\longrightarrow$ (a) is easy, using the absoluteness of the concept of Stability. We now develop the forcing required to prove (a) $\longrightarrow$ (b).

Fix a stable, abstract Skolem system $\vec{F}$ and let M denote $L[\vec{F}]$, $M_\lambda = \langle \widetilde{J}_\lambda^{\vec{F}}, \vec{F} \restriction \lambda \rangle$ for limit λ. The desired forcing $\mathscr{P}$ is a CCC forcing of size ω_1 in M. It is designed so as to produce a generic real R which codes $\vec{F} \restriction \omega_1$ via a careful almost disjoint coding. We will demonstrate that R in fact codes all of $\vec{F}$ using condensation properties of $\vec{F}$.

We begin our description of $\mathscr{P}$. A limit ordinal λ is *small* if for some $x \in \text{Seq}(\lambda)$ and some $n, H_n^\lambda(x) = \lambda$. Let $n(\lambda)$ be the least n for which such an x exists and let p^λ be the least $p \in \text{Seq}(\lambda)$ for which $H_{n(\lambda)}^\lambda(p) = \lambda$. We now define a canonical bijection $\overline{f}_\lambda : \lambda \longrightarrow \omega$. First let $g : \lambda \longrightarrow \omega$ be defined by $g(\delta) =$ least k such that $\delta = F_{n(\lambda)}^\lambda(k, p^\lambda)$. Then $\overline{f}_\lambda(\delta) = m$ if $g(\delta)$ is the mth element of Range(g) under $<$ on ω. Now let $f_\lambda : \omega \longrightarrow M_\lambda$ be $g^* \circ \overline{f}_\lambda^{-1}$, where $g^* : \lambda \longrightarrow M_\lambda$ is a canonical $\underset{\sim}{\Delta}_1(M_\lambda)$ bijection. Now choose $A_\lambda \subseteq \omega$ to code M_λ using f_λ, and let $b_{\lambda+n(\lambda)}$ be a function from ω to ω which is $\Delta_{n(\lambda)+1}\langle M_\omega, A_\lambda\rangle$ yet eventually dominates each function from ω to ω which is $\Delta_{n(\lambda)}\langle M_\omega, A_\lambda\rangle$. Also require that Range $(b_{\lambda+n(\lambda)}) \subseteq_*$ Range$(b_{\overline{\lambda}+n})$ for all $\overline{\lambda} < \lambda$ and $n < \omega$, where we have (inductively) defined $b_{\overline{\lambda}+n}$. ($\subseteq_*$ denotes inclusion except for a finite set.)

We also define $b_{\lambda+n}$ for $n = n(\lambda) + m, m > 0$. For this purpose define $\overline{F}_1^{\lambda+n}(k, \overline{x}) \simeq \overline{y}$ to mean $F_1^{\lambda+n}(k, x) \simeq y$, where $x(i) = \lambda + \overline{x}(i)$ if $\overline{x}(i) < n$, and $\overline{x}(i) = n + x(i)$ otherwise (similarly for y). Let $A_{\lambda+m} \subseteq \omega$ code:

$$\langle M_\lambda, \overline{F}_1^{\lambda+n(\lambda)}, \cdots, \overline{F}_1^{\lambda+n(\lambda)+m-1}, F_{n(\lambda)}^\lambda, \cdots, F_{n(\lambda)+m-1}^\lambda\rangle$$

using f_λ, and let $b_{\lambda+n(\lambda)+m}$ be a function from ω to ω that is $\Delta_{n(\lambda)+m+1}\langle M_\omega, A_{\lambda+m}\rangle$ yet eventually dominates $\Delta_{n(\lambda)+m}\langle M_\omega, A_{\lambda+m}\rangle$ functions. Furthermore, require that Range$(b_{\lambda+n(\lambda)+m}) \subseteq_*$ Range$(b_{\lambda+n(\lambda)+m-1})$. We use the $b_{\lambda+n}, n \geq n(\lambda)$, to facilitate the desired almost disjoint coding.

An *index* is a tuple of one of the forms $\langle \lambda + n, 1, k, \overline{x}, \overline{y}\rangle$, $\langle \lambda, n, k, \overline{x}, \overline{y}\rangle$, where λ is small, $n \geq n(\lambda)$, and $\overline{F}_1^{\lambda+n}(k, \overline{x}) \simeq \overline{y}$ or $F_n^\lambda(k, \overline{x}) \simeq \overline{y}$, respectively. Let $\langle Z_e | e \in \omega\rangle$ be a recursive partition of $\omega - \{0\}$ into infinite pieces. For each index x we define a "code" b_x as follows: If $x = \langle \lambda + n, 1, k, \overline{x}, \overline{y}\rangle$, $\langle \lambda, n, k, \overline{x}, \overline{y}\rangle$, then $b_x = b_{\lambda+n} \restriction Z_e$, where $f_\lambda(e) = \langle n, 1, k, \overline{x}, \overline{y}\rangle$, $\langle 0, n, k, \overline{x}, \overline{y}\rangle$, respectively. A *restraint* is a function of the form b_x, x an index. We sometimes view b_x as a subset of ω by identifying it with $\{\langle n, m\rangle | b_x(n) = m\}$, $\langle \cdot, \cdot\rangle$ a recursive pairing on ω.

A condition in $\mathscr{P}$ is $p = \langle s, \overline{s}\rangle$, where $s : |s| \longrightarrow 2$, $|s| \in \omega$, $\overline{s}$ is a finite set of restraints, and when $i = \langle m, k, x, y\rangle < |s|$ then $s(i) = 1 \longleftrightarrow F_1^m(k, x) \simeq y$.

Extension is defined by: $(s,\bar{s}) \le (t,\bar{t})$ iff $s \supseteq t$, $\bar{s} \supseteq \bar{t}$ and $s(i) = 1 \longrightarrow t(i) = 1$ or $i \notin \bigcup \bar{t}$. (Recall that we can think of $b_x \in \bar{t}$ as a subset of ω.)

This is a CCC forcing, and a generic G is uniquely determined by the real $R = \bigcup\{s|(s,\bar{s}) \in G$ for some $\bar{s}\}$. Fix such a real R.

LEMMA 4. *$\langle F_n^\delta | \delta < \lambda, n < \omega\rangle$ is uniformly $\Delta_1(\widetilde{J}_\lambda^R)$ for limit λ, in the parameter R.*

PROOF. By induction we define F_n^λ, $F_1^{\lambda+n}$ for λ limit or 0, $n \in \omega$. If $\lambda = 0$ then F_1^n can be defined directly from R by the restriction we placed on s for conditions $(s,\bar{s})$. For λ limit, F_1^λ is defined by induction and Continuity. Also, induction and Continuity enable us to define F_n^λ, $F_1^{\lambda+n}$ provided $n \le n(\lambda) \ne 1$ or $n(\lambda)$ is not defined. Thus if λ is not small we are done, and otherwise we can define $f_\lambda, b_{\lambda+n}$, by induction. Let $f_\lambda(e) = \langle n, 1, k, \bar{x}, \bar{y}\rangle$. Then $\overline{F}_1^{\lambda+n}(k,\bar{x}) \simeq \bar{y}$ iff $\langle \lambda+n, 1, k, \bar{x}, \bar{y}\rangle$ is an index iff R is almost disjoint from $b_{\lambda+n} \restriction Z_e$. The definition of F_n^λ is similar, using $\langle 0, n, k, \bar{x}, \bar{y}\rangle$. □

Our next goal is to establish a strong statement of the definability of the forcing relation for $\mathscr{P}$. For any infinite ordinal δ we let $\mathscr{P}(\delta)$ denote those conditions in $\mathscr{P}$ involving restraints with indices $\langle \lambda+n, 1, k, \bar{x}, \bar{y}\rangle$, $\langle \lambda, n, k, \bar{x}, \bar{y}\rangle$, where $\lambda + n < \delta$. For $p \in \mathscr{P}$ we let $p \restriction \delta$ be obtained from p by discarding all restraints which are not of the above form.

LEMMA 5 (Persistence). *Let λ be small and for $p \in \mathscr{P}(\lambda+\omega)$ let p^* be obtained by replacing each of its restraints of the form $b_x, x = \langle \lambda+n, 1, k, \bar{x}, \bar{y}\rangle$, $\langle \lambda, n, k, \bar{x}, \bar{y}\rangle$, by $\langle n, 1, k, \bar{x}, \bar{y}\rangle$, $\langle n, k, \bar{x}, \bar{y}\rangle$, respectively. (Then $p^* \in M_\lambda$.) Suppose $W \subseteq \mathscr{P}(\lambda + n(\lambda) + m)$ and $W^* = f_\lambda^{-1}[\{p^*|p \in W\}]$ is $\Sigma_{n(\lambda)+m}$ over $\langle M_\omega, A_{\lambda+m}\rangle$. Then $D = \{p \in \mathscr{P}(\lambda+n(\lambda)+m)|\exists q \in W(p \le q)$ or $\forall\, q \le p(q \notin W)\}$ is predense on $\mathscr{P}$.*

PROOF. Given $p \in \mathscr{P}$ we must find $q \le p$ such that $q \restriction \lambda + n(\lambda) + m$ belongs to D. Write $p = (s, \bar{s} \cup \bar{t})$, where $p \restriction \lambda + n(\lambda) + m = (s\bar{s})$, $\bar{s} \cap \bar{t} = \emptyset$. For each n let s_n extend s by assigning $\langle m_0, m_1\rangle$ to 0 whenever $\langle m_0, m_1\rangle \notin \mathrm{Dom}(s)$ and $m_0 \le m_1 \le n$. (We intend that $n \longmapsto s_n$ is recursive.) If $(s_n, \bar{s})$ belongs to D for some n then we are done since $(s_n, \bar{s} \cup \bar{t})$ extends p. If not then we can define a $\Sigma_{n(\lambda)+m}$ over $\langle M_\omega, A_{\lambda+m}\rangle$ function $n \longmapsto t_n$ so that for some $\bar{t}_n$, $(t_n, \bar{t}_n) \le (s_n, \bar{s})$, $(t_n, \bar{t}_n) \in W$, using the fact that $A_{\lambda+m}$ codes $\langle M_\lambda, \overline{F}_1^{\lambda+n(\lambda)}, \cdot, \overline{F}_1^{\lambda+n(\lambda)+m-1}\rangle$ and hence "codes" $\mathscr{P}(\lambda + n(\lambda) + m)$. Then $f(m+1) = $ length $(t_{f(m)})$, $f(0) = 0$, defines a $\Sigma_{n(\lambda)+m}$ over $\langle M_\omega, A_{\lambda+m}\rangle$ function, and every such function is eventually dominated by the function $b_{\lambda+n(\lambda)+m}$. Thus there must be infinitely many l such that $[f(l), f(l+1)]$ is disjoint from $\mathrm{Range}(b_{\lambda+n(\lambda)+m})$. As $\mathrm{Range}(b) \subseteq_* \mathrm{Range}(b_{\lambda+n(\lambda)+m})$ for all $b \in \bar{t}$, it follows that for some l, $[f(l), f(l+1)]$ is disjoint from $\bigcup\{\mathrm{Range}(b)|b \in \bar{t}\}$. But then $(t_{f(l)}, \bar{t}_{f(l)} \cup \bar{t}) = q \le q$ and $q \restriction \lambda + n(\lambda) + m$ belongs to $W \subseteq D$. □

COROLLARY 6. *The forcing relation $\{(p,\varphi)|p \in \mathscr{P}(\lambda)$ and $p \Vdash \varphi$ in $\mathscr{P}(\lambda)$, where φ is a ranked sentence in $M_\lambda\}$ is Σ_1 over M_λ, for limit λ.*

PROOF. We argue by induction on λ. Note that if $\bar{\lambda} < \lambda$, $\bar{\lambda}$ limit, then for $p \in \mathscr{P}(\bar{\lambda})$ and φ ranked in $M_{\bar{\lambda}}$ we have $p \Vdash \varphi$ in $\mathscr{P}(\bar{\lambda})$ iff $p \Vdash \varphi$ in $\mathscr{P}(\lambda)$. The reason is that by Lemma 5, every $\mathscr{P}(\lambda)$-generic is $\mathscr{P}(\bar{\lambda})$-generic for ranked sentences, since by induction the $\mathscr{P}(\bar{\lambda})$ forcing relation for ranked sentences is Σ_1 over $M_{\bar{\lambda}}$.

Thus we are done by induction if λ is a limit of limit ordinals. Now suppose that we wish to establish the corollary for $\lambda+\omega$. We may assume that λ is small, as otherwise $\mathscr{P}(\lambda+\omega)$ is a set forcing in $M_{\lambda+\omega}$. Now any ranked sentence φ in M_λ is equivalent to a $\Sigma_{n(\lambda)+M}$ statement about $M_\lambda[\underline{R}]$ for some $m(\underline{R}$ denoting the generic real). But then by Lemma 5, $p \Vdash \varphi$ in $\mathscr{P}(\lambda+\omega)$ iff $p \Vdash \varphi$ in $\mathscr{P}(\lambda+n(\lambda)+m)$ for $p \in \mathscr{P}(\lambda+n(\lambda)+m)$. As the latter is Σ_1-definable over $M_{\lambda+\omega}$, we are done. □

COROLLARY 7. *Suppose λ is small and $W \subseteq \mathscr{P}(\lambda)$ is $\Sigma_{n(\lambda)}$ over M_λ. Let $D = \{p \in \mathscr{P}(\lambda) | \exists q \in W(p \leq q)$ or $\forall\, q \leq p(q \notin W)\}$. Then D is predense on $\mathscr{P}$.*

PROOF. Let $m = 0$ in Lemma 5. □

Now we are prepared to finish the proof of the characterization theorem. Note that the only remaining condition to verify in showing that $\vec{F}$ is a standard Skolem system is condition (E), where Stability is used.

LEMMA 8. *For λ limit and $x \in$ Seq(λ), $H_1^\lambda(x) = \lambda \cap \Sigma_1$ Skolem hull of x in $\widetilde{J}_\lambda^R$. For λ limit and $x \in$ Seq(λ), $H^\lambda(x) = \bigcup_n H_n^\lambda(x) = \lambda \cap$ Skolem hull of x in $\widetilde{J}_\lambda^R$.*

PROOF. We begin with the first statement. The inclusion $H_1^\lambda(x) \subseteq \Sigma_1$ Skolem hull of x in $\widetilde{J}_\lambda^R$ follows from Lemma 4 and Continuity. To prove the converse we make a definition: R is *Σ_n-generic* for $\mathscr{P}(\lambda)$ if for any $\Sigma_n(M_\lambda)$ $W \subseteq \mathscr{P}(\lambda)$ there exists $p \in G \cap \mathscr{P}(\lambda)$, G denoting the generic determined by R, such that either p extends a condition in W or p has no extension in W. By Corollary 7, if λ is small then R is $\Sigma_{n(\lambda)}$-generic for $\mathscr{P}(\lambda)$.

Suppose $\varphi(x,y)$ is a Σ_1 formula with parameter x. Let $\pi : \bar{\lambda} \longrightarrow \lambda$ be the increasing enumeration of $H_1^\lambda(x)$ and let $\pi(\bar{x}) = x$. By Corollary 6 the forcing relation for $\mathscr{P}(\bar{\lambda})$ is $\Sigma_1(M_{\bar{\lambda}})$ for ranked sentences. Since R is Σ_1-generic for $\mathscr{P}(\bar{\lambda})$, there is $p \in G \cap \mathscr{P}(\bar{\lambda})$ such that either $p \Vdash \varphi(\bar{x},\bar{y})$ in $\mathscr{P}(\bar{\lambda})$ for some $\bar{y}$ or $p \Vdash \neg\exists\bar{y}\varphi(\bar{x},\bar{y})$ in $\mathscr{P}(\bar{\lambda})$. Since $\vec{F}$ is stable we have that $p \Vdash \neg\exists y\varphi(x,y)$ in $\mathscr{P}(\lambda)$ or $p \Vdash \varphi(x,y)$, where $y = \pi(\bar{y})$. (Note that π extends to a Σ_1-elementary embedding $\tilde{\pi} : M_{\bar{\lambda}} \longrightarrow M_\lambda$ such that $\tilde{\pi}(p) = p$.) If λ is small then R is Σ_1-generic for $\mathscr{P}(\lambda)$, and thus we have shown that $\lambda \cap \Sigma_1$ Skolem hull of x in $\widetilde{J}_\lambda^R$ is contained in $H_1^\lambda(x)$. But the above shows that if R is Σ_1-generic for $\mathscr{P}(\lambda)$ for all small λ, then R is Σ_1-generic for all λ. So we are done.

To prove the second statement, use Stability for $\vec{F}$. The direction $H^\lambda(x) \subseteq$ Skolem hull of x in $\widetilde{J}_\lambda^R$ follows again from Lemma 4. For the converse, handle each formula $\psi(x,y)$ as in the Σ_1 case, using Stability. □

This completes the proof of Theorem 3.

§3. **Universal combinatorial principles.** Inherent in any abstract Skolem system $\vec{F}$ is its cofinality function cof$^{\vec{F}}$ defined at limit ordinals λ as follows: cof$^{\vec{F}}(\lambda)$ = least ordertype of an unbounded subset of λ of the form

$$H_n^\delta(\gamma \cup \{p\}) = \bigcup\{H_n^\delta(x * p) | x \in \mathrm{Seq}(\gamma)\}$$

for some $\delta \geq \lambda, n \geq 1, \gamma \leq \lambda, p \in$ Seq(δ). For any inner model M let cof^M be the cofinality function of M, and write cof = cof^V.

LEMMA 9. *Suppose $\vec{F}$ is an abstract Skolem system. Then there exists a stable abstract Skolem system $\vec{G}$ such that* cof$^{\vec{G}}$ = cof$^{L[\vec{F}]}$.

PROOF. First note that in the statement of Condensation for $\vec{F}$ we can in fact let $\pi : \bar{\lambda} \longrightarrow \lambda$ be the increasing enumeration of any set of the form $H_n^\lambda(X) = \bigcup\{H_n^\lambda(x) | x$ a finite sequence from $X\}$, $X \subseteq \lambda$: This follows from the usual statement if $H_n^\lambda(X) = H_n^\lambda(x)$ for some $x \in \mathrm{Seq}(\lambda)$ and otherwise follows by induction, using Continuity (see the proof of Lemma 4). Thus, the Σ^* theory relativizes without difficulty to $\vec{F}$. Now let $\vec{G}$ be obtained from $\vec{F}$ just as in the proof of Proposition 2, with R replaced by $\vec{F}$. Then $\vec{G}$ is stable. Since $\vec{G}$ codes $L[\vec{F}]$, $\mathrm{cof}^{\vec{G}}(\lambda) \leq \mathrm{cof}^{L[\vec{F}]}(\lambda)$ all λ. But $\vec{G}$ is $\langle L[\vec{F}], \vec{F}\rangle$-definable, so $\mathrm{cof}^{\vec{G}} = \mathrm{cof}^{L[\vec{F}]}$. □

We now state our universal combinatorial principle P.

PRINCIPLE P. *There is an abstract Skolem system $\vec{F}$ such that* $\mathrm{cof}^{\vec{F}} = \mathrm{cof}$.

We show that P implies all "fine-structural principles" for L.

DEFINITION. A *fine-structural principle* is a statement of the form $\exists \mathscr{A} \psi(\mathscr{A})$, where $\mathscr{A}$ denotes a class and ψ is first-order, such that:

(a) For every real R and every standard Skolem system $\vec{F}$ for $R, L[R] \models \psi(\mathscr{A})$ for some $\mathscr{A}$ which is definable over $\langle L[\vec{F}], \varepsilon, \vec{F}\rangle$.

(b) If M, N are inner models of ZFC, $\mathscr{A}$ is amenable to both M and N, $\mathrm{cof}^M = \mathrm{cof}^N$, and $\langle M, \mathscr{A}\rangle \models \psi(\mathscr{A})$, then $\langle N, \mathscr{A}\rangle \models \psi(\mathscr{A})$.

THEOREM 10. P *implies all fine-structural principles.*

PROOF. Suppose $M \models$ P with witness $\vec{F}$, and let φ be fine-structural. Then $\mathrm{cof}^{\vec{F}} = \mathrm{cof}^M = \mathrm{cof}^{L[\vec{F}]}$, since $L[\vec{F}] \subseteq M$. By Lemma 9 there is $\vec{G}$ amenable to M such that $\mathrm{cof}^{\vec{G}} = \mathrm{cof}^M$ and $\vec{G}$ is stable. By the characterization theorem there is a (generic) real R such that $\vec{G}$ is a standard Skolem system for R and hence $L[R] \models \varphi$ with witness $\mathscr{A}$ definable over $\langle L[\vec{G}], \varepsilon, \vec{G}\rangle$. Then $\mathscr{A}$ is amenable to M and $\mathrm{cof}^M = \mathrm{cof}^{\vec{G}} = \mathrm{cof}^{L[R]}$, so $M \models \varphi$. □

□ and Morass are fine-structural, but ◇ is not. To obtain a universal principle which also implies ◇ we introduce a strengthening of P.

PRINCIPLE P*. $V = L[\vec{F}]$, *where $\vec{F}$ is an abstract Skolem system.*

Note that $\mathrm{P}^* \longrightarrow \mathrm{P}$, in view of Lemma 9. We define an *L-like principle* to be a statement φ which is true in $L[\vec{F}]$ whenever $\vec{F}$ is a standard Skolem system. By Lemma 9 and the characterization theorem, P* implies all L-like principles. But unfortunately P* is not much weaker than $V = L$:

THEOREM 11. P* *holds iff $V = L[A], A \subseteq \omega_1$, where A is L-reshaped* ($\alpha < \omega_1 \longrightarrow \alpha < \omega_1$ *in* $L[A \cap \alpha]$).

PROOF. Suppose $V = L[\vec{F}]$ for some abstract Skolem system $\vec{F}$ and that α is countable in $L[\vec{F}]$. If $\alpha < \lambda$ limit and $\widetilde{J}_\lambda^{\vec{F}} \models \alpha$ uncountable, then $\vec{F} \restriction \lambda$ can be recovered inductively from $\vec{F} \restriction \alpha$, using continuity and condensation for abstract Skolem systems. We can also recover F_n^λ for all $n > 0$ for such λ. Thus if λ is least so that α is countable in $\widetilde{J}_{\lambda+\omega}^{\vec{F}}$, we see that α is countable in $L[\vec{F} \restriction \alpha]$. So $\vec{F} \restriction \omega_1$ is L-reshaped. The same argument shows that $\vec{F}$ is definable over $L[\vec{F} \restriction \omega_1]$, so we have the desired conclusion.

For the converse, note that for L-reshaped $A \subseteq \omega_1$ we can define the canonical Skolem system $\vec{F}^A$ for A as we defined $\vec{F}^R$ for reals R, provided we replace the hierarchy $\widetilde{J}_\delta^R, \delta \in \mathrm{ORD}$, by $\widetilde{J}_\delta^A, \delta \in \mathrm{ORD}$, and we assume that for $\lambda < \omega_1, A \cap [\lambda, \hat{\lambda}] = \emptyset$, where $\hat{\lambda}$ is the least limit so that $\widetilde{J}_{\hat{\lambda}}^{A\restriction\lambda} \models \lambda$ is countable. Then $L[\vec{F}^A] =$

$L[A]$ and $\vec{F}^A$ satisfies the axioms for an abstract Skolem system. (In fact $\vec{F}^A = \vec{F}^R$ for some generic real R coding A.) □

Though P* does not therefore have models which are very far from L, we hope that its analogue in the context of core models will lead to an interesting class of "K-like" models.

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