Descriptive Set Theory and Absoluteness

Let E be an analytic or co-analytic equivalence relation.

E can have

Countably many classes (small)
Uncountably many but no perfect set of classes (medium)
A perfect set of classes (large)

The second case does not occur if E is co-analytic.

In the third case, E can be either smooth, Borel non-smooth or non-Borel.

General Question. How absolute are these properties?

A property is *persistent* if it continues to hold in outer models. It is *absolute* if it and its negation are persistent.

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Proposition

(a) For analytic equivalence relations, smallness, mediumness, largeness, smoothness and Borelness are Σ^1_3, Π^1_3, Σ^1_2, Σ^1_3 *and* Σ^1_3, *respectively. So all except possibly mediumness are persistent and largeness is absolute. For co-analytic equivalence relations, they are* Σ^1_2, *vacuous,* Π^1_2, Σ^1_3 *and* Σ^1_3, *respectively. So all are persistent and both smallness and largeness are absolute.*
(b) For analytic equivalence relations, mediumness is not persistent, and smallness, smoothness and Borelness are not absolute. For co-analytic equivalence relations, smoothness and Borelness are not absolute.
(c) For orbit equivalence relations, smallness is Σ^1_2 *and therefore both smallness and mediumness are absolute.*

(a): Just write it down.

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(b): Here and in other proofs below we use *master codes*.

x is a *master code* if x codes the first-order theory of some L_α. The set of master codes is Π^1_1, wellordered by $\leq_T$ and in L uncountable.

For the analytic case consider

xEy iff x, y compute the same master codes

Then E is analytic and medium in L, but small after ω_1^L is collapsed. So mediumness is not persistent and smallness, smoothness and Borelness are not absolute for analytic equivalence relations.

For the co-analytic case consider:

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xEy iff x, y are both master codes or $x = y$

Then E is co-analytic, non-Borel in L, but smooth after ω_1^L is collapsed. So neither smoothness nor Borelness is absolute for co-analytic equivalence relations.

(c) (Orbit relations): An analytic equivalence relation with only Borel classes is *tame* if there is a function with Σ_2^1 graph that produces a Borel code for $[x]_E$ from x in all outer models. For tame relations, smallness is Σ_2^1 (not just Σ_3^1) and therefore absolute; as largeness is also absolute it follows that mediumness is absolute. Becker observed that orbit equivalence relations are tame.

Question. Are smoothness and Borelness absolute for orbit equivalence relations?

Sizes of classes

An E-class can be

Countable (small)
Uncountable with no perfect subset (medium)
With a perfect subset (large)

The second case does not occur if E is analytic.

In the third case, the E-class can be either Borel or non-Borel.

Regarding possible sizes of classes:

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Theorem

(a) An analytic equivalence relation is either large or has a large class.
(b) There is an analytic equivalence relation which is not large and has only non-Borel classes.
(c) (SDF-Törnquist, inspired by Clinton and helped by Ben) In L there is a co-analytic equivalence relation with only medium classes.

(a): Ben gave me this argument. Let E be an analytic equivalence relation. If E is meager then E is large by Mycielski. Otherwise E has a non-meager class by Kuratowski-Ulam, and this class is large.

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(b): Define a relation E on finite sequences $(x_0, \ldots, x_{n-1})$ of reals as follows: Suppose m is at most n. Then

$(x_0, ..., x_{m-1})E(y_0, ..., y_{n-1})$ iff
$(y_0, ..., y_{n-1})E(x_0, ..., x_{m-1})$ iff

1. For all $i < m$, (x_i, y_i code isomorphic linear orders or neither belongs to WO).
2. For i in $[m, n)$, y_i is not in WO.

Then E is an analytic equivalence relation (must check transitivity).

Each E-class is non-Borel as for any $(x_0, ..., x_{n-1})$

$$\{x \mid (x_0, ..., x_{n-1})E(x_0, ..., x_{n-1}, x)\} = \sim WO$$

Moreover E is has ω_1 classes absolutely, and therefore has no perfect set of classes.

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(c): Suppose G is an uncountable thin Π^1_1 subgroup of $(R, +)$. Then the orbit equivalence relation induced by G on R has the desired property.

To get such a group G, argue as follows:

Let C be a perfect Π^0_1 set of linearly independent reals and (using $V = L$) choose P to be an uncountable thin Π^1_1 subset of C. Let G be the group generated by P under $+$.

Then G is Π^1_1: Any nonzero element of the group generated by C has a unique decomposition as a linear combination (with integer coefficients) of increasing elements of C. So this decomposition is *Hyp* in x and we get: x belongs to G iff $x = 0$ or x is a linear combination of reals *Hyp* in x which belong to P; this is Π^1_1.

G is thin as its cardinality is that of P, at most ω_1 absolutely.

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How absolute is it to have a class of a certain type?

Proposition

(a) For analytic equivalence relations, to say that a class is small, Borel is Δ^1_2, Σ^1_3, *respectively. So having a small class or a large class is absolute and having a Borel class is persistent. For co-analytic equivalence relations, to say that a class is small, medium, large, Borel is* Σ^1_3, Π^1_3, Σ^1_2, Σ^1_3, *respectively. So having a small class or a Borel class is persistent and having a large class is absolute.*
(b) For medium or large analytic equivalence relations, having a Borel class is not absolute and having a non-Borel class is not persistent.
(c) For co-analytic equivalence relations, having a small class or a Borel class is not absolute and having a medium or non-Borel class is not persistent.

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(a): Just write it down, except need Ben's observation that largeness of classes is not only Π^1_2 but also Σ^1_2: $[x]_E$ is uncountable iff there is a continuous injection from a comeager subset of Cantor space into $[x]_E$.

(b): As in the Asger-Ben-Clinton-Sy example, let C be a perfect Π^0_1 set of linearly independent reals but now let G be the group generated by a Σ^1_1 subset A of C whose complement in C is medium in L. For the large case, take the complement of A in C to be the union of a large set and a medium set in L.

(c): Use the Asger-Ben-Clinton-Sy co-analytic relation which in L has only medium classes. After collapsing ω_1^L each of its classes is small.

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Not covered by previous Proposition:

Is having only Borel classes persistent for analytic equivalence relations? Is having only small, only large or only Borel classes persistent for co-analytic equivalence relations?

Finally, we can ask if the notions above which are strictly Σ^1_n or Π^1_n for some n are complete for that projective class. For example:

Question. Is having only countably many classes a Σ^1_3 complete property of a code for an analytic equivalence relation?

Descriptive Set Theory and Absoluteness: Addendum

The Spector relation is $xE^{Spec}y$ iff $\omega_1^x = \omega_1^y$.

The wellorder relation is $xE^{wo}y$ iff x, y code isomorphic linear orders or x, y code illfounded linear orders.

E almost Borel-reduces to F if there is a Borel function which reduces E to F except on the reals of countably many E-classes.

Theorem

(William Chan, independently) (a) E^{wo} almost Borel-reduces to the Spector relation if $0^{\#}$ exists but not in set-generic extensions of L. (b) Isomorphism on the countable models of a counterexample to Vaught's conjecture does not almost Borel-reduce to the Spector relation in set-generic extensions of L.

Thanks! I hope that you find absoluteness interesting!