Cantor's Set Theory from a Modern Point of View

Georg Ferdinand Ludwig Philipp Cantor

Berlin doctorate 1867 (number theory)
Appointed in Halle 1869, habilitation (number theory)
Heine → Study of trigonometric series →
Set theory:
Theory of transfinite numbers and cardinality
Algebraic numbers are countable
Real numbers are not countable
1-1 correspondence between n-dimensional space and the real line

Founder of the DMV 1890
First President of the DMV 1891

Opposition: Kronecker
Support: Dedekind
???: Mittag-Leffler

Transfinite counting
C closed set of reals
$C' =$ limit points of C (Cantor derivative)
$C \supseteq C' \supseteq C'' \supseteq \cdots$
$C^\infty =$ the intersection
$C^\infty \supseteq (C^\infty)'$, maybe strict!
Keep counting: $C^\infty \supseteq C^{\infty+1} \supseteq C^{\infty+2} \supseteq \cdots$!

What is $0, 1, ..., \infty, \infty + 1, ...$?
Wellordering: Linear ordering with no infinite descending sequence

Cantor: Any 2 wellorderings are comparable
Each wellordering isomorphic to an *ordinal*, a special wellordering ordered by $\in$
$0 = \emptyset, 1 = \{0\}, 2 = \{0, 1\}, ..., \omega = \{0, 1, 2, ...\}$, $\omega + 1 = \omega \cup \{\omega\}, \ldots$
Cantor's assumption: Every set can be wellordered
Therefore every set bijective with an ordinal (not unique)

Cardinal = Ordinal not bijective with a smaller ordinal
Every set bijective with a *unique* cardinal, its *cardinality*

Zermelo: Cantor's assumption follows from the Axiom of Choice
So Cantor's theory of cardinality applies to arbitrary sets

One major gap!

What is the cardinality of the continuum?

Continuum Hypothesis (CH):
Every uncountable set of reals has the same cardinality as the set of all reals

Paradoxes

Cantor, Burali-Forti, Russell
$x =$ all y such that $y \notin y$
$x \in x \leftrightarrow x \notin x$!

Zermelo's proposal
Only use established principles of set-formation
Axiomatic theory: Zermelo set theory
ZFC = Zermelo-Fraenkel set theory with the Axiom of Choice

The Universe of Sets V

ZFC gives the following picture:

First picture of V
Reduces V to ordinals and power set operation
Not a canonical description

The Vagueness of Power Set

2 approaches:

Definable sets: descriptive set-theory
Borel sets = smallest σ-algebra containing all open sets

Σ^1_1 = continuous image of a Borel set
Π^1_1 set = complement of Σ^1_1 set
Σ^1_{n+1} set = continuous image of Π^1_n set
Π^1_{n+1} set = complement of Σ^1_{n+1} set
Projective = Σ^1_n or Π^1_n for some n

1930s
Σ^1_1 sets satisfy CH: an uncountable Σ^1_1 set has the cardinality of the reals
Π^1_1 sets?

Constructibility (Gödel)
Replace power set operation by a weak power set operation:
$V_{\alpha+1} =$ all subsets of V_α
$L_{\alpha+1} =$ all "simple" subsets of L_α
$L =$ union of the L_α's
L satisfies ZFC
First canonical model ($=$ interpretation) of ZFC
CH holds in L!

Gödel:
L is *not* the correct intepretation of ZFC
Only a tool for showing that statements are consistent with ZFC

There are other interpretations of ZFC:
Cohen's Forcing method

Add new sets to L, preserving ZFC
R is *Cohen over* L iff
R belongs to every open dense set of reals which L can "describe"
Add many Cohen reals to L, obtain model where CH fails

Another use of forcing: R in $[0,1]$ is *random over* L iff
R belongs to every measure one subset of $[0,1]$ which L can "describe"
Using random reals: Model where every projective set of reals is Lebesgue measurable

Thus CH is undecidable using the ZFC axioms

Dilemma: Different universes with different kinds of mathematics?

Canonical Universes

Find canonical, acceptable interpretation of V
Correct answers to undecidable problems given by this interpretation

Gödel's L is canonical, but not acceptable:
Too easily changed using forcing
Universes constructed using forcing are not canonical:
If there is one Cohen (random) real over L, then there are many

How does one obtain canonical universes larger than L?

Answer from measure theory
Countably additive extension of Lebesgue measure to all sets of reals $\to V$ is not L

Model of ZFC with such a measure $\leftrightarrow$
Model of ZFC with a *measurable cardinal*

Silver:
Measurable cardinal $\to$ Canonical inner model (= subuniverse) with a measurable cardinal

First canonical interpretation larger than L
Acceptable?

Measurable cardinal: example of a "large cardinal hypothesis"
These hypotheses have a crucial role in set theory:

φ is *consistency-equivalent* to ψ:
ZFC $+\varphi$ has a model iff ZFC $+\psi$ has a model
Empirical fact:
For any natural set-theoretic assertion φ, φ is consistency-equivalent to $0 = 0$, $0 = 1$ or a large cardinal hypothesis
Large cardinal hypotheses measure the strength of set-theoretic assertions

Silver's model = desired canonical interpretation of V?
Too small!
More than a measurable cardinal is needed to measure strength:

A is *Wadge reducible* to B iff
For some continuous f, $x \in A$ iff $f(x) \in B$

WP_n: If A, B are Σ^1_n but not Π^1_n then
A is Wadge reducible to B and vice-versa

We have:

WP_1 is consistency equivalent to #'s, a large cardinal hypothesis below a measurable cardinal.
WP_2 is consistency equivalent to the existence of a Woodin cardinal, much larger than a measurable cardinal!
WP_n requires $n-1$ Woodin cardinals

Desired canonical model for ZFC should allow Woodin cardinals

Ongoing project: Construction of canonical inner models for large cardinals

Cannot be built in ZFC!

Instead: If there is a certain large cardinal then there is a canonical inner model with this large cardinal

Circular? Why should these large cardinals exist? Maybe WP_n is simply false for $n > 1$!

Important challenge: Justification of large cardinal hypotheses

One approach: self-embeddings of models

M is *rigid* iff there is no embedding $M \to M$ preserving basic operations (union, product, difference, ...)

Smallest large cardinal axiom ($0^\#$ exists) equivalent to: L is not rigid

L not rigid $\to$ there is a canonical $L^\#$ which satisfies "L is not rigid"

Repeat this:
$L^{\#}$ not rigid $\rightarrow$ there is a canonical $L^{\#\#}$ where this is true
$L^{\#\#}$ not rigid $\rightarrow$ $L^{\#\#\#}$, etc.
Fact: There is a canonical such #-iteration which leads to Woodin cardinals

Analogous to Gödel's construction of L (iteration of a weak power set operation)
ZFC justifies use of Gödel's operation
Here one must argue that models in a canonical #-iteration are not rigid
Justifies existence of inner models with Woodin cardinals

However: No canonical # iteration is known past Woodin cardinals
Finding such an operation remains an important problem and would give:

1. A satisfying picture of the set-theoretic universe
2. Numerous further applications of set theory
3. Justify the use of large cardinal hypotheses
4. Substantiate the claim that the paradoxes that worried Cantor in the infancy of set theory have been definitively resolved.