AN IMMUNE PARTITION OF THE ORDINALS

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A κ-partition of a set or class X is a function f from X onto κ. The sets $f^{-1}(\alpha), \alpha < \kappa$ are the cells of the κ-partition f. We shall concentrate on κ-partitions for $1 < \kappa \leq \omega$. In that case we say that f is immune if no infinite constructible set is contained in the union of fewer than κ cells. Our goal in this paper is to study the existence of immune κ-partitions of ORD = class of all ordinals.

Theorem 1. (Woodin) If $0^\#$ exists then there is an immune ω-partition of ORD. Moreover this partition is Δ_1-definable with parameter $0^\#$.

Theorem 2. If there is an immune ω-partition f and R is Cohen-generic over $L[f]$ then there is an immune 2-partition.

Theorem 3. There are immune ω-partitions and 2-partitions which are class-generic over L. They can also be taken to be definable over $L[0^\#]$ (if $0^\#$ exists).

In all cases we obtain immune 2-partitions from immune ω-partitions using Cohen forcing. We do not know if it is provable in class theory that the existence of an immune ω-partition implies that of an immune 2-partition.

Note that it is easy to obtain an immune 2-partition of X for a set X: If $\mathrm{card}(X) = \lambda$ then add a subset G of λ using finite conditions (equivalently add λ' Cohen reals, $\lambda = \omega \cdot \lambda'$). Then the characteristic function of G is an immune 2-partition. But notice that 2^ω is enlarged (if $\lambda > \omega_1$) so this is not a useful approach to Theorem 3.

The partitions f constructed in Theorem 3 have the property that GCH holds in $L[f]$. However cardinals are not preserved:

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ω_1 in $L[f] = \omega_2^L$. (L-cardinals $> \omega_1^L$ are preserved.) The reason is that we must make use of the technique used to prove the following.

Theorem 4. (Mack Stanley) There is a closed unbounded $C \subseteq ORD$ such that C is class-generic over L and $\alpha \in C \longrightarrow$ L-cofinality$(\alpha) > \omega$.

The idea for proving Theorem 3 is to add an immune ω-partition of λ for each L-cardinal λ by means of iterated forcing. However to establish the distributivity of this forcing at λ it is necessary to have a CUB subset of λ consisting of ordinals of uncountable L-cofinality. Thus we simultaneously add such CUB sets to obtain the desired partition. By using a Cohen real we can reduce this ω-partition to a 2-partition. Carrying this out in $L[0^\#]$ requires the use of the "backwards Easton" nature of the iteration.

Proof of Theorem 1. First list all L-definable terms $t_0, t_1, \ldots$ so that every $x \in L$ can be written as $t_j(i_1, \ldots, i_n)$ for some j, where $i_0 < \ldots < i_n$ belong to I = the Silver indiscernibles. For any j let $X_j = \{t_j(i_1, \ldots, i_n) | i_1 < \ldots < i_n$ in I, t_j n-ary$\} \cap ORD$ and set $f'(k) = X_{j(k)} - (X_{j(0)} \cup \ldots \cup X_{j(k-1)})$ where $j(k)$ is least so that this is nonempty. We claim that f is the desired partition where $f(\alpha)$ = least k such that $\alpha \in f'(k)$. Clearly f maps ORD onto ω. We must show that no infinite constructible set $y \subseteq ORD$ is contained in $X_{j_1} \cup \ldots \cup X_{j_k}$ for any finite $\{j_1, \ldots, j_k\} \subseteq \omega$.

We prove the last statement by induction on $\cup y$. First note that $\cup y$ cannot be less than $\alpha_0 = \min(I)$ as $t(i_1, \ldots, i_n) < \alpha_0 \longrightarrow t$ is constant on I^n and so $\alpha_0 \cap (X_{j_1} \cup \ldots \cup X_{j_k})$ is finite. Now choose a consecutive pair $i < i^*$ in I so that $i < \cup y < i^*$; this is possible as we can assume $\cup y \notin y$ and thus $\cup y$ has L-cofinality ω. We can also assume that $y \subseteq (i, i^*)$ and hence for some $\ell, m < \omega$: $\alpha \in y \longrightarrow \alpha = t(\beta, i, i_1 \ldots i_m)$ for some Σ_ℓ term t, some $\beta < i$ and $i^* \leq i_1 < \ldots < i_m$ in I. (The fact that ℓ, m exist follows from $y \subseteq X_{j_1} \cup \ldots \cup X_{j_k}$.) Now define new terms $t_1', \ldots, t_k'$ as follows: If $t_{j_h} = t_{j_h}(i_1 \ldots i_u, j_1 \ldots j_v)$ where $i_1 < \ldots < i_u \leq t_{j_h}(\vec{i}, \vec{j}) < j_1 < \ldots < j_v$ then let $t_h'(i_1, \ldots, i_u, j_1 \ldots j_v \ldots j_m)$ = least $\langle \beta, p \rangle$ such that $t_{j_h}(i_1 \ldots i_u, j_1 \ldots j_v) = t_p(\beta, i_u, j_1 \ldots j_m)$ and t_p is Σ_ℓ. Then also let $y' = \{\langle \beta, p \rangle |$ For some $\alpha \in y, \langle \beta, p \rangle$ is least so that t_p is Σ_ℓ, $\alpha = t_p(\beta, i, i_1 \ldots i_m)\}$. Thus y' is constructible and infinite since y is and moreover $\cup y' < i$. Finally note that y' is a subset of $X_{j_1'} \cup \ldots \cup X_{j_k'}$ where $t_{j_h'} = t_h'$. By induction we are done. It is clear that f is Δ_1-definable from $0^\#$. ⊣

Proof of Theorem 2. Suppose f is an immune ω-partition and define $g(\alpha) = 1$ or 0, depending on whether or not $f(\alpha)$ belongs to R (R Cohen-generic over $L[f]$). If $y \subseteq ORD$ is infinite and constructible then y must intersect infinitely many cells of f. But then y must intersect both cells of g as otherwise either R or $\bar{R}$ has an infinite subset in $L[f](y) = L[f]$, contradicting Cohen genericity. ⊣

We now turn to the main result.

Proof of Theorem 3. We use "backward Easton" forcing (see Jech [78]). By induction on $\kappa \in$ L-Card = {uncountable L-cardinals} we define a forcing P_κ. For successor κ^+, $P_{\kappa^+} = P_\kappa * Q_{\kappa^+}$, where Q_κ is defined below. For limit κ, P_κ = Direct Limit $\langle P_{\kappa'} | \kappa' < \kappa\rangle$ if κ is regular, P_κ = Inverse Limit $\langle P_{\kappa'} | \kappa' < \kappa\rangle$ otherwise.

Begin by defining $P_{\omega_1^L} = \mathrm{Coll}(\omega_1^L)$ where $\mathrm{Coll}(\omega_1^L)$ is the Lévy collapse of ω_1^L to ω with finite conditions.

Suppose $\kappa > \omega$ is a singular cardinal or a successor cardinal. We define Q_{κ^+}, assuming P_κ has been defined. First add a CUB subset of κ^+ consisting of ordinals of uncountable L-cofinality: Conditions are bounded closed subsets of κ^+ with this property. We claim that this forcing is κ^+-distributive (the intersection of κ dense open sets is dense open), in the ground model $L[G_\kappa]$ where G_κ is P_κ-generic. Suppose $\langle D_i | i < \lambda\rangle$ is a λ-sequence of dense open sets and $\lambda \le \kappa$ is regular in $L[G_\kappa]$ (without loss of generality). Suppose p is a condition and let $C \subseteq \kappa^+$ be the closed unbounded subset consisting of all $\alpha < \kappa^+$ such that $L_\alpha[G_\kappa, D] \prec L_{\kappa^+}[G_\kappa, D]$ where $D = \{(q,i) | q \in D_i\}$. Let $\alpha_0 < \alpha_1 < \ldots$ enumerate the first $\lambda + 1$ elements of C and using a CUB subset X of λ consisting of ordinals of uncountable L-cofinality, thin this to a closed subsequence $\beta_0 < \beta_1 < \ldots$ of ordertype $\lambda + 1$ also consisting of ordinals of uncountable L-cofinality. Then inductively define $p_0 = p$, p_{i+1} = least $q \le p_i$ in D_i such that $\cup q \ge \beta_i$, $p_{\lambda'} = \cup\{p_i | i < \lambda'\}$ for limit $\lambda' \le \lambda$. As we can assume that X belongs to $L_{\beta_0}[G_\kappa]$ we see that for limit $\lambda' \le \lambda$, $\cup p_{\lambda'} = \beta_{\lambda'}$. Thus $q = p_\lambda$ is the desired extension of p. Note that in case $\kappa^+ = \omega_2^L$ we must use the fact that ω_1^L is countable in $L[G_\kappa]$ to obtain X (in this case $\lambda = \omega$).

The second part of Q_{κ^+} in this case consists of adding an immune ω-partition of κ^+. The conditions are immune ω-partitions of an ordinal $\alpha < \kappa^+$. To show that conditions can be extended it suffices to produce an immune ω-partition of κ in $L[G_\kappa]$. If $\kappa = \omega_1^L$ then this is easy, using the fact that ω_1^L is countable. If $\kappa > \omega_1^L$ is a successor cardinal then this follows by induction. Finally suppose that κ is singular. Let $\kappa_0 < \kappa_1 < \ldots$ in L be a closed subsequence of κ of ordertype $\mathrm{cof}(\kappa) = \lambda$. For each $i < \lambda$ let $f_i\colon [\kappa_i, \kappa_{i+1}) \longrightarrow \omega$ be an immune ω-partition of $[\kappa_i, \kappa_{i+1})$ and let $g\colon \lambda \longrightarrow \omega$ be an immune ω-partition of λ. Define $f(\alpha) = \langle f_i(\alpha), g(i)\rangle$ where $\kappa_i \leq \alpha < \kappa_{i+1}$. Then f is a partition of κ into countably many cells. It suffices to show that no infinite constructible set y is contained in the union of finitely many of these cells. If y were a counterexample then in fact we can assume $y = y_1 \cup \ldots \cup y_n$ where $y_k \subseteq [\kappa_{i_k}, \kappa_{i_k+1})$ for some $i_k < \lambda$, since g is an immune ω-partition. But some y_k must be infinite and hence intersect infinitely many cells of f_{i_k}. So y intersects infinitely many cells of f and we are done.

We must also show that this forcing is κ^+-distributive. This argument is identical to that used for the preceding forcing: extend $p = p_0$ to $p_1 \subseteq p_2 \subseteq \ldots$ successively and arrange that for limit $\lambda' \leq \lambda$, $\mathrm{Dom}(p_{\lambda'}) = \beta_{\lambda'}$ has uncountable L-cofinality. Then it is clear that $p_{\lambda'}\colon \beta_{\lambda'} \longrightarrow \omega$ is immune as any countably infinite constructible $y \subseteq \beta_{\lambda'}$ must in fact be contained in β_i for some $i < \lambda'$. This completes the discussion of Q_{κ^+} when κ is a successor cardinal or is singular.

When κ is inaccessible we proceed exactly in the same way except first add a CUB subset C of κ consisting of ordinals of uncountable L-cofinality and an immune ω-partition of κ. The proof that these forcings are extendible and κ-distributive is as before. Then add the CUB subset of κ^+ and the immune ω-partition of κ^+ as before and use the set C to show that $\cap\{D_i \mid i < \kappa\}$ is open dense if each D_i is open dense. This completes the description of Q_{κ^+} and hence the definition of $\mathcal{P}_\kappa$ for all L-cardinals $\kappa > \omega$.

It is now easy to obtain the desired immune ω-partition of ORD. Let G be $\mathcal{P}$-generic over L where $\mathcal{P} = \text{Direct Limit } \langle \mathcal{P}_\kappa \mid \kappa \in \text{L-Card}\rangle$. The fact that G preserves cardinals $> \omega_1^L$ and the GCH follows from the "backward Easton" nature of our forcing: $\mathcal{P} \simeq \mathcal{P}(\leq\kappa) * \mathcal{P}(>\kappa)$ where $\mathcal{P}(>\kappa)$ is κ^+-distributive and $\mathrm{card}(\mathcal{P}(\leq\kappa)) = \kappa$, for regular $\kappa > \omega_1^L$. So cardinals are preserved above ω_1^L and the GCH holds. Now force over $L[G]$ with conditions $p\colon \alpha \longrightarrow \omega$ which are immune ω-partitions. Extendibility and ORD-distributivity follow as before, using the existence in $L[G]$ of immune ω-partitions of each $\alpha \in$ ORD and the existence of CUB subsets of each regular $\kappa > \omega_1^L$ consisting of ordinals of uncountable L-cofinality. This completes the proof of the first statement of Theorem 3, as to obtain an immune 2-partition of ORD we need only add a Cohen real.

We must finally argue that G can be obtained definably over $L[0^\#]$. (The final Cohen real can then be chosen in $L[0^\#]$ using the fact that $\omega_1^{L[G]} = \omega_2^L < \omega_1^{L[0^\#]}$.) This is again a consequence of the "backward Easton" nature of the forcing P. It suffices to build $H \subseteq L_{i_\omega}$ in $L[0^\#]$ which is P_{i_ω}-generic over L_{i_ω} and such that $t(j_1 \ldots j_n) \in H \leftrightarrow t(j_1' \ldots j_n) \in H$ whenever $j_1 < \ldots < j_n, j_1' < \ldots < j_n'$ belong to $I \cap i_\omega$, $i_\omega = \omega^{\text{th}}$ indiscernible. For then define $t(k_1 \ldots k_n) \in G$ iff $t(i_1 \ldots i_n) \in H$, $i_1 < \ldots < i_n$ the first n indiscernibles. This is well-defined due to the above property of H. To see that G is P-generic, choose $D = s(\ell_1 \ldots \ell_m)$ to be predense on P. Then $\bar{D} = s(i_1 \ldots i_m)$ is also predense on P and hence some $\bar{p} = t(i_1 \ldots i_n)$ belongs to G and extends an element of $\bar{D}$. By definition $p = t(\ell_1 \ldots \ell_m, \ell_{m+1} \ldots \ell_n)$ belongs to G (where $\ell_m < \ell_{m+1} < \ldots < \ell_n$ belong to I) and by indiscernibility p extends an element of D. As any L-definable open dense subclass of P contains a predense $D \in L$, we have shown that G is P-generic over L.

Now we build H. Let $H_2 \subseteq L_{i_2}$ be the $L[0^\#]$-least P_{i_2}-generic and $H_1 = H_2 \cap L_{i_1}$. We must now define $H_3 \subseteq L_{i_3}$ so as to be P_{i_3}-generic and so that $t(i_1, j_1 \ldots j_n) \in H_2$ iff $t(i_2, j_1 \ldots j_n) \in H_3$ where $i_\omega \leq j_1 < \ldots < j_n$ belong to I. Recall that a $Q_{i_1}^+$-generic consists first of a generic CUB $C_{i_1} \subseteq i_1$ consisting of ordinals of uncountable L-cofinality and also a generic ω-partition $f_{i_1} : i_1 \longrightarrow \omega$; then similar $C_{i_1^+}, f_{i_1^+}$ are added. We must define a $Q_{i_2^+}$-generic consisting of $C_{i_2}, f_{i_2}, C_{i_2^+}, f_{i_2^+}$. But notice that C_{i_1}, f_{i_1} are conditions in the forcings for adding C_{i_2}, f_{i_2} over the ground model $L[H_2]$; choose C_{i_2}, f_{i_2} so as to extend C_{i_1}, f_{i_1}. Then clearly $t(i_1, j_1 \ldots j_n)$ belongs to the generic determined by $H_1 * C_{i_1}, f_{i_1}$ iff $t(i_2, j_1, \ldots, j_n)$ belongs to the generic determined by $H_2 * C_{i_2}, f_{i_2}$ as in this case $t(i_1, j_1 \ldots j_n) \in L_{i_1}$ and so $t(i_1, j_1 \ldots j_n) = t(i_2, j_1 \ldots j_n)$. To define $C_{i_2^+}, f_{i_2^+}$ consider $K = \{t(i_2, j_1 \ldots j_n) \mid t(i_1, j_1 \ldots j_n) \in$ generic determined by $H_1 * C_{i_1}, f_{i_1}, C_{i_1^+}, f_{i_1^+}\} \subseteq P_{i_2^+} * Q_{i_2^+}$. Now for each n and $i \in I$ consider $Y_{i,n} = L_{i^*} \cap$ Skolem hull$^2(i \cup \{i, j_1 \ldots j_n\})$ where $i < j_1 < \ldots < j_n$ belong to I and $i^* = \min(I - (i+1))$. Then $L_{i^*} = \cup\{Y_{i,n} \mid n \in \omega\}$. As the forcing P_1 (P_2, respectively) which adds $C_{i_1^+}, f_{i_1^+}$ ($C_{i_2^+}, f_{i_2^+}$, respectively) is i_1+-distributive (i_2+-distributive, respectively) it follows that for each n there exists $p_n \in K$ such that $p_n = (p_n^0, p_n^1)$ where $p_n^0 \Vdash p_n^1$ meets all open dense $D \subseteq P_2$, $D \in Y_{i_2,n}[H_2 * C_{i_2}, f_{i_2}]$. Thus we obtain a $P_{i_2} * Q_{i_2}$+-generic $H_2 * G$ by defining $H_2 * G = \{p \mid \exists q \in K, q \leq p\}$. This defines $C_{i_2}+, f_{i_2}+$. But note that the same reasoning can be applied to $P(>i_1) \cap L_{i_2}, P(>i_2) \cap L_{i_3}$ to obtain a P_{i_3}-generic H_3 such that $t(i_1, j_1 \ldots j_n) \in H_2$ iff $t(i_2, j_1 \ldots j_n) \in H_3$.

Now H_4 is uniquely determined by P_{i_4}-genericity and the requirement that $t(i_1, i_2, j_1 \ldots j_n) \in H_3$ iff $t(i_2, i_3, j_1 \ldots j_n) \in H_4$, as the forcing to add C_{i_2}, f_{i_2} is i_2-distributive, $L_{i_2} = \cup\{Y_{i_1,n} \mid n \in \omega\}$ and the forcing to add $H_3(>i_2)$ is i_2+-distributive, $L_{i_3} = \cup\{Y_{i_2,n} \mid n \in \omega\}$. We must check that $t(i_1, i_3, j_1 \ldots j_n) \in H_4$ iff $t(i_2, i_3, j_1 \ldots j_n) \in H_4$. First suppose that $t(i_1, i_3, j_1 \ldots j_n) \in H_4(\leq i_3)$, so that $t = (t_0, t_1)$ where $t_0 \in P(<i_3)$ and $t_0 \Vdash t_1 \in$ Forcing to add C_{i_3}, f_{i_3}. Now note that by construction of H_3, C_{i_2}, f_{i_2} extends C_{i_1}, f_{i_1}. So by definition of H_4 we have that C_{i_3}, f_{i_3} extends C_{i_2}, f_{i_2}. Now actually $t_0 \in P(<i_2)$ so we have that $t(i_1, i_2, j_1 \ldots j_n) \in H_3(\leq i_2)$ and so $t(i_2, i_3, j_1 \ldots j_n) \in H_4$ by definition of H_4. This argument is reversible, so the equivalence is proved in this case. Now to handle the general case note that any condition in H_4 can be extended to a condition in H_4 of the form (t_0, t_1) where $t_0 \in H_4(\leq i_3)$ and $t_1 = t_1(i_3, j_1 \ldots j_n)$; to see this just note that this is true for $H_2(\leq i_1)$ and there is by definition an elementary embedding $H_2 \longrightarrow H_4$ sending i_1 to i_3. So it is enough to consider such (t_0, t_1). But we have already shown that $t_0(i_1, i_3, j_1 \ldots j_n) \in H_4(\leq i_3)$ iff $t_0(i_2, i_3, j_1 \ldots j_n) \in H_4(\leq i_3)$ and as t_1 does not mention i_1, i_2 we are done.

In general define H_{m+3} by the condition $t(i_m, i_{m+1}, j_1 \ldots j_n) \in H_{m+2}$ iff $t(i_{m+1}, i_{m+2}, j_1 \ldots j_n) \in H_{m+3}$. This uniquely determines H_{m+3} as a $P_{i_{m+3}}$-generic set. As in the preceding argument we can show that $t(i_1, \ldots, i_{m+1}, j_1 \ldots j_n) \in H_{m+2}$ iff $t(i_1, \ldots, i_m, i_{m+2}, j_1 \ldots j_n) \in H_{m+3}$. Finally let H = Direct Limit $\{H_m \mid m < \omega\}$. Then H is P_{i_ω}-generic. We have arranged that for any $k_1 < \ldots < k_{\ell+2} < j_1 < \ldots < j_n$ in $I \cap i_\omega$ that $t(k_1, \ldots, k_\ell, k_{\ell+1}, j_1 \ldots j_n) \in H$ iff $t(k_1, \ldots, k_\ell, k_{\ell+2}, j_1 \ldots j_n) \in H$. But now it is easy to argue that for any $k_1 < \ldots < k_\ell < j_1 < \ldots < j_\ell$ that $t(k_1, \ldots, k_\ell) \in H \leftrightarrow t(j_1, \ldots, j_\ell) \in H$ by applying the preceding ℓ times. We have constructed the desired H and thus completed the proof of Theorem 3. ———┤

Remark Mack Stanley pointed out to me that there is a simpler construction for producing G inside $L[0^\#]$: Just define $G(<i)$ by induc-

tion on $i \in I$ so that C_i, f_i extends C_j, f_j for $j \in I \cap i$. It appears though that the above is necessary to control the indiscernibles relative to G. Thus for example one can then code G by a real R in $L[0^\#], I^R = I$.

Some Open Questions

(a) The obvious question is if Theorem 3 can be proved using a forcing which preserves cardinals. Some progress was made by Shelah, who showed that an immune 2-partition of $\aleph_\omega$ can be obtained by a cardinal-preserving forcing.

(b) Clearly immune 2-partitions yield immune ω-partitions. How about the converse?

(c) Clearly immune k-partitions, $2 \leq k < \omega$ yield immune 2-partitions. But what if we weaken immunity to say that no infinite constructible set is contained in just one cell? Then does the existence of a weakly immune k-partition imply that of an immune 2-partition?

Reference

Jech [78] Set Theory, Academic Press, 1978.