The Current State of the Foundations of Set Theory

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Gödel's work on incompleteness still casts a long shadow on the foundations of set theory.

Gödel's First Incompleteness Theorem: There is no complete system of axioms for mathematics: For any system, there will be a statement that can neither be proved nor disproved using the axioms of that system.

However the fact remains that there is a system of axioms, called ZFC and formulated in the language of set theory, which does a pretty good job: it seems strong enough to answer about 90% of the statements of mathematical interest.

Thus there are two sides of set theory:

1. The *Mathematics* of set theory: that which can be proved in ZFC.

Examples: All of your favourite theorems. The Poincaré Conjecture, the 4-colour theorem, Fermat's last theorem (probably[1]).

2. The *Meta-Mathematics* of set theory: the study of statements that ZFC cannot prove or disprove (i.e., which are *undecidable* in ZFC).

a. The most famous example:

The Continuum Hypothesis (CH): If X, Y are uncountable sets of real numbers then there is a bijection between X and Y.

Gödel: ZFC does not refute CH, i.e. ZFC + CH is consistent.

[1] This is an interesting case. Wiles' proof uses ideas which trace back to Grothendieck's work, which formally speaking uses "universes" whose existence requires *inaccessible infinities* (which we introduce below) and therefore cannot be proved in ZFC. However experts in the area are all convinced that Grothendieck's work can be formulated without inaccessible infinities and indeed within ZFC without difficulty. A discussion of this appears in [5].

Cohen: ZFC does not prove CH, i.e., ZFC + $\sim$ CH is consistent.

b. Another example:

The projective sets of reals are defined as follows:

i. Open sets are projective.
ii. The complement of a projective set is projective.
iii. If f is a continuous function and X is projective then so is $f[X]$, the image of X under f.

Projective Measurability (PM): All projective sets are Lebesgue measurable.

Gödel: ZFC does not prove PM.
Solovay: ZFC does not refute PM.

However there is an important difference between these two examples, CH and PM:

In both examples, when we say that ZFC cannot prove or refute something, we are of course assuming that ZFC is a consistent theory! Otherwise ZFC proves a contradiction and from a contradiction we can derive anything at all. And the assumption that ZFC is consistent is a nontrivial assumption:

Gödel's Second Incompleteness Theorem (for ZFC): ZFC cannot prove that ZFC is consistent (of course assuming that ZFC is consistent).

So the CH example is really the following statement:

Assuming ZFC is consistent, ZFC does not refute CH.
Assuming ZFC is consistent, ZFC does not prove CH.

But the PM example is actually the following:

Assuming ZFC is consistent, ZFC does not prove PM.
Assuming that the theory (ZFC + There is an *inaccessible infinity*)
is consistent, ZFC does not refute PM.

I'll say something about inaccessible infinities in a moment. But we cannot get rid of them! The consistency of the theory (ZFC + There is an inaccessible infinity) does not follow from the consistency of ZFC alone. And we have a converse:

Shelah: If ZFC does not refute PM (i.e., if ZFC + PM is consistent) then (ZFC + There is an inaccessible infinity) is consistent!

So we really are forced to deal with inaccessible infinities if we want to understand the undecidability of PM in ZFC.

Axioms of Infinity (Large Cardinal Axioms)

What is an inaccessible infinity?

First note the following obvious facts:

i. If A is a finite set then so is $\mathcal{P}(A)$, the set of subsets of A (the *power set* of A).
ii. If A is a finite set and for each element a of A, B_a is a finite set then the union of the B_a's is also finite.

Therefore we can say that the size (cardinality) of the set of natural numbers is *inaccessible*, as it cannot be reached using only finite sets.

We say that an uncountable set has *inaccessible* size (cardinality) if it cannot be reached using sets of smaller size in a similar way.

In set theory, the sizes of sets are measured by special numbers called *cardinal numbers*. The cardinal number that measures the size of a set X is called the *cardinality of* X. Thus an *inaccessible cardinal* is a cardinal number which is the cardinality of an uncountable set of inaccessible size.

Can we prove that inaccessible cardinals exist? We cannot. It turns out that in the theory ZFC + There is an inaccessible cardinal, one can prove that ZFC is consistent. It then follows from Gödel's Second Incompleteness Theorem that in ZFC, one cannot prove that inaccessible cardinals exist.

Now that I have introduced large infinities, I can describe:

The Modern Meta-Mathematics of Set Theory

The result

If (ZFC + There is an inaccessible cardinal) is consistent
then so is (ZFC + PM)

is an example of a *Consistency Upper Bound* result. It establishes the consistency of ZFC together with a statement of interest, in this case PM, assuming the consistency of ZFC together with the existence of a large infinity, in this case an inaccessible cardinal.

But this is just the beginning. A huge number of statements in set theory have been shown to be consistent with ZFC in this way, using various kinds of large cardinals. Without going into details, here is a brief list of some of these large cardinal notions:

Inaccessible
Mahlo
Weakly compact
Ramsey
Measurable
Hypermeasurable
Woodin
Superstrong
Hyperstrong
Supercompact
n-Superstrong, $n > 1$
ω-Superstrong

The above notions of infinity get stronger and stronger (as you go down the list) and go all the way "to the top": the natural extension to ω+1-Superstrong is inconsistent!

Now the result

If (ZFC + PM) is consistent
then so is (ZFC + There is an inaccessible cardinal)

is an example of a *Consistency Lower Bound* result. It shows that a certain large infinity is required for establishing the consistency with ZFC of a statement of interest. With PM we have the ideal situation:

Con(ZFC + Inaccessible) $\rightarrow$ Con(ZFC + PM) $\rightarrow$ Con(ZFC + Inaccessible) so we have exactly "measured" the *consistency strength* of PM. More often, however, we just get upper and lower bounds which don't quite match; for example, if PFA stands for the Proper Forcing Axiom we have:

$$\text{Con(ZFC + Supercompact)} \rightarrow \text{Con(ZFC + PFA)} \rightarrow \text{Con(ZFC + Woodin)}$$

It is conjectured that Con(ZFC + PFA) → Con(ZFC + Supercompact), but this remains open.

To summarise: Large cardinals provide the tools needed for establishing the consistency of statements in set theory (Consistency Upper Bounds). We have made some progress toward showing that large cardinals are necessary for such consistency results (Consistency Lower Bounds), but techniques for obtaining the consistency of Superstrong cardinals and beyond are still missing.

Extending ZFC

We now come to the most controversial topic in the contemporary foundations of set theory.

It was Gödel's hope that by adding large cardinal axioms (the assumption that large infinities exist) to ZFC one would obtain a theory that would resolve the major questions of set theory, like CH. But this turned out to be true only in a very limited way:

Good news: Large cardinals not only show that PM is consistent but in fact show that PM and many other nice properties of projective sets are true ([4]).

Bad news: Large cardinals do not help with CH: they do not imply CH and they do not refute CH.

Despite the bad news, should we add large cardinal axioms to ZFC and adopt them as part of the "true" axioms of set theory anyway? Set-theorists differ in their opinions about this.

1. Yes! (Woodin [7], for example). The "real" universe should be as large as possible ("maximal" in some sense) and therefore should include all conceivable large infinities.

2. No! (Shelah [6], for example). ZFC summarises our intuitions about sets and therefore any interpretation of the ZFC axioms is as good as any other. We can hope to show that a statement is consistent with ZFC, but never claim that it is "true" unless ZFC proves it.

3. Yes and No! (My position, see [1]). Adding axioms based on "maximality" principles for the universe of sets does lead to "true" statements, including the existence of the smaller of the large infinities (below a Ramsey cardinal), but *not* to the existence of the larger infinities (Ramsey and above).

When formulated carefully (see [1]), maximality principles come in two varieties. There are the *ordinal (or vertical) maximality* principles which say that the extension of the sequence of natural numbers given by the ordinal number sequence $0, 1, \ldots, \omega, \omega+1, \ldots$ is as long as possible, and the *power set (or horizontal) maximality* principles, which say, among other things, that the set of real numbers is as large as possible. Ordinal maximality leads to the smaller of the large cardinals, the ones favoured by Gödel, such as the inaccessible, Mahlo and weakly compact cardinals. But power set maximality, although it has some important consequences, seems incapable of yielding the existence of Ramsey cardinals [2].

As I said earlier, even if we adopt large cardinal axioms as new axioms, the fact remains that we still have not resolved many important questions in set theory, like CH. Attempts have been made to answer such questions by supplementing large cardinal axioms using the power set maximality principles. But this has not yet succeeded because it is very difficult to find consistent and convincing maximality principles which both decide CH and

[2] Ordinal maximality principles are closely related to the "reflection principles" favoured by Gödel (see [3] for a discussion of these principles). Power set maximality, in its basic form, fixes the ordinal numbers and asserts that if one enlarges the universe, then any statement that becomes true was already true in a subuniverse of the original universe. This is also called the inner model hypothesis (IMH), introduced in [2]. The IMH contradicts the existence of inaccessible cardinals; however if it is applied only to ordinal maximal universes (suitably defined), one obtains a consistent version which embodies the smaller of the large cardinals, but fails to imply the existence of the larger ones, such as Ramsey cardinals. Of course one could restrict the IMH to universes containing very large cardinals, but unlike for the smaller large cardinals, there is no convincing principle similar to "reflection" which implies that such very large cardinals exist, and therefore such a version of the IMH would be artificial.

are compatible with the stronger of the large cardinal axioms[3]. There is a good candidate[4] for such a principle; what remains is the challenging task of showing that it is consistent.

A New Proposal

It is too early to say if set-theorists will ever agree on what new axioms to add to ZFC. At present, most but not all set-theorists would be happy to add the smaller large cardinal axioms, up to a weakly compact cardinal. That is of course a long way from agreeing on an extension of ZFC that will decide questions like CH.

But maybe set-theorists should look outside of set theory. I said at the beginning that ZFC is not a bad theory in the sense that it is sufficient to decide 90% of the statements of mathematical interest. But 90% is not 100% and there are indeed areas of mathematics (and of mathematical logic outside of set theory) where one exceeds the capabilities of ZFC. This has happened in point-set topology, functional analysis and homological algebra, for example. Within mathematical logic, this is currently a pressing issue in model theory, where one tries to understand classes of structures which are not described by properties expressible in standard, first-order logic.

[3]Although large cardinals do not settle CH, they do imply that statements like PM ("projective statements") which are true in an enlargement of the universe via "set-forcing" are already true without such an enlargement. ("Set-forcing" is the method that Cohen used to prove the consistency of ZFC with $\sim$ CH.) This can be viewed as an "invariance" result, a cousin of "maximality". Building on this, Woodin [7] has explored, via his Ω-logic, a similar phenomenon for a larger class of statements, including CH; he shows that any statement which together with large cardinals achieves a similar effect for this larger class must also imply $\sim$ CH. Unfortunately, the large cardinals required for this work are not small and therefore their existence cannot be derived from a convincing principle such as "reflection". More seriously, the artificial restriction to "set-forcing" is essential for Woodin's work: for any large cardinal property there are methods beyond "set-forcing" which can make a true projective statement false while preserving the truth of that large cardinal property. In fact one can easily decide CH via a simple and consistent power set maximality principle, provided one is willing to restrict to a special type of set-forcing extension. Thus the main goal is to obtain such a principle without any set-forcing restriction at all.

[4]This is called the strong inner model hypothesis (see [2]), a version of the inner model hypothesis which allows statements with "absolute" parameters. In [2] it is formulated without ordinal maximality, but a formulation restricted to ordinal maximal universes is also possible.

Thus I pose the following question: Could it be that to successfully generalise what is known about first-order model theory to larger logics, one needs a particular extension of ZFC? Could there be a similar situation in other fields, where a particular extension of ZFC is required to successfully generalise current theory? And after discovering all of this, could it be that one particular extension of ZFC is optimal for success across all of these fields?

I really don't know how things will go. But I do think it is worthwhile for set-theorists to keep the possibility in mind that future judgments about which axioms of set theory are to be embraced may come not from within set theory itself, but from other areas of mathematics and logic. Set theory has for many decades provided a very useful foundation for mathematics. It would be both satisfying and appropriate if these other areas would return the favour by offering set theory new criteria for the choice of axioms, thereby helping to resolve the difficulties posed by Gödel incompleteness.

Literatur

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