The Silver Dichotomy for Generalised Baire Space

Classical Silver Dichotomy: If E is a Borel (or even co-analytic) equivalence relation on ω^ω with uncountably many classes, then E has a perfect set of classes.

The *Generalised Baire space* κ^κ has basic open sets

$$N_\sigma = \{\eta : \kappa \to \kappa \mid \eta \text{ extends } \sigma\}$$

where σ belongs to $\kappa^{<\kappa}$.

We assume $\kappa^{<\kappa} = \kappa$, so there is a dense set of size κ.

Borel sets: Close the basic open sets under complements and intersections of size κ.

The Silver Dichotomy for Generalised Baire Space

$T \subseteq \kappa^{<\kappa}$ is a *perfect* tree iff T is $< \kappa$-closed and every node extends to a splitting node.

$X \subseteq \kappa^\kappa$ is *perfect* if $X = [T]$ for some perfect tree T.

Silver Dichotomy for κ^κ: If E is a Borel equivalence relation on κ^κ with more than κ classes then E has a perfect set of classes.

An equivalent form:

Silver Dichotomy for κ^κ: If E is a Borel equivalence relation on κ^κ with more than κ classes then id is Borel-reducible to E, where id is equality on 2^κ.

The Generalised Silver Dichotomy: A negative result

Theorem

(SDF-Hyttinen-Kulikov) Assume $V = L$. Then the Silver dichotomy for κ^κ fails for all uncountable regular κ. There are Borel equivalence relations with more than κ classes which lie strictly below id and also ones which are incomparable with id (with respect to Borel-reducibility).

The problem in L is caused by *weak Kurepa trees*.

$T \subseteq 2^{<\kappa}$ is *weak Kurepa* if every node of T splits, T has more than κ branches but T_α (the set of nodes in T of length α) has size at most card(α) for stationary-many $\alpha < \kappa$.

T is *Kurepa* if the previous holds for all infinite $\alpha < \kappa$.

The Generalised Silver Dichotomy: A negative result

Fact. If $T \subseteq 2^{<\kappa}$ is perfect then for some (infinite) α, T_α has size greater than card(α); if κ is inaccessible then this holds for club-many $\alpha < \kappa$.

It follows that Kurepa trees do not contain perfect subtrees and if κ is inaccessible then weak Kurepa trees do not contain perfect subtrees.

Now for any tree T define:
$xE_T y$ iff x, y are not branches of T or $x = y$.
If T has more than κ branches then the Silver Dichomomy would yield a perfect subtree of T.

It follows that Kurepa trees kill the Silver Dichotomy and weak Kurepa trees kill the Silver Dichotomy at inaccessibles.

The Generalised Silver Dichotomy: A negative result

Lemma

(Jensen, essentially) Suppose $V = L$ and κ is regular and uncountable. Then there exists a weak Kurepa tree on κ. If κ is a successor cardinal then there exists a Kurepa tree on κ.

Corollary

If $V = L$ then the Silver Dichotomy fails at all uncountable κ.

The Generalised Silver Dichotomy: Silver's hint

Fortunately Silver showed us how to get rid of Kurepa trees.

If $\kappa < \lambda$, λ regular then $\mathrm{Coll}(\kappa, < \lambda)$ is the partial order that forces $\lambda = \kappa^+$ using conditions of size $< \kappa$.

Silver: If $\kappa < \lambda$, λ inaccessible then $\mathrm{Coll}(\kappa, < \lambda)$ forces that there are no Kurepa trees on κ.

The same proof shows:

If $\kappa < \lambda$ are both inacessible then $\mathrm{Coll}(\kappa, < \lambda)$ forces that there are no weak Kurepa trees on κ.

So maybe if $\kappa < \lambda$, λ inaccessible then $\mathrm{Coll}(\kappa, < \lambda)$ forces the Silver Dichotomy at κ?

I will return to this question later.

The Generalised Silver Dichotomy: Another worry

More bad news about the Silver Dichotomy at an uncountable κ.

Fact. The Silver Dichotomy provably fails at uncountable κ for Δ^1_1 equivalence relations: Define $xE^{\mathrm{rank}}y$ iff x, y do not code wellorders of κ or they code wellorders of the same length. Then E^{rank} is Δ^1_1, has more than κ classes but no perfect set of classes.

E^{rank} is Δ^1_1 because wellfoundedness is Δ^1_1 (indeed closed: x codes a wellorder iff $x \restriction \alpha$ codes a wellorder for all $\alpha < \kappa$).

If T were a perfect set of codes for wellorders of distinct lengths then let x be a sufficiently generic branch through T and let $\beta < \kappa$ be the length of the wellorder coded by x.
Then for some $\alpha < \kappa$, all sufficienitly generic branches through T extending $x \restriction \alpha$ code a wellorder of length β, contradiction.

The Generalised Silver Dichotomy: Another worry

Fortunately, not every Δ^1_1 set is Borel: There is a Δ^1_1 set $D(x, y)$ such that the D_x's are exactly the Borel sets. By diagonalisation, D is not Borel.

In fact:

Theorem

E^{rank} is not Borel.

The proof of this result points the way toward a consistency proof for the Silver Dichotomy.

The Generalised Silver Dichotomy: The second hint

Theorem

E^{rank} is not Borel.

Proof. For $\alpha < \kappa^+$, $\mathrm{Coll}(\kappa, \alpha)$ denotes the forcing to collapse α to κ using conditions of size less than κ.

If $g : \kappa \to \alpha$ is $\mathrm{Coll}(\kappa, \alpha)$-generic then g^* denotes the subset of κ defined by $i \in g^*$ iff $g((i)_0) \leq g((i)_1)$ where $i \mapsto ((i)_0, (i)_1)$ is a bijection between κ and $\kappa \times \kappa$.

The Generalised Silver Dichotomy: The second hint

By induction on Borel rank we show that if B is Borel then there is a club C in κ^+ such that:

$(*)$ For $\alpha \leq \beta$ in C of cofinality κ and (p_0, p_1) a condition in $\mathrm{Coll}(\kappa, \alpha) \times \mathrm{Coll}(\kappa, \alpha)$, (p_0, p_1) forces that (g_0^*, g_1^*) belongs to B in the forcing $\mathrm{Coll}(\kappa, \alpha) \times \mathrm{Coll}(\kappa, \alpha)$ iff it forces this in the forcing $\mathrm{Coll}(\kappa, \alpha) \times \mathrm{Coll}(\kappa, \beta)$.

It follows that E^{rank} is not Borel, as otherwise we have $g_0^* E^{\mathrm{rank}} g_1^*$ where g_0, g_1 are sufficiently generic for $\mathrm{Coll}(\kappa, \alpha) \times \mathrm{Coll}(\kappa, \beta)$ with $\alpha < \beta$. $\square$

The Generalised Silver Dichotomy: Main Result

We now return to the earlier question:

Question. If $\kappa < \lambda$, λ inaccessible then does Coll$(\kappa, < \lambda)$ force the Silver Dichotomy at κ?

I don't know the answer.
But if we require more of λ we get a positive result:

Theorem

(Main Theorem) Suppose that $0^{\#}$ exists, κ is regular in L and $\kappa < \lambda$ where λ is a Silver indiscernible. Then after forcing over L with Coll$(\kappa, < \lambda)$ the Silver Dichotomy holds for the Generalised Baire Space κ^{κ}.

Silver indiscernibles are very large in L, indeed much more than inaccessible. Indeed any conceivable large cardinal property consistent with $V = L$ holds for the Silver indiscernibles.

The Generalised Silver Dichotomy: Main Result

Note that to show that $\text{Coll}(\kappa, < \lambda)$ forces the Silver Dichotomy for *all* Silver indiscernible $\lambda > \kappa$, it suffices to verify it for *some* Silver indiscernible $\lambda > \kappa$, by indiscernibility.

We verify it when λ is κ^+ of V. The proof works as long as λ is a fixed point in the enumeration of the Silver indiscernibles.

Let G be $\text{Coll}(\kappa, < \lambda)$-generic.
We assume that our Borel equivalence relation in $L[G]$ has a Borel code in L and therefore has Borel rank less than $(\kappa^+)^L$.

Suppose that E has λ classes in $L[G]$ and let p force that the names $(\sigma_\alpha \mid \alpha < \lambda)$ are pairwise E-inequivalent.
We can assume that the σ_α's have size $< \lambda$ and choose $f : \lambda \to \lambda$ in L so that σ_α is a $\text{Coll}(\kappa, < f(\alpha))$-name for each $\alpha < \lambda$.
To simplify notation let $\mathcal{L}_\gamma$ denote $\text{Coll}(\kappa, < \gamma)$ for any ordinal γ.

The Generalised Silver Dichotomy: Main Result

We may also assume that for each $\alpha < \lambda$, the E-equivalence class of σ_α does not depend on the choice of $\mathcal{L}_{f(\alpha)}$-generic, as otherwise we get a perfect set of E-equivalence classes by building a perfect set of mutual generics.

Let I consist of the Silver indiscernibles between κ and λ and for $i < j$ in I let π_{ij} be an elementary embedding from L to L with critical point i, sending i to j.

The Generalised Silver Dichotomy: Main Result

In vague analogy to the previous proof we show that for each Borel B there is a club C contained in I such that:

$(*)$ Suppose that $i_0 < i_1 < \cdots < i_n = j < i_{n+1} = k$ belong to C, $(p_0, p_1) \leq (p, p)$ belongs to $\mathcal{L}_{f(j)} \times \mathcal{L}_{f(j)}$ and is L-definable from parameters in $i_0 \cup \{i_0, i_1, \ldots, i_n\}$ together with indiscernibles $> j$. Then (p_0, p_1) forces that $(\sigma_j^{g_0}, \sigma_j^{g_1})$ belongs to B in the forcing $\mathcal{L}_{f(j)} \times \mathcal{L}_{f(j)}$ iff $(p_0, \pi_{i_0 i_1}\pi_{i_1 i_2} \cdots \pi_{i_{n-1} i_n}\pi_{i_n i_{n+1}}(p_1))$ forces that $(\sigma_j^{g_0}, \sigma_k^{g_1})$ belongs to B in the forcing $\mathcal{L}_{f(j)} \times \mathcal{L}_{f(k)}$.

Now apply $(*)$ to the Borel set E, producing a club C.
We know that (p, p) $\mathcal{L}_{f(i)} \times \mathcal{L}_{f(i)}$-forces $\sigma_i^{\dot{g}_0} E \sigma_i^{\dot{g}_1}$.
It follows from $(*)$ that for $i < j$ in C, (p, p) also $\mathcal{L}_{f(i)} \times \mathcal{L}_{f(j)}$-forces $\sigma_i^{\dot{g}_0} E \sigma_j^{\dot{g}_1}$.
But this contradicts the fact that $\sigma_\alpha^{\dot{g}_0}$, $\sigma_\beta^{\dot{g}_1}$ are forced by (p, p) in $\mathcal{L}_\alpha \times \mathcal{L}_\beta$ to be E-inequivalent when p belongs to $\mathcal{L}_\alpha$ and $\alpha < \beta$. $\square$

Final remark

Another important dichotomy from the classical setting is:

Harrington-Kechris-Louveau Dichotomy: If E is Borel and not smooth then E_0 Borel-reduces to E.

This is *provably false* for uncountable κ:

Theorem

(SDF-Hyttinen-Kulikov) In κ^κ for uncountable κ there is a Borel equivalence relation E_0' which is strictly above id and strictly below E_0 with respect to Borel reducibility.

E_0' is defined as follows:

$xE_0'y$ iff
xE_0y and $\{i < \kappa \mid x(i) \neq y(i)\}$ is a finite union of intervals.

Final remark

But maybe there is still some hope; the following is open:

Question. Suppose that a Borel equivalence relation E is not Borel reducible to id. Then is E_0' Borel reducible to E?

Thanks for your attention!

Final remark

Claim 1. id $\leq_B E_0' \leq_B E_0$.
For the first reduction use $f(x) =$ the set of codes for proper initial segments of x; then $x = y \to f(x)E_0'f(y)$ and
$x \neq y \to \sim f(x)E_0f(y) \to \sim f(x)E_0'f(y)$.
For the second reduction: for each $\alpha < \kappa$ choose $f_\alpha : 2^\alpha \to 2^\alpha$ such that for $x, y \in 2^\alpha$, $\{i < \kappa \mid x(i) \neq y(i)\}$ is a finite union of intervals iff $f_\alpha(x) = f_\alpha(y)$ and for $x \in 2^\kappa$ define $f(x) =$ the set of codes for the pairs $(f_\alpha(x|\alpha), x(\alpha))$, $\alpha < \kappa$; then
$xE_0'y \to f(x)E_0f(y)$ and $\sim xE_0'y \to \sim f(x)E_0f(y)$.

Final remark

Claim 2. $E_0' \not\leq_B \mathrm{id}$.
Otherwise let M be a transitive model of ZFC$^-$ of size κ containing all bounded subsets of κ as well as a code for the Borel reduction f.
Let $x \in 2^\kappa$ be κ-Cohen generic over M and define $\bar{x}(i) = 1 - x(i)$ for each $i < \kappa$.
Then as $\sim x E_0 \bar{x}$ there is $\alpha < \kappa$ such that $f(x) \neq f(y)$ whenever y is κ-Cohen generic over M and extends $\bar{x}|\alpha$. But then $f(x) \neq f((\bar{x}|\alpha) * (x|[\alpha,\kappa)))$, contradicting $x E_0' ((\bar{x}|\alpha) * (x|[\alpha,\kappa)))$.

Final remark

Claim 3. $E_0 \not\leq_B E_0'$.
As in the previous argument choose a reduction f, a transitive model M of ZFC$^-$ of size κ containing all bounded subsets of κ as well as a Borel code for f and $x \in 2^\kappa$ which is κ-Cohen over M.
Choose α_0 so that $\sim f(x)E_0'f(y)$ whenever y is κ-Cohen over M and extends $\bar{x}|\alpha_0$; we can further demand that for some ordinal $i_0 < \alpha_0$, $f(x)(i_0) \neq f(y)(i_0)$ for such y.
Then choose $\alpha_1 > \alpha_0$ so that $f(x)E_0'f(y)$ whenever y is κ-Cohen over M and extends $(\bar{x}|\alpha) * (x|[\alpha_0, \alpha_1))$; we can further demand that for some ordinal $i_1 \in [\alpha_0, \alpha_1)$, $f(x)(i_1) = f(y)(i_1)$ for such y.
After ω steps we obtain $\sim f(x)E_0'f(y)$ whenever y is κ-Cohen over M and extends $(\bar{x}|\alpha_0) * (x|[\alpha_0, \alpha_1)) * (\bar{x}|[\alpha_1, \alpha_2)) * \cdots$, contradicting the fact that there is such a y which is E_0 equivalent to x. $\square$