Forcings which Preserve Large Cardinals

Summary:

1. What are large cardinals?

Forcings which preserve large cardinals:

2. Failures of GCH.

3. Cardinal characteristics at large cardinals.

4. L-like universes and large cardinals.

Not covered: Forcings which use large cardinals, but destroy largeness (Singular Cardinal Hypothesis)

What are large cardinals?

κ is *inaccessible* iff:
$\kappa > \aleph_0$
κ is regular
$\lambda < \kappa \to 2^\lambda < \kappa$

κ inaccessible implies V_κ is a model of ZFC

κ is *measurable* iff:
$\kappa > \aleph_0$
$\exists$ nonprincipal, κ-complete ultrafilter on κ

What are large cardinals?

Embeddings:

V = universe of all sets
M an inner model (transitive class satisfying ZFC, containing Ord)

$j : V \to M$ is an *embedding* iff:
j is not the identity
j preserves the truth of formulas with parameters

Critical point of j is the least κ, $j(\kappa) \neq \kappa$

Idea: κ is "large" iff κ is the critical point of an embedding $j : V \to M$ where M is "large"

What are large cardinals?

Suppose that κ is the critical point of $j : V \to M$

κ is $H(\lambda)$-strong iff $H(\lambda) \subseteq M$

κ is λ-supercompact iff $M^\lambda \subseteq M$

Fact: Measurable $= H(\kappa^+)$-strong $= \kappa$-supercompact.

Kunen: No $j : V \to M$ witnesses $H(\lambda)$-strength for all λ, i.e., M cannot equal V

However: κ could be $H(\lambda)$-strong for all λ (i.e., the critical point of embeddings with arbitrary degrees of strength)

What are large cardinals?

Larger cardinals:

Again suppose κ is the critical point of $j : V \to M$

κ is superstrong iff $H(j(\kappa)) \subseteq M$

κ is hyperstrong iff $H(j(\kappa)^+) \subseteq M$

κ is n-superstrong iff $H(j^n(\kappa)) \subseteq M$ (n finite)

κ is ω-superstrong iff $H(j^\omega(\kappa)) \subseteq M$

Kunen: More than ω-superstrong is inconsistent
(cannot have $H(j^\omega(\kappa)^+) \subseteq M$)

Why study large cardinals?

First Reason:

Set theory, even with large cardinals, is *incomplete*:

For many φ, both ZFC $+ \varphi$ and ZFC $+ \ \varphi$ are consistent

But set theory with large cardinals seems to be *consistency complete*:

For almost all φ, if φ is consistent then we have

$\text{Con}(\text{ZFC} + LC) \to \text{Con}(\text{ZFC} + \varphi)$

for some large cardinal axiom LC; moreover we often get:

$\text{Con}(\text{ZFC} + \varphi) \to \text{Con}(\text{ZFC} + \text{lc})$

where lc is almost as strong as LC

Conclusion: We need large cardinals to show consistency.

Why study large cardinals?

Second reason: Forcing is interesting when there are large cardinals!

Examples:

a. Failure of GCH at a measurable

Increasing 2^κ with κ-Cohen is painful, with κ-Laver regrettable, but with κ-Sacks perfect!

b. Cardinal characteristics at a measurable (new area)

$$\mathfrak{a}, \mathfrak{b}, \mathfrak{d}, \mathfrak{e}, \mathfrak{g}, \mathfrak{h}, \mathfrak{i}, \mathfrak{m}, \mathfrak{p}, \mathfrak{r}, \mathfrak{s}, \mathfrak{t}, \mathfrak{u}$$

at κ. Iterated forcing with uncountable supports

c. Forcing combinatorial principles at a measurable (surprises with Jensen's $\Box$ Principle)

d. Singular cardinal problems (Prikry-type forcings)

Forcings that preserve large cardinals: Silver lifting

Question: Suppose κ is a large cardinal and G is P-generic over V. Is κ still a large cardinal in $V[G]$?

Lifting method (Silver):

Given $j : V \to M$ and G which is P-generic over V

Let P^* be $j(P)$

Goal: Find a G^* which is P^*-generic over M such that $j[G] \subseteq G^*$

Then $j : V \to M$ lifts to $j^* : V[G] \to M[G^*]$, defined by $j^*(\sigma^G) = j(\sigma)^{G^*}$ (well-defined: $\sigma_0^G = \sigma_1^G \to p \Vdash \sigma_0 = \sigma_1$ some $p \in G \to j(p) \Vdash j(\sigma_0) = j(\sigma_1)$ some $p \in G \to j(\sigma_0)^{G^*} = j(\sigma_1)^{G^*}$ as $j[G] \subseteq G^*$; elementary by similar argument)

If G^* belongs to $V[G]$ then κ is still measurable (and maybe more) in $V[G]$

Remark: The lifting method is the most common, but *not* the only way to preserve large cardinals

Forcings that preserve large cardinals: Ultrapowers

To apply the lifting method often need a special $j : V \to M$:

Theorem

(Ultrapower Theorem) Suppose that κ is $H(\lambda)$-strong, i.e., there is $j : V \to M$ with critical point κ such that $H(\lambda) \subseteq M$.
(a) (Extender ultrapower) If $\lambda \leq j(\kappa)$ then j can be modified so that: $M = \{j(f)(a) \mid f : H(\kappa) \to V,\ a \in H(\lambda)\}$.
(b) (Hyperextender ultrapower) If $\lambda = j(\kappa)^+$ then j can be modified so that: $M = \{j(f)(a) \mid f : H(\kappa^+) \to V,\ a \in H(j(\kappa)^+)\}$.
(c) (2-Hyperextender ultrapower) If $\lambda \leq j^2(\kappa)$ then j can be modified so that: $M = \{j(f)(a) \mid f : H(j(\kappa)) \to V,\ a \in H(\lambda)\}$.
(d) $n+1$-Hyperextender ultrapower uses $f : H(j^n(\kappa)) \to V$;
ω-Hyperextender ultrapower uses $f : H(j^\omega(\kappa)) \to V$.

Proof (a): Define $H = \{j(f)(a) \mid f : H(\kappa) \to V,\ a \in H(\lambda)\} \prec M$, $k : H \simeq M'$ the transitive collapse, $j' : V \to M'$ by $j' = k \circ j$. $\square$

Forcings that preserve large cardinals: Easy cases

Sometimes it is easy to lift $j : V \to M$ to $j^* : V[G] \to M[G^*]$.

Recall: $j : V \to M$ has critical point κ, G is P-generic over V, $P^* = j(P)$ and we want a G^* which is P^*-generic over M satisfying $j[G] \subseteq G^*$. We say that j *lifts for* P.

Small forcing

Suppose that P belongs to $H(\kappa)$ (P is small). Then j lifts for P.

Proof: $P^* = j(P) = P$. Take $G^* = G$. Then G^* is P^*-generic over $M \subseteq V$ and $j[G] = G \subseteq G^*$, trivially!

κ^+ distributive forcing

P is κ^+ distributive iff the intersection of κ-many open dense sets is always nonempty.

Forcings that preserve large cardinals: Easy cases

Theorem

Suppose that $j : V \to M$ is given by an extender ultrapower, i.e., $M = \{j(f)(a) \mid f : H(\kappa) \to V,\ a \in H(\lambda)\}$ for some $\lambda \leq j(\kappa)$, $H(\lambda) \subseteq M$.
Suppose that P is κ^+ distributive in V. Then j lifts for P.

Proof: Suppose that $D \in M$ is open dense on $P^* = j(P)$. Write $D = j(f)(a)$ where $f : H(\kappa) \to V$, $a \in H(\lambda)$. We can assume that $f(x)$ is open dense on P for each $x \in H(\kappa)$. By the κ^+ distributivity of P there is $p \in G$ which belongs to each $f(x)$. It follows that $j(p)$ belongs to each $j(f)(y)$, $y \in H(j(\kappa))^M$ and therefore to $j(f)(a)$. So $j[G]$ "generates" the P^*-generic $G^* = \{p^* \in P^* \mid j(p) \leq p^*$ for some p in $G\}$. $\Box$

So P-lifting is nontrivial only when P has size at least κ and adds κ-sequences. A good example is κ-Cohen forcing.

An embedding which lifts for κ-Cohen?

Goal: Make GCH fail at a measurable cardinal

Obvious approach: Let P be Cohen(κ, κ^{++})
Adds κ^{++}-many κ-Cohen sets
Conditions are partial functions of size $< \kappa$ from $\kappa \times \kappa^{++}$ to 2

Want $j : V \to M$ that lifts for P. Then for P-generic G we have $j^* : V[G] \to M^*$, witnessing that κ is measurable in $V[G]$, and moreover GCH fails at κ in $V[G]$.

Easier lifting problem: $P =$ Cohen$(\kappa, 1)$, i.e. κ-Cohen forcing.
Bad news!

Theorem

Let P be κ-Cohen forcing. Then no $j : V \to M$ lifts for P.

An embedding which lifts for κ-Cohen?

Here is the problem:

Suppose that $C \subseteq \kappa$ is generic for κ-Cohen

Want to lift $j : V \to M$ to $j^* : V[C] \to M[C^*]$

Want to find C^* which is $j(\kappa)$-Cohen generic over M and "extends" C, i.e., such that $C = C^* \cap \kappa$

Impossible! Proper initial segments of C^* must belong to M, but C does not even belong to V!

Need the forcing to add C^* to be defined not in M but in a model that *already has* C

Solution: Force not just at κ, but at all inaccessible $\alpha \leq \kappa$, via an iteration

$$P = P(\alpha_0) * P(\alpha_1) * \cdots * P(\kappa)$$

where $P(\alpha)$ denotes α-Cohen forcing.

Lift not just $P(\kappa) = \kappa$-Cohen forcing, but the entire iteration P ("Prepare below κ")

Preparing κ-Cohen

What is the iteration

$$P = P(\alpha_0) * P(\alpha_1) * \cdots * P(\kappa) \ ?$$

Use *Easton support*, i.e., for p in $P = P(\alpha_0) * P(\alpha_1) * \cdots * P(\kappa)$, Support$(p) = \{i \mid p \restriction i \nVdash p(\alpha_i)$ is trivial$\}$ has bounded intersection with each inaccessible. Then for regular λ, P factors as:

$$P(\leq \lambda) * P(> \lambda)$$

where $P(\leq \lambda)$ has "size" λ and $P(> \lambda)$ is λ^+-closed (descending sequences of length λ have lower bounds). As in Easton's theorem, this gives cofinality preservation.

Preparing κ-Cohen

Theorem

Assume GCH. Let $P = P(\leq \kappa) = P(\alpha_0) * P(\alpha_1) * \cdots * P(\kappa)$ be the iteration of α-Cohen for inaccessible $\alpha \leq \kappa$ described above. Suppose that $j : V \to M$ is an extender ultrapower witnessing the $H(\lambda)$-strength of κ for some regular λ less than the least inaccessible above κ. Then j lifts for P.

Preparing κ-Cohen

Let $C(\leq \kappa) = C(\alpha_0) * C(\alpha_1) * \cdots * C(\kappa)$ denote the P-generic and $V^* = V[C(\leq \kappa)]$

We want to lift $j : V \to M$ to

$j^* : V[C(\leq \kappa)] \to M[C^*(\leq \kappa) * C^*(\beta_0) * C^*(\beta_1) * \cdots * C^*(j(\kappa))]$

where the β_i's are the inaccessibles of M between κ and $j(\kappa)$ and the C^*'s are chosen in $V^* = V[C(\leq \kappa)]$

Set $C^*(\leq \kappa) = C(\leq \kappa)$
Middle part: Take $\langle C^*(\beta) \mid \kappa < \beta < j(\kappa) \rangle = C^*(\kappa, j(\kappa))$ to be any generic in V^* (why are there any ???)

Last lift: Take $C^*(j(\kappa))$ to be any generic in V^* for $j(\kappa)$-Cohen forcing of $M[C^*(\leq \kappa) * C^*(\kappa, j(\kappa))]$
containing the condition $C(\kappa) = C^*(\kappa)$ (why are there any ???).

Preparing κ-Cohen

Explaining the two ???'s

$j^* : V[C(\leq \kappa)] \to M[C(\leq \kappa) * C^*(\kappa, j(\kappa))??? * C^*(j(\kappa))???]$

Middle part: We want a generic $C^*(\kappa, j(\kappa))$ in $V^* = V[C(\leq \kappa)]$ for $P^*(\kappa, j(\kappa)) = P^*(\beta_0) * P^*(\beta_1) * \cdots$, a forcing which is β_0-closed and has $j(\kappa)$-many maximal antichains in $M[C(\leq \kappa)]$.

Recall that the original $j : V \to M$ was an extender ultrapower witnessing $H(\lambda)$-strength for some regular $\lambda < \beta_0$.

Claim.

(a) $M^\kappa \cap V \subseteq M$.

(b) $j(\kappa)$ can be written in V as the union of κ^+-many subsets, each an element of M of size λ in M.

Preparing κ-Cohen

Claim.
(a) $M^\kappa \cap V \subseteq M$.
(b) $j(\kappa)$ can be written in V as the union of κ^+-many subsets, each an element of M of size λ in M.

Given (a) and (b): The κ^+-cc of $P(\leq \kappa)$ implies that (a) also holds for the models $M[C(\leq \kappa)]$, $V[C(\leq \kappa)]$:

$$M[C(\leq \kappa)]^\kappa \cap V[C(\leq \kappa)] \subseteq M[C(\leq \kappa)]$$

Therefore $P^*(\kappa, j(\kappa))$ is κ^+-closed in $V[C(\leq \kappa)]$. But then (b) and the λ^+ closure of $P^*(\kappa, j(\kappa))$ in $M[C(\leq \kappa)]$ implies that we can build a $P^*(\kappa, j(\kappa))$-generic in κ^+ steps.

Preparing κ-Cohen

Proof of (a): $M^\kappa \cap V \subseteq M$

Given $j(f_0)(a_0), j(f_1)(a_1), \cdots$ of length κ define $f : H(\kappa) \to V$ by $f(\langle x_0, x_1, \cdots \rangle = \langle f_0(x_0), f_1(x_1), \cdots \rangle$; then $j(f)(\langle a_0, a_1, \cdots \rangle)$ is the κ-sequence of the $j(f_i)(a_i)$'s and $\langle a_0, a_1, \cdots \rangle$ is an element of $H(\lambda)$.

Proof of (b): $j(\kappa)$ can be written in V as the union of κ^+-many subsets, each an element of M of size λ in M

Every ordinal less than $j(\kappa)$ is of the form $j(f)(a)$ where $f : H(\kappa) \to V$ and $a \in H(\lambda)$; but we may assume $f : H(\kappa) \to \kappa$ (simply redefine $f(x)$ to be 0 if $f(x)$ is not an ordinal $< \kappa$; this won't affect $j(f)(a)$). So $j(\kappa)$ is the union of the sets $A(f) = \{j(f)(a) \mid a \in H(\lambda)\}$, $f : H(\kappa) \to \kappa$, each of which has size λ in M by GCH, and again by GCH there are only κ^+-many such sets.

Preparing κ-Cohen

The second ???:

$j^* : V[C(<\kappa) * C(\kappa)] \to M[C(\leq \kappa) * C^*(\kappa, j(\kappa)) * C^*(j(\kappa))???]$

We need a generic in V^* for $P^*(j(\kappa))$ = the $j(\kappa)$-Cohen forcing of $M[C(\leq \kappa) * C^*(\kappa, j(\kappa))]$ containing the condition $C(\kappa)$.

This is similar to the previous case. We have:

(a) $M[C^*(<j(\kappa))]^\kappa \cap V^* \subseteq M[C^*(<j(\kappa))]$.
(b) $P^*(j(\kappa))$ has $(j(\kappa)^+)^{M[C^*(<j(\kappa))]} = j(\kappa^+)$ many maximal antichains in $M[C^*(<j(\kappa))]$ and $j(\kappa^+)$ can be written in V^* as the union of κ^+ many subsets, each an element of M of size λ in M.

For (a) we need only show $\mathrm{Ord}^\kappa \cap V^* \subseteq M[C^*(<j(\kappa))]$, which follows from $\mathrm{Ord}^\kappa \cap V^* \subseteq M[C^*(\leq \kappa)]$.

For (b), note that every $\alpha < j(\kappa^+)$ can be written as $j(f)(a)$ with $f : H(\kappa) \to \kappa^+$, $a \in H(\lambda)$, and there are still only κ^+-many such f's. So we can build a $P^*(j(\kappa))$-generic in V^* containing $C(\kappa)$.

Failure of GCH at a measurable

So we have succeeded in lifting $j : V \to M$ to $j : V^* = V[C(\leq \kappa)] \to M[C^*(\leq j(\kappa))]$ in V^*, where $C(\leq \kappa)$ results by iterating α-Cohen forcing for inaccessible $\alpha \leq \kappa$.

Now we would like to make this work with α-Cohen forcing replaced by Cohen(α, α^{++}), a forcing that kills the GCH at α

It doesn't work! Here is the problem:

Failure of GCH at a measurable

Assuming that the original $j : V \to M$ witnessed $H(\kappa^{++})$-strength (to allow $C^*(\kappa) = C(\kappa)$), all goes well until the last lift: we *can* choose $C^*(\gamma)$ for M-inaccessible $\gamma < j(\kappa)$ and lift $j : V \to M$ to

$j' : V[C(< \kappa)] \to M[C^*(< j(\kappa)]$

We then need to find a generic $C^*(j(\kappa))$ for $P^*(j(\kappa)) =$ the Cohen$(j(\kappa), j(\kappa^{++}))$-forcing of $M[C^*(< j(\kappa)]$ which contains $j'[C(\kappa)]$ to get:

$j^* : V[C(\leq \kappa)] \to M[C^*(< j(\kappa)) * C^*(j(\kappa))???]$

But $P^*(j(\kappa)) =$ Cohen$(j(\kappa), j(\kappa^{++}))$ is a big forcing: it has size κ^{++} and won't have a generic in $V[C(\leq \kappa)]$!

Even worse, whereas before $j'[C(\kappa)]$ was equal to $C(\kappa)$, now $j'[C(\kappa)]$ is a complicated set of conditions!

Failure of GCH at a measurable

Here is the solution: Use Sacks(κ, κ^{++}) instead of Cohen(κ, κ^{++})

Now we want to lift $j : V \to M$ to

$j^* : V[S(\leq \kappa)] \to M[S(\leq \kappa) * S^*(\kappa, j(\kappa)) * S^*(j(\kappa))]$

The nice thing now is that we don't have to build a generic $S^*(j(\kappa))$ for $P^*(j(\kappa)) =$ Sacks$(j(\kappa), j(\kappa^{++}))$ containing $j'[S(\kappa)]$, because in fact $j'[S(\kappa)]$ (almost) generates one for us!

Illustrate this with just Sacks$(\kappa, 1) = \kappa$-Sacks: A condition is a κ-tree, i.e. a subtree T of $2^{<\kappa}$ such that:

i. T has no terminal nodes and is $< \kappa$-closed, i.e., the union of a $(< \kappa)$ increasing sequence of nodes in T is a node in T.
ii. T has "CUB splitting": For some CUB $C(T) \subseteq \kappa$, $\sigma \in T$ "splits" in T iff the length of σ belongs to $C(T)$.

If G is generic then the intersection of the κ-trees in G gives us a function $g : \kappa \to 2$, which uniquely determines G.

Failure of GCH at a measurable

Now prepare as before, iterating for $\kappa+1$ steps, but with α-Sacks instead of α-Cohen. Then as before we obtain an embedding

$$j' : V[S(<\kappa)] \to M[S^*(< j(\kappa))]$$

To extend j' further we want to find a generic $S^*(j(\kappa))$ for the $j(\kappa)$-Sacks of $M[S^*(< j(\kappa))]$ which contains $j'[S(\kappa)]$.

But in fact there are only two possible choices for $S^*(j(\kappa))$:

Claim: The intersection of the $j(C)$, C CUB in κ, is $\{\kappa\}$.

Assume this Claim. For any CUB C in κ there are κ-trees T in the generic $S(\kappa)$ which only split on C. Thus by the Claim the intersection of the $j(T)$, $T \in S(\kappa)$ splits only at κ and is therefore the union of exactly two $b_0, b_1 : j(\kappa) \to 2$ which first disagree at κ (a "Tuning Fork"). As $S^*(j(\kappa))$ must contain each $j(T)$, $T \in S(\kappa)$, b_0, b_1 are the only candidates for the desired $j(\kappa)$-Sacks generic! It can be shown that both b_0, b_1 are indeed $j(\kappa)$-Sacks generic.

Failure of GCH at a measurable

Proof of

Claim: The intersection of the $j(C)$, C CUB in κ, is $\{\kappa\}$.

We assume that $j : V \to M$ is an extender ultrapower witnessing the $H(\kappa^{++})$-strength of κ, so $M = \{j(f)(a) \mid f : H(\kappa) \to V$, $a \in H(\kappa^{++})\}$. We must show that if α does not equal κ then α fails to belong to $j(C)$ for some CUB C in κ. We may assume that α lies between κ and $j(\kappa)$; write $\alpha = j(f)(a)$ for some $f : H(\kappa) \to \kappa$, $a \in H(\kappa^{++})$. We take C to be $\{\beta < \kappa \mid \beta$ is a limit cardinal and $H(\beta)$ is closed under $f\}$, a CUB subset of κ. Then $j(C) = \{\beta < j(\kappa) \mid \beta$ is a limit cardinal of M and $H(\beta)^M$ is closed under $j(f)\}$. If $\beta > \kappa$ belongs to $j(C)$ then $j(f)(b) < \beta$ for all $b \in H(\kappa^{++})^M = H(\kappa^{++})$, so in particular $\kappa < \alpha = j(f)(a) < \beta$. Thus α does not belong to $j(C)$.

Failure of GCH at a measurable

A similar result holds for Sacks(κ, κ^{++}) (joint work with Katie Thompson). A condition is a function $p : \kappa^{++} \to \kappa$-Sacks which is trivial on all but κ many $i < \kappa^{++}$.

Prepare as before, iterating for $\kappa + 1$ steps, but with Sacks(α, α^{++}) at inaccessible stages $\alpha \leq \kappa$. As before we obtain an embedding

$$j' : V[S(< \kappa)] \to M[S^*(< j(\kappa))]$$

To extend j' further we want to find a generic $S^*(j(\kappa))$ for the Sacks($j(\kappa), j(\kappa^{++})$ of $M[S^*(< j(\kappa))]$ which contains $j'[S(\kappa)]$, where $S(\kappa)$ is the Sacks(κ, κ^{++})-generic, yielding:

Failure of GCH at a measurable

$$j^* : V[S(\leq \kappa)] \to M[S^*(< j(\kappa))][S^*(j(\kappa)]$$

Now what happens is this:

For $i < j(\kappa^{++})$ in the range of j, the intersection of the $j(p)(i)$ is a tuning fork $b_0^i, b_1^i : j(\kappa) \to 2$.

For $i < j(\kappa^{++})$ not in the range of j, the intersection of the $j(p)(i)$ is a single $b^i : j(\kappa) \to 2$.

And if for $i < j(\kappa^{++})$ we take the b_0^i for i in the range of j and the b^i for i not in the range of j then we obtain a Sacks($j(\kappa), j(\kappa^{++})$)-generic. This generic contains $j'[S(\kappa)]$ by its definition (and is almost generated by it).

Conclusion: The fusion property for κ-Sacks is a good substitute for κ^+-distributivity, and therefore works better than κ-Cohen.

Other applications

Some other applications of “fusion lifting”:

(with Magidor) Assume GCH, let κ be measurable and let α be any cardinal at most κ^{++}. Then there is a cofinality-preserving forcing extension in which there are exactly α normal measures on κ. If κ is $H(\kappa^{++})$-strong, then there is a cofinality-preserving forcing extension in which GCH fails at κ and there is a unique normal measure on κ.

Uses variants of κ-Sacks, tuning forks and nonstationary support iterations.

(with Dobrinen) Assume GCH and let κ be $H(\kappa^{++})$-strong. Then there is a forcing extension in which κ is still measurable and the tree property holds at κ^{++}.

Extends the tuning fork method form a κ-Sacks product to κ-Sacks iteration (of length κ^{++}).

Forcings that preserve large cardinals

(with Honzìk) (Special Case) Assume GCH and F is an Easton function such that $F \restriction \kappa$ is definable over $H(F(\kappa))$ uniformly for all regular κ. Then there is a cofinality-preserving forcing extension in which $2^\gamma = F(\gamma)$ for all regular γ and every κ which is $H(F(\kappa))$-strong in the ground model remains measurable.

Uses the tuning fork method and matrices of conditions to lift an embedding.

Cardinal characteristics at large cardinals

New area; we consider three examples:

$$\mathfrak{d}(\kappa),\ \mathrm{CofSym}(\kappa),\ \mathfrak{s}(\kappa)$$

Generalised dominating number $\mathfrak{d}(\kappa)$

Cummings and Shelah proved an Easton-type theorem for the function $\kappa \mapsto \mathfrak{d}(\kappa)$. In particular:

Cardinal characteristics at large cardinals

Theorem

(Cummings-Shelah) Assume GCH and κ regular. Then in a cofinality-preserving extension, $\kappa^+ = \mathfrak{d}(\kappa) < 2^\kappa$.

Their proof goes as follows: First apply Cohen(κ, κ^{++}) to make $2^\kappa = \kappa^{++}$.

Then iterate κ-Hechler forcing for κ^+ steps, adding at each step a function $f : \kappa \to \kappa$ which eventually dominates all ground model functions.

A condition in κ-Hechler is a pair (s, f) where

$s : |s| \to \kappa$, $|s| < \kappa$

$f : \kappa \to \kappa$

$(t, g) \leq (s, f)$ iff $t \supseteq s$, g dominates f, t dominates f on $|t| \setminus |s|$.

This is κ-closed and κ^+-cc.

In the resulting model $\mathfrak{d}(\kappa) = \kappa^+$

Cardinal characteristics at large cardinals

Question: Can one have $\mathfrak{d}(\kappa) < 2^\kappa$ for a measurable κ?

Assume GCH, κ is $H(\kappa^{++})$-strong and $j : V \to M$ witnesses the latter via an extender ultrapower.

Strategy: Prepare up to κ using Cohen(α, α^{++}) followed by an α^+ iteration of α-Hechler, and lift the embedding:

$$V[CH(\leq \kappa)] \to M[CH(< j(\kappa)) * CH(j(\kappa))]$$

Doesn't work!

We already saw the problems with lifting for Cohen(κ, κ^{++}); but κ-Hechler presents even more serious difficulties:

Cardinal characteristics at large cardinals

Consider

$$j^* : V[H(\leq \kappa)] \to M[H^*(< j(\kappa)) * H^*(j(\kappa))]$$

where the $H(\alpha), H^*(\alpha)$ are generic for α-Hechler forcing. Now we want the $j(\kappa)$-Hechler generic $H^*(j(\kappa))$ to extend the κ-Hechler generic $H(\kappa)$. Let $h^* : j(\kappa) \to j(\kappa)$ be the $j(\kappa)$-Hechler generic function associated with $H^*(j(\kappa))$ and $h : \kappa \to \kappa$ the κ-Hechler generic function associated with $H(\kappa)$. Then:

For any $f : \kappa \to \kappa$ in V, h dominates f beyond some $\alpha < \kappa$; so

For any $f : \kappa \to \kappa$ in V, h^* dominates $j(f)$ beyond (the same) ordinal $\alpha < \kappa$, and in particular $j(f)(\kappa) < h^*(\kappa)$.

But we have seen that the intersection of the $j(C)$, C club in κ is $\{\kappa\}$ and from this it follows that the $j(f)(\kappa)$ for $f : \kappa \to \kappa$ are cofinal in $j(\kappa)$. So $h^*(\kappa)$ cannot be defined!

Cardinal characteristics at large cardinals

But note that we have already solved this problem:

We showed that κ remains measurable after iterating Sacks(α, α^{++}) for inaccessible $\alpha \leq \kappa$. This factors as

(Iteration of Sacks(α, α^{++}) below κ) $*$ Sacks(κ, κ^{++}).

A forcing is κ^κ bounding iff every function $f : \kappa \to \kappa$ that it adds is dominated by such a function from the ground model.
Any κ-cc forcing is κ^κ bounding, and fusion shows that Sacks(κ, κ^{++}) is also κ^κ bounding.
It follows that the above iteration is κ^κ bounding and therefore over a model of GCH forces $\mathfrak{d}(\kappa) = \kappa^+ < 2^\kappa = \kappa^{++}$.

Cardinal characteristics at large cardinals

Remark: With enough supercompactness, it can be shown that the κ-Cohen with κ-Hechler strategy does work, and indeed one can get κ measurable with any reasonable values for $\mathfrak{d}(\kappa)$, $\mathfrak{b}(\kappa)$ and 2^κ, where $\mathfrak{b}(\kappa)$ is the bounding number at κ, i.e., the smallest size of a subset of ${}^\kappa\kappa$ which is not bounded in ${}^\kappa\kappa$ under the order of eventual domination.

Question: Is it consistent relative to a strong cardinal (i.e., a cardinal κ which is $H(\lambda)$-strong for all λ) to have a measurable κ with $\mathfrak{b}(\kappa) = \kappa^{++}$?

The Cardinal Characteristic CofSym(κ)

Let κ be regular.
Sym(κ) = group of permutations of κ under composition.
CofSym(κ) = least λ such that Sym(κ) is the union of a strictly increasing λ-chain of subgroups.

Macpherson and Neumann: CofSym(κ) $> \kappa$

Cardinal characteristics at large cardinals

Sharp and Thomas: For any regular κ, can force CofSym(κ) to be greater than κ^+.

Theorem

(F-Zdomskyy) Suppose that κ is $H(\kappa^{++})$-strong. Then in a forcing extension, κ is measurable and CofSym(κ) $= \kappa^{++}$.

The Sharp-Thomas proof (based on a forcing of Mekler-Shelah) does not appear to work; instead one uses an iteration of Miller(κ) (a version of Miller forcing at κ with continuous club-splitting) mixed with a variant of κ-Sacks forcing.
It is another lifting argument using fusion.

Question: Is it consistent that CofSym(κ) $= \kappa^{+++}$ for a measurable κ?

The Cardinal Characteristic $\mathfrak{s}(\kappa)$

Fix κ regular. For x, y subsets of κ of size κ, x *splits* y iff both $y \setminus x$ and $y \cap x$ have size κ. $\mathfrak{s}(\kappa)$ is the least size of a splitting family of subsets of κ, i.e., a family sufficient to split every size κ subset of κ.

Facts. For κ regular and uncountable:
κ is inaccessible iff $\mathfrak{s}(\kappa) \geq \kappa$
κ is weakly compact iff $\mathfrak{s}(\kappa) > \kappa$
Relative to a supercompact, it is consistent to have a measurable κ with $\mathfrak{s}(\kappa) = \kappa^{++}$.
(Zapletal) $\mathfrak{s}(\kappa) > \kappa^{+}$ for an uncountable regular κ requires an α of Mitchell order α^{++} (slightly weaker than $H(\alpha^{++})$-strong)

Question: Can one obtain a measurable κ with $\mathfrak{s}(\kappa) = \kappa^{++}$ from an α which is $H(\alpha^{++})$-strong?

Large Cardinals and L-like Universes

Question: Can we have the advantages of both $V = L$ and large cardinals?

2 approaches:

Inner model programme: A universe with large cardinals has an *inner model* which is L-like and has large cardinals

Outer model programme: A universe with large cardinals has an *outer model* which L-like and has large cardinals

1st approach uses fine structure theory and iterated ultrapowers
2nd approach uses forcing: much easier

Large Cardinals and L-like Universes

Examples of L-like properties:
GCH
Definable Wellorders of the Universe
Jensen's $\diamondsuit$, $\square$ and Morass Principles
Condensation Principles

Recall: $j : V \to M$ with critical point κ is
Superstrong iff $H(j(\kappa)) \subseteq M$
We may assume $M = \{j(f)(a) \mid f : H(\kappa) \to V,\ a \in H(j(\kappa))\}$
Hyperstrong iff $H(j(\kappa)^+) \subseteq M$
We may assume $M = \{j(f)(a) \mid f : H(\kappa^+) \to V,\ a \in H(j(\kappa)^+)\}$
$n+1$-superstrong iff $H(j^{n+1}(\kappa)) \subseteq M$
We may assume $M =$
$\{j(f)(a) \mid f : H(j(\kappa)^{+n}) \to V,\ a \in H(j^{n+1}(\kappa))\}$
ω-superstrong iff $H(j^\omega(\kappa)) \subseteq M$
We may assume $M = \{j(f)(a) \mid f : H(j^\omega(\kappa)) \to V,\ a \in H(j^\omega(\kappa))\}$

Forcing GCH

We simply iterate α^+-Cohen for regular α

ω_1-Cohen forces CH, collapses $2^{\aleph_0}$ to ω_1
Then ω_2-Cohen forces GCH at ω_1, collapses 2^{ω_1} to ω_2
Etc.

Preserving a superstrong: Want a lifting of $j : V \to M$ to

$j^* : V[G(<\kappa) * G[\kappa, \infty)] \to$
$M[G^*(<\kappa) * G^*[\kappa, j(\kappa)) * G^*[j(\kappa), \infty)]$
The forcings P (to add G) and $P^* = j(P)$ (to add G^*) agree strictly below $j(\kappa)$ since $j : V \to M$ is superstrong;
but they may take different limits at $j(\kappa)$:

$P^*(<j(\kappa)) =$ DirLim of $P^*(<\alpha)$, $\alpha < j(\kappa)$
$P(<j(\kappa)) =$ InvLim of $P(<\alpha)$, $\alpha < j(\kappa)$, if $j(\kappa)$ singular (???)

Large Cardinals and L-like Universes: Forcing GCH

Fact: If $j : V \to M$ is superstrong with $j(\kappa)$ least then $j(\kappa)$ has cofinality κ^+.

So we have to deal with a singular $j(\kappa)$.
But it is easy to show:

$G(< j(\kappa)) \cap P^*(< j(\kappa))$ is generic over M for $P^*(< j(\kappa))$

so we can simply take this to be $G^*(< j(\kappa))$.

Now we are done, as $P[\kappa, \infty)$ is κ^+-distributive and this implies that the image of $G[\kappa, \infty)$ generates a $P^*[j(\kappa), \infty)$-generic

Large Cardinals and L-like Universes: Forcing GCH

Preserving a Hyperstrong: Want a lifting of $j : V \to M$ to

$j^* : V[G(<\kappa) * G(\kappa) * G[\kappa^+, \infty)] \to$
$M[G^*(<\kappa) * G^*[\kappa, j(\kappa)) * G^*(j(\kappa)) * G^*[j(\kappa)^+, \infty)]$

Now P and P^* agree up to $j(\kappa)$, so we would like to take $G^*(\leq j(\kappa))$ to be $G(\leq j(\kappa))$; we must however ensure that this contains $j[G(\leq \kappa)]$.
We first lift j to $j' : V[G(<\kappa)] \to M[G^*(<j(\kappa))]$ and then observe that $j'[G(\kappa)]$ has a greatest lower bound in the forcing $P^*(j(\kappa))$. So we simply assume that $G(j(\kappa))$ was chosen below this greatest lower bound.

Finally in analogy to the superstrong case, the κ^{++}-distributivity of $P[\kappa^+, \infty)$ implies that the image of $G[\kappa^+, \infty)$ generates a $P^*[j(\kappa)^+, \infty)$-generic.

Large Cardinals and L-like Universes: Forcing GCH

Preserving a 2-superstrong: Want a lifting of $j : V \to M$ to

$j^* : V[G(< \kappa) * G[\kappa, j(\kappa)) * G[j(\kappa), \infty)] \to$
$M[G^*(< \kappa) * G^*[\kappa, j^2(\kappa)) * G^*[j^2(\kappa), \infty)]$

This time P^* and P agree strictly below $j^2(\kappa)$, P^* takes a direct limit at $j^2(\kappa)$ and P possibly takes an inverse limit there, as $j^2(\kappa)$ may be singular. This singularity can occur:

Fact: If $j : V \to M$ is 2-superstrong with $j^2(\kappa)$ least then j is continuous at $j(\kappa)$ and therefore $j^2(\kappa)$ has cofinality $j(\kappa)$.

So as before we take $G^*(< j^2(\kappa))$ to be $G(< j^2(\kappa)) \cap P^*(< j^2(\kappa))$. We can ensure that $j[G(< j(\kappa))]$ is contained in $G(< j^2(\kappa))$, as the former has a greatest lower bound in the forcing $P(< j^2(\kappa))$.

And the $j(\kappa)^+$-distributivity of $P[j(\kappa), \infty)$ implies that the image of $G[j(\kappa), \infty)$ generates a $P^*[j^2(\kappa), \infty)$-generic.

Finally, for the ω-superstrong case we choose $G(< j^\omega(\kappa))$ to contain a condition forcing $j[G(< j^n(\kappa))] \subseteq G(< j^{n+1}(\kappa))$ for each n, and show:

Claim. $G(< j^\omega(\kappa)) \cap P^*(< j^\omega(\kappa))$ is $P^*(< j^\omega(\kappa))$-generic over M.

The proof of the Claim uses an argument regarding the "reduction" of dense sets.

Forcing Definable Wellorders

We have:

Lemma

(Asperó-F) Preserving a proper class of ω-superstrongs it is possible to force GCH together with a wellorder of V whose restriction to $H(\kappa^+)$ is definable over $H(\kappa^+)$ for uncountable regular κ, uniformly.

Thus one gets a wellorder of $H(\aleph_{\omega+1})$ which is only definable over $H(\aleph_{\omega+2})$, not over $H(\aleph_{\omega+1})$, as one might hope.
This gives a nice open problem:

Question: With set-forcing, can one always add a definable wellorder of $H(\aleph_{\omega+1})$?

Large Cardinals and L-like Universes: Definable Wellorders

Note: One cannot expect to force a definable wellorder of $H(\omega_1)$; this is not possible if there is a proper class of Woodin cardinals, for example, as then Projective Determinacy holds in all set-generic extensions.

Another note: It is definitely not always possible to force a definable wellorder of $H(\lambda^+)$ for singular λ:
This is contradicted by an elementary embedding from $L[H(\lambda^+)]$ to itself with critical point less than λ, using Kunen's proof that there is no nontrivial elementary embedding of V to itself.

Forcing $\diamondsuit$

In this case we iterate α-Cohen forcing for all regular α.
It is easy to see that this forces $\diamondsuit_\alpha$ for all regular α and preserves cofinalities, GCH.

Preserving a superstrong: We want to lift $j : V \to M$ to:

$j^* : V[G(<\kappa) * G(\kappa) * G(\kappa, j(\kappa)) * G[j(\kappa), \infty)] \to M[G^*(<j(\kappa)) * G^*(j(\kappa)) * G^*[j(\kappa)^+, \infty)]$

As before we can take $G^*(<j(\kappa))$ to be $G(<j(\kappa))$.
The new concern is:

How do we choose $G^*(j(\kappa))$?

Note that we can't set $G^*(j(\kappa)) = G(j(\kappa))$ as $j(\kappa)$ is in general singular in V, so $G(j(\kappa))$ is not even defined!

Large Cardinals and L-like Universes: Forcing $\diamondsuit$

The solution is to use a minimal $j(\kappa)$ (of cofinality κ^+):

Each relevant dense D in M is of the form $j(f)(a)$ for some $f : H(\kappa) \to H(\kappa^+)$, some $a \in H(j(\kappa))$.
We choose:
$\alpha_0 < \alpha_1 < \cdots$ cofinal in $j(\kappa)$ of length κ^+
A list $f_0, f_1, \ldots$ of all relevant f's.
Then for each $i < \kappa^+$ consider the collection

$\mathcal{S}_i = \{D \mid D$ is dense and of the form $j(f_i)(a)$ for some $a \in H(\alpha_i^+)\}$

Each $\mathcal{S}_i$ has size $< j(\kappa)$ and $P^*(j(\kappa))$ is $j(\kappa)$-distributive.
Also M is κ-closed in V.
So we can build a $P^*(j(\kappa))$-generic in κ^+ steps, hitting the dense sets in $\mathcal{S}_i$ at step i.

Large Cardinals and L-like Universes: Forcing $\diamondsuit$

Preserving Hyperstrength: This is easier, as $P^*(j(\kappa))$ now equals $P(j(\kappa))$.
One only needs to guarantee that the image of $G(\kappa) * G(\kappa^+)]$ is contained in $G^*(j(\kappa)) * G^*(j(\kappa)^+)$, which is possible as this image has a greatest lower bound in the forcing $P^*(j(\kappa)) * P^*(j(\kappa)^+)$.

Preserving 2-superstrength: The new task here is to build $G^*(j^2(\kappa))$.
As observed before, for a minimal $j^2(\kappa)$, j is continuous at $j(\kappa)$; from this it follows using the $j(\kappa)$-distributivity of $P(j(\kappa))$ that the image of $G(j(\kappa))$ will in fact generate the desired generic $G^*(j^2(\kappa))$.

Forcing $\square$

$\square$ asserts that one can assign CUB subsets C_α of ordertype $< \alpha$ to singular limit ordinals α which cohere: If $\bar{\alpha}$ is a limit point of C_α then $C_{\bar{\alpha}}$ is just an initial segment of C_α.
Global $\square$ is the conjunction of two weaker properties:

$\square$ on the Singular Cardinals: This is $\square$ where C_α is only defined for singular *cardinals* α.

$\square_\kappa$ for all (uncountable cardinals) κ, where $\square_\kappa$ is $\square$ restricted to ordinals between κ and κ^+.

Forcing $\square$, preserving superstrength:
Very similar to forcing $\diamondsuit$. At regular stage α force $\square$ below α in the natural way. The main problem is to build $C(j(\kappa))$, as $j(\kappa)$ can be singular. Again the trick is to minimise $j(\kappa)$ so that it will have cofinality κ^+, enabling a construction of $C(j(\kappa))$ in κ^+ steps.

But now something unexpected happens: Solovay (later improved by Jensen) showed that $\square$ contradicts large cardinals!
A weakening of Jensen's result can be stated as follows:

Lemma

(Jensen) If κ is hyperstrong then $\square_\kappa$ fails.

Jensen's argument is essentially that if $\vec{C}$ witnesses $\square_\kappa$ and $j : V \to M$ witnesses hyperstrength, then there is a problem with the $\square_{j(\kappa)}$-sequence $j(\vec{C})$ in M at the ordinal $\alpha = \sup \pi[\kappa^+]$.

Large Cardinals and L-like Universes: Forcing $\Box$

In fact Jensen shows that $\Box_\kappa$ fails for all κ which are *subcompact*, a property weaker than hyperstrength. κ is *subcompact* iff for any $A \subseteq H(\kappa^+)$ there are $\bar\kappa < \kappa$, $\bar A \subseteq H(\bar\kappa)$ and an elementary embedding $\pi : (H(\bar\kappa^+), \bar A) \to (H(\kappa^+), A)$ with critical point $\bar\kappa$. More generally, we can define n-subcompact in the same way, with κ^+, $\bar\kappa^+$ replaced by κ^{+n}, $\bar\kappa^{+n}$.

I conjecture that Jensen's result is optimal:

Conjecture. There is a forcing that preserves n-subcompactness for all n such that in the extension, $\Box_\alpha$ holds unless α is of the form κ^{+n} where κ is $n+1$-subcompact.

Large Cardinals and L-like Universes: Forcing $\Box$

$\Box$ on the Singular Cardinals is also contradicted by large cardinals, but now the large cardinal strength is greater.
$j : V \to M$ is *inaccessibly hyperstrong* iff $H(\lambda) \subseteq M$ for some inaccessible greater than κ; we say *almost inaccessibly hyperstrong* if λ is only required to be inaccessible in M.

Theorem

(Cummings-F) (a) If κ is inaccessibly hyperstrong then $\Box$ fails on the singular cardinals below κ.
(b) One can force $\Box$ on the singular cardinals preserving almost inaccessible hyperstrength.

Forcing Morasses

The only work so far on forcing morasses in the presence of large cardinals is for the Gap 1 case.

I showed that one can do this for a single ω-superstrong and with A. Brooke-Taylor for all ω-superstrongs simultaneously.

We also force *universal* morasses, which by an observation of Donder implies the consistency of "tree-like continuous scales" at very large cardinals.

Forcing Condensation

There are different formulations of Condensation.

Club-Condensation, which holds in L, is very strong and contradicts the existence of an ω_1-Erdős cardinal.

Stationary Condensation can be forced preserving ω-superstrongs.

Better is Strong Condensation, which holds in the known core models and can also be forced preserving ω-superstrength.

But the best of all is Strong Condensation with Acceptability, which better captures the condensation properties of core models.

Peter Holy and I show that one can force this preserving ω-superstrongs; this is especially important when combined with some work of Neeman-Schimmerling:

Large Cardinals and *L*-like Universes: Condensation

(Neeman-Schimmerling) Given a Σ_1^2 indescribable 1-Gap the Proper Forcing Axiom for c^+ linked forcings holds in a proper forcing extension

The above hypothesis is between a subcompact and a 2-subcompact in strength.

(Neeman) The previous result is optimal if there is a "sufficiently *L*-like" model with a Σ_1^2 indescribable 1-Gap.

(F-Holy) One can force a "sufficiently *L*-like" model with a Σ_1^2 indescribable 1-Gap. Therefore:

(F-Holy) It is consistent with the existence of a proper class of subcompacts that the Proper Forcing Axiom for c^+ linked forcings fails in all proper set-forcing extensions.

This gives a "quasi lower bound" on the consistency strength of PFA(c^+ linked).