Large cardinals and locally defined well–orders of the universe

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Abstract

By forcing over a model of $ZFC + GCH$ (above $\aleph_0$) with a class–sized partial order preserving this theory we produce a model in which there is a locally defined well–order of the universe; that is, one whose restriction to all levels $H(\kappa^+)$ ($\kappa \geq \omega_2$ a regular cardinal) is a well–order of $H(\kappa^+)$ definable over the structure $\langle H(\kappa^+), \in\rangle$ by a parameter–free formula. Further, this forcing construction preserves all supercompact cardinals as well as all instances of regular local supercompactness. It is also possible to define variants of this construction which, in addition to forcing a locally defined well–order of the universe, preserve many of the n–huge cardinals from the ground model (for all n).

1 Introduction and statement of the main result

This article is a contribution to the *outer model programme*, whose aim is to show that large cardinal properties can be preserved when forcing desirable features of Gödel's constructible universe. The properties GCH, $\diamondsuit$, $\square$ and gap-1 morass were discussed in [C-S] and [F2]. Globally defined well–orders were considered in [B]. In this article we consider *locally defined* well-orders in the sense of the following theorem.

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Theorem 1.1 *(GCH above $\aleph_0$) There is a formula $\varphi(x, y)$ without parameters and there is a definable ω_2–directed closed class–sized partial order $\mathcal{P}$ preserving ZFC, GCH above $\aleph_0$ and cofinalities, and such that*

(1) $\mathcal{P}$ forces that there is a well–order $\leq$ of the universe such that

$$\{(a, b) \in H(\kappa^+)^2 : \langle H(\kappa^+), \in\rangle \models \varphi(a, b)\}$$

is the restriction $\leq \restriction H(\kappa^+)^2$ and is a well–order of $H(\kappa^+)$ whenever $\kappa \geq \omega_2$ is a regular cardinal, and

(2) for all regular cardinals $\kappa \leq \lambda$, if κ is a λ–supercompact cardinal in V, then κ remains λ–supercompact after forcing with $\mathcal{P}$.

The reason why $\kappa = \omega_1$ has been excluded from the formulation of this theorem is that we do not know how to force over a model of GCH a well–order of $H(\omega_2)$, definable over $\langle H(\omega_2), \in\rangle$ from no parameters, without collapsing any cardinals and preserving GCH.[1] By results in [A-S], such a well–order of $H(\omega_2)$ can be added without collapsing cardinals over any model of $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$, but $2^{\aleph_0} = \aleph_2$ holds in the extension. On the other hand, if there is an inaccessible cardinal κ, then by [As] we can add such a definable well–order while preserving GCH. However, in the extension κ becomes ω_2. The definition, over $\langle H(\omega_2), \in\rangle$, of the resulting well–order in the result from [As] is similar, but not identical, to the definition over $\langle H(\kappa^+), \in\rangle$ of the well–order of $H(\kappa^+)$, for regular $\kappa \geq \omega_2$, that we are going to force here.[2]

There are some limitations to extending Theorem 1.1. to the context of very strong large cardinal assumptions: Suppose $j : L(V_{\lambda+1}) \longrightarrow L(V_{\lambda+1})$ is a nontrivial elementary embedding, for some ordinal λ, with critical point below λ. Then there is no well–order $\leq$ of $H(\lambda^+)$ definable over $\langle H(\lambda^+), \in\rangle$ from parameters. Otherwise, since $H(\lambda) = V_\lambda$, $H(\lambda^+)$ can be computed in $L(V_{\lambda+1})$, which implies that in $L(V_{\lambda+1})$ there is a well–order of $V_{\lambda+1}$. This contradicts the fact – which follows from Kunen's proof ([Ku1]) of the non-existence of a nontrivial elementary embedding $j : V \longrightarrow V$ if ZFC holds

[1] However, by results of Friedman it is possible to force over any GCH–model with a cofinality–preserving and GCH–preserving poset of size $\aleph_2$ adding a well–order of $H(\omega_2)$ which is definable over $\langle H(\omega_2), \in\rangle$ from some parameter.

[2] The well–order forced in [A-S] has a completely different definition.

– that the Axiom of Choice necessarily fails in $L(V_{\lambda+1})$ under the present assumption.[3]

There are also limitations towards extending Theorem 1.1 in the direction of requiring the co-existence of large cardinals with certain other types of objects[4] $\mathcal{W} \subseteq H(\kappa^+)$ definable over $\langle H(\kappa^+), \in\rangle$. For example, it is not possible to have a κ–complete non-principal ultrafilter $\mathcal{U}$ on a measurable cardinal κ such that $\mathcal{U}$ is definable over $\langle H(\kappa^+), \in\rangle$ (even allowing parameters). Otherwise, since $H(\kappa^+)^{Ult(V,\mathcal{U})} = H(\kappa^+)^V$, $\mathcal{U}$ would be a member of the ultrapower $Ult(V, \mathcal{U})$, which is impossible.

In a related direction, note that if Projective Determinacy holds,[5] then there can be no well–order of $H(\omega_1)$ definable over $\langle H(\omega_1), \in\rangle$, even allowing parameters, as the restriction of such a well–order to $\mathbb{R}$ would yield a projective well–order of $\mathbb{R}$.

In the rest of this section we will fix some pieces of notation and definitions.

Given an infinite cardinal α, a partial order $\mathcal{P}$ is α–distributive if it does not add new sequences of ordinals of length less than α, and it has the stronger property of being α–directed closed in case every directed set $X \subseteq \mathcal{P}$ of size less than α has a lower bound in $\mathcal{P}$.[6]

We will mostly, but not always, use the standard notation from [Ku2] in contexts of forcing. In particular, given a partial order $\mathcal{P}$, a $\mathcal{P}$–name τ and a $\mathcal{P}$–generic filter G, the interpretation of τ by G will be denoted by τ_G or also by $(\tau)_G$.

Suppose $\mathcal{P}$ is a partial order and $\dot{X}$ is a $\mathcal{P}$–name for a subset of some ordinal α. We will say that $\dot{X}$ is a nice $\mathcal{P}$–name (for a subset of α) in case it consists of pairs of the form $\langle p, \check{\xi}\rangle$, with $p \in \mathcal{P}$ and $\check{\xi}$ the canonical name for an ordinal $\xi \in \alpha$. When dealing with set–forcing, the following slightly nonstandard notion of two–step iteration will simplify several statements and parts of proofs: Suppose $\mathcal{P}$ is a poset. In an AC–context, if $\dot{\mathcal{Q}}_0$ is a $\mathcal{P}$–name for a poset, then it is clear that, for some ordinal α, the two–step iteration $\mathcal{P} * \dot{\mathcal{Q}}_0$ (in the standard sense) is isomorphic to one of the form $\mathcal{P} * \dot{\mathcal{Q}}$ in which $\dot{\mathcal{Q}}$ is forced to consist of subsets of α. And furthermore, it is clear that this second iteration has a dense suborder consisting of pairs $\langle p, \dot{q}\rangle$ such that $\dot{q}$ is a nice $\mathcal{P}$–name for a subset of α. When $\dot{\mathcal{Q}}$ is a $\mathcal{P}$–name for a subset of

[3] Note, however, that λ is singular.

[4] Other than well–orders.

[5] Which follows from the existence of infinitely many Woodin cardinals ([M-St]).

[6] $X \subseteq \mathcal{P}$ is directed if every finite subset of X has a lower bound in X.

some ordinal α, we will define the two–step iteration $\mathcal{P} * \dot{\mathcal{Q}}$ as the suborder of the corresponding two–step iteration, taken in the standard sense, consisting precisely of the pairs $\langle p, \dot{q}\rangle$ such that $\dot{q}$ is a nice $\mathcal{P}$–name for a subset of α. The above remark shows that we do not lose any generality by doing so. When $\dot{\mathcal{Q}}$ denotes the definition of a class-forcing in $V^{\mathcal{P}}$, the expression $\mathcal{P} * \dot{\mathcal{Q}}$ will retain the standard meaning.

Suppose $\mathcal{P}$ is a partial order, G is a generic filter for $\mathcal{P}$ over a ground model V and H is a $\dot{\mathcal{Q}}_G$–generic filter over $V[G]$. Then $G * H$ denotes the generic filter over V for $\mathcal{P} * \dot{\mathcal{Q}}$ consisting of all $\mathcal{P} * \dot{\mathcal{Q}}$–conditions $\langle p, \dot{q}\rangle$ such that $p \in G$ and $\dot{q}_G \in H$. Also, $\dot{G}$ denotes the canonical $\mathcal{P}$–name such that $(\dot{G})_G = G$ for every generic filter G for $\mathcal{P}$.

In the context of a forcing iteration $\langle \mathcal{P}_\xi : \xi \leq \lambda\rangle$, if G is a generic filter for $\mathcal{P}_\lambda$ and $\xi \leq \lambda$, then $G_\xi = \{p \restriction \xi : p \in G\}$. If in fact $\xi < \lambda$, then $G(\xi) = \{p(\xi)_{G_\xi} : p \in G\}$. Given $p \in \mathcal{P}_\lambda$, the support of p is the set of $\xi < \lambda$ such that $p \restriction \xi$ does not force that $p(\xi)$ is the weakest condition. It will be denoted here by *supp*(p).

Also, in an abuse of notation we will sometimes assume $\mathcal{P}_\xi \subseteq \mathcal{P}_{\xi'}$ (and consequently $G_\xi \subseteq G_{\xi'}$ when referring to the corresponding generic filters) if $\xi < \xi' \leq \lambda$. $\dot{G}_\xi$ will denote the canonical $\mathcal{P}_\xi$–name for the corresponding generic object.

If $\xi < \lambda$ and G is a $\mathcal{P}_\xi$–generic filter over V, then $\mathcal{P}_\lambda / G$ denotes, in $V[G]$, the quotient forcing $\{q \restriction [\xi, \lambda) : q \in \mathcal{P}_\lambda \text{ and } q \restriction \xi \in G\}$, where $[\xi, \lambda)$ denotes the interval of ordinals α with $\xi \leq \alpha < \lambda$, ordered by setting $q_1 \restriction [\xi, \lambda) \leq q_0 \restriction [\xi, \lambda)$ iff there is some $q \in G$ such that $q \cup q_1 \restriction [\xi, \lambda) \leq q \cup q_0 \restriction [\xi, \lambda)$. Of course $q \cup q_i \restriction [\xi, \lambda)$ denotes the $\mathcal{P}_\lambda$–condition r whose restriction to $\mathcal{P}_\xi$ is q and such that $r \restriction \zeta \Vdash_\zeta r(\zeta) = q_i(\zeta)$ for all $\zeta < \lambda$, $\zeta \geq \xi$.

Finally, if $\langle \dot{\mathcal{Q}}_\xi : \xi < \lambda\rangle$ is a sequence of names on which the iteration $\langle \mathcal{P}_\xi : \xi \leq \lambda\rangle$ is built,[7] then we may identify the $\mathcal{P}_0$–name $\dot{\mathcal{Q}}_0$ with the set $\mathcal{Q}_0$ of q such that $\{\emptyset\} \Vdash_0 \check{q} \in \dot{\mathcal{Q}}_0$.[8]

A *reverse Easton iteration* is any forcing iteration which has been built taking direct limits at regular stages and taking inverse limits everywhere else.

The following easy general fact will be useful.

Lemma 1.2 *Let α be a regular cardinal, let λ be Ord or a member of it, let $\xi_0 \in \lambda$, and let $\langle \mathcal{P}_\xi : \xi \leq \lambda\rangle$ be a forcing iteration, based on a sequence*

[7]That is, each $\dot{\mathcal{Q}}_\xi$ is a $\mathcal{P}_\xi$–name for a poset and $\mathcal{P}_{\xi+1} = \mathcal{P}_\xi * \dot{\mathcal{Q}}_\xi$.

[8]$\mathcal{P}_0 = \{\emptyset\}$ by convention.

$\langle \dot{\mathcal{Q}}_\xi : \xi < \lambda \rangle$ of names such that, for every $\xi \in (\xi_0, \lambda)$, $\dot{\mathcal{Q}}_\xi$ is α–directed closed in $V^{\mathcal{P}_\xi}$. Suppose that $\mathcal{P}_{\xi_0}$ has the α–chain condition. Suppose also that $\{supp(q)\backslash \xi_0 : q \in \mathcal{P}_\lambda\}$ is closed under unions of $\subseteq$–increasing sequences of length less than α.

Then, $\mathcal{P}_\lambda/G$ is α–directed closed in $V[G]$ for every $\mathcal{P}_{\xi_0}$–generic filter G over V. In fact, in $V[G]$ it holds that for every directed $X \subseteq \mathcal{P}_\lambda/G$ of size less than α there is a condition $q \in \mathcal{P}_\lambda/G$ extending all conditions in X and such that $supp(q) = \bigcup\{supp(p) : p \in X\}$.

The following hereditary notion of internal approachability will be useful in the proof of Lemma 2.2 in the next section. Suppose θ is an infinite cardinal and Δ is a well–order of $H(\theta)$. We can define by recursion the notion of being a *hereditarily internally approachable* (HIA) elementary substructure of $\langle H(\theta), \in, \Delta\rangle$ by saying that $N \preccurlyeq \langle H(\theta), \in, \Delta\rangle$ has this property in case $N = \bigcup_{i<cf(|N|)} N_i$ for a $\subseteq$–continuous $\in$–chain $(N_i)_{i<cf(|N|)}$ of sets of size less than $|N|$ such that N_i is an HIA elementary substructure of $\langle H(\theta), \in, \Delta\rangle$ whenever N_i is infinite and i is either 0 or a successor ordinal. It is easy to see that the set of HIA elementary substructures of $\langle H(\theta), \in, \Delta\rangle$ of size μ is a stationary subset of $[H(\theta)]^\mu$ for every infinite cardinal $\mu \leq |H(\theta)|$.

Let λ be a cardinal and let $\mathcal{P}$ momentarily denote either $\mathcal{P}_\kappa(\lambda)$ (for some $\kappa \leq \lambda$) or $\mathcal{P}(\lambda)$. Recall that an ultrafilter $\mathcal{U}$ on $\mathcal{P}$ is fine if the set of $x \in \mathcal{P}$ such that $\alpha \in x$ is in $\mathcal{U}$ for all $\alpha \in \lambda$, and that it is normal if the diagonal intersection $\Delta_{\alpha<\lambda} X_\alpha$ is in $\mathcal{U}$ whenever $(X_\alpha)_{\alpha<\lambda}$ is a sequence of members of $\mathcal{U}$, where $\Delta_{\alpha<\lambda} X_\alpha = \{x \in \mathcal{P} : x \in X_\alpha \text{ for all } \alpha \in x\}$.

Recall that a cardinal κ is λ–supercompact if and only if there is an elementary embedding $j : V \longrightarrow M$ with critical point κ and with M closed under λ–sequences (${}^\lambda M \subseteq M$); equivalently, κ is λ–supercompact if and only if there is a fine and normal κ–complete ultrafilter on $\mathcal{P}_\kappa(\lambda)$. κ is λ–compact if and only if the weaker condition holds that there is a fine κ–complete ultrafilter on $\mathcal{P}_\kappa(\lambda)$. Also, given a positive integer n, a cardinal κ is n–huge if and only if there is an elementary embedding $j : V \longrightarrow M$ with ${}^{\kappa_n} M \subseteq M$, where $(\kappa_i)_{i\leq n}$ is defined by $\kappa_0 = crit(j)$ and by $\kappa_{i+1} = j(\kappa_i)$ for all $i < n$; equivalently, κ is n–huge if and only if there are cardinals $\kappa = \lambda_0 < \lambda_1 < \ldots \lambda_n$ and there is a fine and normal κ–complete ultrafilter on $\mathcal{P}(\lambda_n)$ such that $\{x \in \mathcal{P}(\lambda_n) : ot(x \cap \lambda_{i+1}) = \lambda_i \text{ for all } i < n\} \in \mathcal{U}$. The reader may find it useful to consult [K] for other standard facts in large cardinal theory.

Finally, if $\mathcal{X}$ is a definable class in V and $j : V \longrightarrow M$ is an elementary embedding, then $j(\mathcal{X})$ denotes the class in M defined over $\langle M, \in\rangle$ by $\varphi(j(p), x)$ whenever $p \in V$ and $\varphi(p, x)$ is a formula defining $\mathcal{X}$ over $\langle V, \in\rangle$. $j(\mathcal{X})$ is well–defined thanks to the elementarity of j.

The rest of the paper is structured as follows: In Section 2 we review several aspects of the poset from [As] for adding a definable, over $\langle H(\kappa^+), \in\rangle$, well–order of $H(\kappa^+)$ ($\kappa \geq \omega_2$ a regular cardinal). Instances of this forcing construction will be the main building blocks of the forcing $\mathcal{P}$ for Theorem 1.1. In Section 3 we define $\mathcal{P}$ and prove Theorem 1.1 except for (2). The large cardinal preservation part of the proof is done in Section 4. First we prove a somewhat general lifting theorem, and then we use this theorem to prove preservation of all relevant instances of supercompactness by $\mathcal{P}$. We finish this section with a minor result on preservation of λ–compactness for λ singular. Finally, in Section 5 we define a variant of $\mathcal{P}$ suited for preserving many instances of n–hugeness and list some open questions.

2 The one–step construction

The forcing notion witnessing Theorem 1.1 will be a two–step iteration $\mathcal{B} * \dot{\mathcal{C}}$ of class forcings. $\mathcal{B}$ will be an Ord–length forcing iteration adding a system of "bookkeeping functions" for all $H(\alpha^+)$ ($\alpha \geq \omega_2$ a regular cardinal). $\dot{\mathcal{C}}$, which will add the required well–order of the universe, will be a certain forcing iteration, also of length Ord, built by always using the same type of forcing. The building blocks of $\dot{\mathcal{C}}$, which here we will call $(Code^*)^{f,\mathcal{W}}_\kappa$, where $f : \kappa^+ \longrightarrow H(\kappa^+)$ is a bookkeeping function[9] for $H(\kappa^+)$ and $\mathcal{W}$ is the union of all previously added well–orders, are essentially the posets from [As] which force, for a given regular cardinal $\kappa \geq \omega_2$, a definable (over $\langle H(\kappa^+), \in\rangle$) well–order of $H(\kappa^+)$ without parameters. Each $(Code^*)^{f,\mathcal{W}}_\kappa$ will be a dense suborder of the direct limit $Code^{f,\mathcal{W}}_\kappa$ of a certain forcing iteration. The proof of Theorem 1.1 will be best presented after describing in some detail these posets $Code^{f,\mathcal{W}}_\kappa$, $(Code^*)^{f,\mathcal{W}}_\kappa$ and the coding they force.[10] The reader is referred to [As] for all missing proofs.

[9] f will be defined from the generic object for $\mathcal{B}$.

[10] Many of the forthcoming technicalities are (quite) inessential to the present construction. One reason we are presenting them here is to convince the reader that they are indeed inessential.

2.1 The coding

For this section, let us fix a regular cardinal $\kappa \geq \omega_2$ and let us assume both $2^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$. The following notions are involved in the coding of the well–orders we are going to construct. They appear in [As] with some changes in notation.

A *club–sequence on* κ is a sequence of the form $\vec{C} = \langle C_\alpha : \alpha \in S\rangle$, for some set $S \subseteq \kappa$ with supremum κ, such that C_α is a club of α for every $\alpha \in S$. We will call S the *domain of* $\vec{C}$, which we may also denote by $dom(\vec{C})$. $range(\vec{C})$ will denote $\bigcup\{C_\delta : \delta \in dom(\vec{C})\}$. A club–sequence $\vec{C}$ on κ is *coherent* if it can be extended to a coherent club–sequence in the usual sense; that is, if there is a club–sequence $\vec{C}'$ on κ such that $dom(\vec{C}') \supseteq dom(\vec{C})$ and $C_\delta = C'_\delta$ for every $\delta \in dom(\vec{C})$ and such that $\gamma \in dom(\vec{C}')$ and $C'_\gamma = C'_\delta \cap \gamma$ whenever $\delta \in dom(\vec{C}')$ and γ is a limit point of C'_δ. Given an ordinal τ, a club–sequence $\vec{C}$ will be said to have *height* τ if C_δ has order type τ for every $\delta \in dom(\vec{C})$. In case it exists, we will denote the height of $\vec{C}$ by $ht(\vec{C})$.

Given two sets of ordinals, X and Y, we can define the (non–symmetric) operation $X \cap^* Y = \{\alpha \in X \cap Y : \alpha \text{ is not a limit point of } X\}$. A club–sequence $\vec{C}$ on κ is *type–guessing* in case for every club $D \subseteq \kappa$ there is some $\delta \in dom(\vec{C}) \cap D$ such that $ot(C_\delta \cap^* D) = ot(C_\delta)$. The following strong form of this weak guessing principle for club–sequences is central in the coding from [As] (and also here): A club–sequence $\vec{C}$ is *strongly type–guessing* if for every club $D \subseteq \kappa$ there is a club $D' \subseteq \kappa$ with $ot(C_\delta \cap^* D) = ot(C_\delta)$ for all $\delta \in D' \cap dom(\vec{C})$.

Another technical notion occurring in the coding is that of perfect ordinal: Given a set of ordinals X and an ordinal δ, the *Cantor–Bendixson rank of* δ *with respect to* X, $rnk_X(\delta)$, is defined by specifying that $rnk_X(\delta) = 0$ if and only if δ is not a limit point of X and, for each ordinal $\eta > 1$, that $rnk_X(\delta) > \eta$ if and only if δ is a limit ordinal and there is a sequence $(\delta_\xi)_{\xi < ot(\delta)}$ converging to δ such that $rnk_X(\delta_\xi) \geq \eta$ for every ξ. An ordinal δ will be said to be *perfect* if $rnk_\delta(\delta) = \delta$.[11] Note that $rnk_\delta(\delta) \leq \delta$ for every ordinal δ and that, given any uncountable cardinal μ, the set of perfect ordinals below μ forms a club of μ of order type μ. We will let $(\eta_\xi)_{\xi<\kappa}$ denote the strictly increasing enumeration of all perfect ordinals below κ of countable cofinality.[12]

[11] Thus, with this definition, the first perfect ordinal is 0 and the second is $\epsilon_0 = sup\{\omega, \omega^\omega, \omega^{(\omega^\omega)}, \omega^{(\omega^{(\omega^\omega)})}, \ldots\}$.

[12] $(\eta_\xi)_{\xi<\kappa}$ is obviously definable without parameters over $\langle H(\kappa^+), \in\rangle$.

Let F be a function from κ into $\mathcal{P}(\kappa)$, and let $\mathcal{S} = \langle S_i : i < \kappa \rangle$ be a sequence of pairwise disjoint stationary subsets of κ. Given $B \subseteq \kappa$ and an ordinal $\delta < \kappa^+$, we will say that δ *codes* B *with respect to* F *and* $\mathcal{S}$ if there is a club $E \subseteq \mathcal{P}_\kappa(\delta)$ such that for every $X \in E$ and every $i < \kappa$, if $X \cap \kappa \in S_i$, then $ot(X) \in F(X \cap \kappa)$ if and only if $i \in B$.

It is easy to see, for F and $\mathcal{S}$ as before, that every ordinal in κ^+ can code at most one subset of κ.

Let us fix some canonical and definable way of coding members of $H(\kappa^+)$ by subsets of κ.[13] For technical reasons, this coding should have the property that if $F \in H(\kappa^+)$ is a function with domain κ, $\langle S_i : i < \kappa \rangle$ is a sequence of subsets of κ, $A \subseteq \kappa$ is a set coding $(F, \mathcal{S})$, and $\delta \in \kappa$, then the object $(A \cap \delta, \delta)$ codes (partial) information only about $F \restriction \delta$ and about $\langle S_i \cap \delta : i < \delta \rangle$.

Finally, given a set $B \subseteq \kappa$ and an ordinal $\delta < \kappa^+$, we will say that δ *codes* B if the following holds:

Let I_κ be the set of all ordinals $\xi < \kappa$ for which there exists a strongly type–guessing coherent club–sequence on κ with stationary domain and of height η_ξ. Then I_κ codes a pair $(F, \mathcal{S})$ such that F is a function from κ into $\mathcal{P}_\kappa(\kappa)$ and $\mathcal{S} = \langle S_i : i < \kappa \rangle$ is a sequence of pairwise disjoint stationary subsets of κ, and δ codes B with respect to F and $\mathcal{S}$.

2.2 $Code_\kappa^{f,\mathcal{W}}$ and $(Code^*)_\kappa^{f,\mathcal{W}}$

Let us say that $f : \kappa^+ \longrightarrow H(\kappa^+)$ is a *bookkeeping function for* $H(\kappa^+)$ if $f^{-1}(a)$ is unbounded in κ^+ for every $a \in H(\kappa^+)$.[14]

Now, let f be a bookkeeping function for $H(\kappa^+)$ and suppose $\mathcal{W}$ is a well–order of a subset of $H(\kappa^+)$. $Code_\kappa^{f,\mathcal{W}}$ is then a natural forcing aimed at the following four tasks:

(1) Adding a function $F : \kappa \longrightarrow \mathcal{P}_\kappa(\kappa)$ and a sequence $\mathcal{S} = \langle S_i : i < \kappa \rangle$ of pairwise disjoint stationary subsets of κ with $cf(\delta) > \omega$ for all $i < \kappa$ and all $\delta \in S_i$.

(2) Making sure that every subset of κ is coded by some ordinal in κ^+ with respect to F and $\mathcal{S}$.

[13] The phrase 'A codes X', where A is a subset of κ (but not an ordinal) and $X \in H(\kappa^+)$, will be understood to refer to this fixed coding.

[14] The existence of a bookkeeping function for $H(\kappa^+)$ certainly follows from $2^\kappa = \kappa^+$.

(3) Making sure that $\mathcal{W}$ is an initial segment of the well–order of $H(\kappa^+)$ consisting of all pairs $\langle a, b\rangle$ such that the first ordinal coding, with respect to F and $\mathcal{S}$, a subset of κ coding a is less than the first ordinal coding, with respect to F and $\mathcal{S}$, a subset of κ coding b.

(4) Making sure that the set I_κ defined in Subsection 2.1 codes $(F, \mathcal{S})$.

Thus, $Code_\kappa^{f,\mathcal{W}}$ will not only force that every subset of κ is coded by some ordinal in κ^+ with respect to F and $\mathcal{S}$, for some pair $(F, \mathcal{S})$ added generically (and for a choice of ordinals such that the prescribed relation $\mathcal{W}$ is an initial segment of the corresponding well–order of $H(\kappa^+)$), but will also make the decoding device $(F, \mathcal{S})$ definable from no parameters. It is clear then that in the extension by $Code_\kappa^{f,\mathcal{W}}$ there is a well–order of $H(\kappa^+)$ extending $\mathcal{W}$ and definable over $\langle H(\kappa^+), \in\rangle$ by a formula without parameters.

$Code_\kappa^{f,\mathcal{W}}$ will be the limit $\mathcal{Q}_{\kappa^+}$ of a certain $< \kappa$–support forcing iteration $\langle \mathcal{Q}_\xi : \xi \leq \kappa^+\rangle$. The sequence $\langle \dot{\mathcal{R}}_\xi : \xi < \kappa^+\rangle$ of names on which this iteration will be built is chosen in the following way:

In a first step, we force with a natural forcing $\widetilde{\mathcal{P}_0}$ for adding, by initial segments, a function $F : \kappa \longrightarrow \mathcal{P}_\kappa(\kappa)$ and a sequence $\mathcal{S} = \langle S_i : i < \kappa\rangle$ of mutually disjoint sets of ordinals in κ of uncountable cofinality. By a standard density argument, each S_i is forced to be a stationary subset of κ, and so is $\kappa \setminus \bigcup_{i<\kappa} S_i$. The second step is to pick a subset I of κ coding the parameter $(F, \mathcal{S})$, and to force with a natural forcing $\widetilde{\mathcal{P}_1}$ for adding, also by initial segments, a sequence $(\vec{C}^\nu)_{\nu<\kappa}$ of coherent club–sequences on κ with $(dom(\vec{C}^\nu))_{\nu<\kappa}$ a sequence of mutually disjoint sets such that $dom(\vec{C}^\nu) \cap (\nu + 1) = \emptyset$ for all ν, and such that $(ht(\vec{C}^\nu))_{\nu<\kappa}$ is $(\eta_{\xi_\nu})_{\nu<\kappa}$ for the strictly increasing enumeration $(\xi_\nu)_{\nu<\kappa}$ of I. Moreover we make sure that the intersection $(\bigcup_{i<\kappa} S_i) \cap (\bigcup_{\nu<\kappa} range(\vec{C}^\nu))$ is empty (for technical reasons).[15]

We can construe both $\widetilde{\mathcal{P}_0}$ and $\widetilde{\mathcal{P}_1}$ as posets consisting of functions with domain some ordinal in κ. It will be convenient to let $\dot{\mathcal{R}}_0$ be the suborder of $\widetilde{\mathcal{P}_0} * \dot{\widetilde{\mathcal{P}_1}}$ consisting of all pairs $(p, \dot{q})$ such that, for some ordinal $\alpha < \kappa$ and some sequence x of length α, $dom(p) = \alpha$ and $\widetilde{\mathcal{P}_0} \restriction p$ forces $\dot{q} = \check{x}$. Using the fact that $\widetilde{\mathcal{P}_0}$ is κ–closed it is not difficult to see that this set of pairs is dense in $\widetilde{\mathcal{P}_0} * \dot{\widetilde{\mathcal{P}_1}}$.

By another density argument, each $\vec{C}^\nu$ is forced to be type–guessing (which in particular implies that $\vec{C}^\nu$ has stationary domain).

[15]Since $cf(ht(\vec{C}^\nu)) = \omega$ for all ν, $(\bigcup_{i<\kappa} S_i) \cap (\bigcup_{\nu<\kappa} dom(\vec{C}^\nu))$ will be empty as well.

All subsequent $\dot{\mathcal{R}}_\xi$ will be chosen to be a $\mathcal{Q}_\xi$–name for either

(A) a natural forcing for shooting a club by initial segments through the set

$$\{\delta < \kappa \,:\, (\forall \nu)(\delta \in dom(C^\nu) \rightarrow ot(C^\nu_\delta \cap^* \dot{D}_\xi) = ot(C^\nu_\delta))\}$$

(for some $\mathcal{Q}_\xi$–name $\dot{D}_\xi$ for a club subset of κ), or

(B) a natural forcing for shooting a club, also by initial segments, through

$$\{X \in \mathcal{P}_\kappa(\delta_\xi) \,:\, (\forall i < \kappa)(X \cap \kappa \in S_i \rightarrow (ot(X) \in F(X \cap \kappa) \text{ iff } i \in \dot{B}_\xi))\},$$

where δ_ξ and $\dot{B}_\xi$ are, respectively, an ordinal less than κ^+ and a $\mathcal{Q}_\xi$–name for a subset of κ.

The names $\dot{D}_\xi$ and $\dot{B}_\xi$ (and the ordinals δ_ξ) will be chosen according to the bookkeeping function f, making sure that the requirement expressed in task 3 is respected.[16] Further, we make sure that all δ_ξ's (for ξ at which we force with a forcing of the second type) are distinct[17] and that, for all $\zeta < \kappa^+$, each $\mathcal{Q}_\zeta$–name in $H(\kappa^+)$ for a subset of κ (resp., for a club of κ) is picked as $\dot{D}_\xi$ (resp., as $\dot{B}_\xi$) for some $\xi \geq \zeta$.

Given any f and $\mathcal{W}$, let $(Code^*)^{f,\mathcal{W}}_\kappa$ denote the collection of all $q \in Code^{f,\mathcal{W}}_\kappa$ for which there is some successor ordinal $\alpha + 1 < \kappa$, which for simplicity we will represent by $D(q)$, and some sequence $\langle x_\xi \,:\, \xi \in supp(q)\rangle$ such that $dom(q \restriction 1) = \alpha + 1$ and such that, for every $\xi \in supp(q)$, $\xi > 0$, x_ξ is a sequence of length $\alpha + 1$ with $x_\xi(\alpha) = \alpha$ (if $\dot{\mathcal{R}}_\xi$ falls under case (A)) or $x_\xi(\alpha) \cap \kappa = \alpha$ (if $\dot{\mathcal{R}}_\xi$ falls under case (B)) such that $q \restriction \xi$ forces $q(\xi) = \check{x}_\xi$.

For the rest of the section, let us denote for simplicity $Code^{f,\mathcal{W}}_\kappa$ and $(Code^*)^{f,\mathcal{W}}_\kappa$ by, respectively, *Code* and *Code**. We will use the fact that the suborder *Code** is dense in *Code* (Lemma 2.2) to argue that *Code* is κ–distributive (since *Code** is κ–directed closed by Lemma 2.1) and has the κ^+–chain condition.[18]

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[16] That is, making sure that for all $\langle a, b\rangle \in \mathcal{W}$, the first δ_ξ for which, at some stage, we apply a forcing as in (B) for δ_ξ and for a name $\dot{B}_\xi$ for some subset of κ coding a is less than the first δ_ξ for which we force at some stage with a forcing as in (B) for δ_ξ and for a name $\dot{B}_\xi$ for some subset of κ coding b.

[17] In fact we can make sure that the set of all such δ_ξ's is precisely the interval $[\kappa, \kappa^+)$.

[18] For the chain condition we also use $\kappa^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$.

Lemma 2.1 *$Code^*$ is κ–directed closed. In fact, if $\mathcal{D}$ is a directed subset of $\mathcal{Q}_1 \cap Code^*$ of size less than κ, then there is some $p^* \in \mathcal{Q}_1 \cap Code^*$ extending all conditions in $\mathcal{D}$ and such that $\mathcal{E} \cup \{p^*\}$ has a greatest lower bound whenever $\mathcal{E}$ is a directed subset of $Code^*$ such that $|\bigcup\{supp(q) : q \in \mathcal{E}\}| < \kappa$ and such that $\{q \restriction 1 : q \in \mathcal{E}\} = \mathcal{D}$.*

Proof: Suppose $\mathcal{D} \subseteq \mathcal{Q}_1 \cap Code^*$ is a nonempty directed set of size less than κ. Note that $\delta := sup\{D(p) : p \in \mathcal{D}\} < \kappa$. We may assume that $\delta > D(p)$ for all $p \in \mathcal{D}$, as otherwise there is some $p^* \in \mathcal{D}$ extending all other conditions in $\mathcal{D}$, and then it is straightforward to see that $\mathcal{E}$ has a greatest lower bound q^* whenever $\mathcal{E} \subseteq Code^*$ satisfies the hypothesis with $\mathcal{D}$: q^* is the condition with support $\bigcup\{supp(q) : q \in \mathcal{E}\}$ such that, for all $\xi \in supp(q^*)$, $q^*(\xi) = q(\xi)$ for any $q \in \mathcal{E}$ with $D(q) = \delta$ and $\xi \in supp(q)$.

That $\delta > D(p)$ for all $p \in \mathcal{D}$ means in particular that δ is a limit ordinal. Let p^* be a sequence of length $\delta + 1$ whose restriction to $\xi + 1$, for any $\xi < \delta$, is $p \restriction \xi + 1$ for some (equivalently, for any) $p \in \mathcal{D}$ with $D(p) \geq \xi + 1$. As to the choice of the top member $p^*(\delta)$ of p^*, we put "blank" information there; that is, we pick p^* so that it forces δ to be outside $\bigcup_{i<\kappa}(S_i \cup dom(\vec{C}^i))$.

Now let $\mathcal{E}$ be a directed subset of $Code^*$ satisfying the hypothesis with $\mathcal{D}$. We can find a condition q^* in $Code^*$ with support $\bigcup\{supp(q) : q \in \mathcal{E}\}$ and extending all conditions in $\{p^*\} \cup \mathcal{E}$ as follows.

The first component of q^* is just p^*. The choice of $q^*(\xi)$ for all further $\xi \in \bigcup\{supp(q) : q \in \mathcal{E}\}$ is completely determined by p^* and by $\mathcal{E}$: We let $q^*(\xi)$ be the canonical name for the unique closed sequence[19] of length $\delta + 1$ extending $q(\xi)$ for all $q \in \mathcal{E}$ such that $\xi \in supp(q)$.[20]

To see that the sequence q^* is indeed a condition in $Code^*$, note that, for all $\xi \in \bigcup\{supp(q) : q \in \mathcal{D}\}$, $q^*(\xi)(\delta) = \delta$ (for forcings of type (A)) and $(q^*(\xi)(\delta)) \cap \kappa = \delta$ (for forcings of type (B)), and note also that there is no local constraint for the clubs added by $\dot{\mathcal{R}}_\zeta$, for $\zeta \geq 1$, as to what should happen at points outside $\bigcup_{\nu<\kappa} dom(\vec{C}^\nu)$ (for forcings of type (A)) and at points outside $\bigcup_{i<\kappa} S_i$ (for forcings of type (B)). The only conflict could appear at δ, but this will not happen since this ordinal has been put outside $\bigcup_{i<\kappa}(S_i \cup dom(\vec{C}^i))$.

It is also clear that q^* is in fact the greatest lower bound of $\{p^*\} \cup \mathcal{E}$. $\square$

[19] In either κ or $\mathcal{P}_\kappa(\kappa^+)$.

[20] Remember that $\dot{\mathcal{R}}_\xi$ is forced to be a poset for adding a suitable club, so such a condition is in fact unique.

Lemma 2.2 *$Code^* \cap \mathcal{Q}_\xi$ is a dense suborder of $\mathcal{Q}_\xi$ for each $\xi \leq \kappa^+$.*

Proof: Let us prove by induction on $\xi < \kappa^+$, $\xi \neq 0$, that for every $q \in \mathcal{Q}_\xi$ there is a condition $q^* \in \mathcal{Q}_\xi \cap Code^*$ extending q.

For $\xi = 1$ the result holds trivially. For the successor case, suppose $\xi = \xi_0 + 1$. By extending q if necessary we may assume $\xi_0 \in supp(q)$. Let N be a countable elementary substructure of some large enough $H(\theta)$ containing q and let $(q_n)_{n<\omega}$ be an $(N, \mathcal{Q}_\xi)$–generic sequence of conditions extending q such that $q_n \restriction \xi_0 \in Code^*$ for all n. Let $\delta = sup(N \cap \kappa)$. We are going to describe a lower bound q^* of $\{q_n\}_{n<\omega}$ with $supp(q^*) = \bigcup_n supp(q_n)$. q^* is going to be a condition in $Code^*$ with $D(q^*) = \delta + 1$.

The restriction of q^* to ξ_0 is a lower bound for the directed set $\{q_n \restriction \xi_0\}_{n<\omega}$ obtained as before, that is, obtained by putting the top element δ outside of $\bigcup_{i<\kappa}(S_i \cup dom(\vec{C}^i))$.[21].

The choice of $q^*(\xi_0)$ is also easy. It will be a canonical $\mathcal{Q}_{\xi_0}$–name for the unique condition extending all $q_n(\xi_0)$ with domain equal to the least successor ordinal above $dom(q_n(\xi_0))$ for all n, which of course is exactly $\delta + 1$ since $(q_n)_{n<\omega}$ is an $(N, \mathcal{Q}_\xi)$–generic sequence and since $\xi_0 \in N$. Using once again the $(N, \mathcal{Q}_\xi)$–genericity of $(q_n)_{n<\omega}$ it also follows that $q^* \restriction \xi_0$ forces that the intersection with κ of the union of the top elements of all $q_n(\xi_0)$ is exactly δ. Also, by the induction hypothesis we have that $\mathcal{Q}_{\xi_0}$ is κ–distributive. In particular, using again the fact that $(q_n)_{n<\omega}$ is $(N, \mathcal{Q}_\xi)$–generic, we get that $q^* \restriction \xi_0$ decides the interpretation of each of the names $q_n(\xi_0)$, which is going to be some object in V. Hence, the unique closed set of length $\delta + 1$ extending all $q_n(\xi_0)$ is an object in V.

Finally, by an argument as in the verification that $Code^*$ is κ–directed closed we can conclude that q^* is a legal condition in $Code$ and therefore, by the above observation, it is a condition in $Code^*$.

For the limit case ξ of the induction, let $\mu = cf(\xi)$. Since our iteration has been built with supports of size less than κ, we can in fact assume that $\mu < \kappa$. Let $(\xi_\tau)_{\tau<\mu}$ be a strictly increasing sequence converging to ξ. Again we pick an elementary substructure N of some large enough $H(\theta)$ containing all relevant objects, but this time we make sure in addition that $H(\theta)$ comes equipped with a well–order Δ and that N is an HIA substructure of $\langle H(\theta), \in, \Delta \rangle$ of size μ. Let $(N_\tau)_{\tau<\mu}$ be a $\subseteq$–continuous $\in$–chain of elemen-

[21] Of course, since the S_i's are to consist of ordinals of uncountable cofinality, this just means that we put δ outside $\bigcup_{i<\kappa} dom(\vec{C}^i)$. This observation does not apply to the limit case of the induction, in which we will also have to deal with δ's of uncountable cofinality.

tary substructures of $\langle H(\theta), \in, \Delta\rangle$ of size less than μ witnessing that N is HIA and such that N_0 contains q and $(\xi_\tau)_{\tau<\mu}$. Let $\delta_\tau = sup(N_\tau \cap \kappa)$ and $\tilde{\mu}_\tau = sup(N_\tau \cap \mu)$ for all τ.

We build by recursion an $(N, \mathcal{Q}_\xi)$–generic sequence $(q_\tau)_{\tau<\mu}$ of conditions extending q as follows.

Suppose $\tau = 0$. We let $(q_{0,\sigma})_{\sigma<|N_0|}$ be the Δ–least $(N_0, \mathcal{Q}_\xi)$–generic sequence of conditions extending q. We have that each dense subset of $\mathcal{Q}_\xi$ in N_0 is met by $q_{0,\sigma}$ for unboundedly many σ in $|N_0|$. Hence, by our induction hypothesis we have that for every $\tilde{\mu} < \tilde{\mu}_0$ in N_0 there are unboundedly many σ in $|N_0|$ such that $q_{0,\sigma} \restriction \xi_{\tilde{\mu}} \in Code^*$. It follows as in the successor case of the induction that we can extend all $q_{0,\sigma}$ to a condition q_0 with $q_0 \restriction \xi_{\tilde{\mu}_0} \in Code^*$ and $D(q_0 \restriction \xi_{\tilde{\mu}_0}) = \delta_0 + 1$ by putting blank information on the top element, namely δ_0.

Now let $\tau < \mu$, $\tau > 0$, be given and assume $q_{\tau'}$ has been built for all $\tau' < \tau$. Assume in addition that for all such τ' there is an $(N_{\tau'}, \mathcal{Q}_\xi)$–generic sequence $(q_{\tau',\sigma})_{\sigma<|N_{\tau'}|}$ of conditions extending q such that $q_{\tau'}$ is a lower bound of $\{q_{\tau',\sigma}\}_{\sigma<|N_{\tau'}|}$ with $q_{\tau'} \restriction \xi_{\tilde{\mu}_{\tau'}} \in Code^*$ and $D(q_{\tau'} \restriction \xi_{\tilde{\mu}_{\tau'}}) = \delta_{\tau'} + 1$ obtained by always putting blank information on the top element and such that $q_{\tau'} \restriction \zeta$ forces, for each nonzero $\zeta \in supp(q_{\tau'}) \cap \xi_{\tilde{\mu}_{\tau'}}$, that $q_{\tau'}(\zeta)$ is the unique condition with domain equal to $\delta_{\tau'} + 1$ extending $q_{\tau',\sigma}(\zeta)$ for all σ such that $\zeta \in supp(q_{\tau',\sigma})$. Assume as well that $q_{\tau'}$ extends $q_{\tau''}$ for all $\tau'' < \tau'$.

If $\tau = \tau_0 + 1$, then we obtain q_τ from q_{τ_0} exactly as in the construction for $\tau = 0$, replacing q by q_{τ_0}.

Finally, suppose τ is a limit ordinal. Since each $q_{\tau'}$ (for $\tau' < \tau$) is an $(N_{\tau'}, \mathcal{Q}_\xi)$–generic condition and the sequence $(q_{\tau'})_{\tau'<\tau}$ is decreasing, we have that this sequence is $(N_\tau, \mathcal{Q}_\xi)$–generic. Now we can build a condition q_τ with $q_\tau \restriction \xi_{\tilde{\mu}_\tau} \in Code^*$ and $D(q_\tau \restriction \xi_{\tilde{\mu}_\tau}) = \delta_\tau + 1$ by putting δ_τ outside $\bigcup_{i<\kappa}(S_i \cup dom(\vec{C}^i))$ as in the construction for the $\tau = 0$–case. For every nonzero $\zeta \in supp(q_\tau) \cap \xi_{\tilde{\mu}_\tau}$ we also make sure, as in that construction, that each $q_\tau \restriction \zeta$ forces $q_\tau(\zeta)$ to be the unique condition extending $q_{\tau'}(\zeta)$ for all τ' such that $\zeta \in supp(q_{\tau'})$ with domain the least successor ordinal above $dom(q_{\tau'}(\zeta))$ for all such τ'. This ordinal will be exactly $\delta_\tau + 1$.

Note that the way in which we are building our generic sequence, always choosing the relevant objects to be minimal with respect to our fixed well–order Δ, ensures that $(q_{\tau'})_{\tau'<\tau}$, being definable over $\langle H(\theta), \in, \Delta\rangle$ from $(N_\tau)_{\tau<\mu}$, has each of its proper initial segments in the relevant model $N_{\tau'}$, so it can always be continued. Finally we take our desired condition q^* to be an $(N, \mathcal{Q}_\xi)$–generic condition obtained from the $(N, \mathcal{Q}_\xi)$–sequence $(q_\tau)_{\tau<\mu}$ as in the above limit case of the construction. q^* is indeed a condition as its support is a union of μ–many sets of size less than κ, and therefore itself of size less than κ as $\mu < \kappa$. $\square$

Using $\kappa^{<\kappa} = \kappa$ and Lemma 2.2 we have that all proper initial segments $\mathcal{Q}_\xi$ of the iteration leading to $Code$ have a dense subset (namely $Code^* \cap \mathcal{Q}_\xi$) of size κ. From this it follows, by $2^\kappa = \kappa^+$, that $Code$ has the κ^+–chain condition. In particular this guarantees that all $B \subseteq \kappa$ and all clubs $D \subseteq \kappa$ appearing in V^{Code} are dealt with along the iteration. From this it is not difficult to check that $Code$ forces the following two statements.

(a) Every set in the range of $\mathcal{S}$ is stationary and every subset of κ is coded by some ordinal in κ^+ with respect to $(F, \mathcal{S})$ (where $(F, \mathcal{S})$ is the pair of objects added in the first stage of the iteration). Moreover, the corresponding well–order $\leq$ of $H(\kappa^+)$ added by $Code$ has $\mathcal{W}$ as an initial segment.

(b) Each $\vec{C}^\nu$ (for $\nu < \kappa$) is a strongly type–guessing club–sequence on κ with stationary domain.

In fact, that is what $Code$ has been designed to do. What requires some more work is to prove the following lemma.

Lemma 2.3 *Code forces that the set $\{ht(\vec{C}^\nu) : \nu < \kappa\}$ is precisely the collection of all ordinals of the form η_ξ for which there is a coherent strongly type–guessing club–sequence on κ with stationary domain and of height η_ξ.*

Lemma 2.3 says, in other words, that if a coherent club–sequence $\vec{C}$ on κ with stationary domain has height η_ξ for a "wrong" ξ (i.e., one not belonging to the set coding the parameter $(F, \mathcal{S})$), then $\vec{C}$ is not strongly type–guessing. This will happen "by accident" and will be witnessed by all clubs D' added by forcings of type (A) at sufficiently high stages of the iteration.[22] We are not going to prove this result here, and instead refer the reader to [As]. The purpose of some of the technicalities leading up to the definition of the coding – e.g. the operation $\cap^*$ or the consideration of perfect ordinals – is precisely to make Lemma 2.3 hold true.

[22]That is, $\{\delta \in dom(\vec{C}) \cap D' : ot(C_\delta \cap^* D') < ot(C_\delta)\}$ will be forced to be stationary for all such D'.

3 The class forcing

Let us assume GCH above $\aleph_0$ throughout this section. As we have already anticipated, the forcing $\mathcal{P}$ for Theorem 1.1 is a two–step iteration $\mathcal{B} * \dot{\mathcal{C}}$ of class forcings. To start with, we let $Reg^* = \{\alpha \geq \omega_2 : \alpha \text{ a regular cardinal}\}$.

3.1 Defining $\mathcal{B}$

$\mathcal{B}$ is the direct limit of a reverse Easton iteration of length Ord, to be denoted by $\langle \mathcal{B}_\alpha : \alpha \in Ord \rangle$, based on a certain sequence $\langle \dot{\mathbb{Q}}_\alpha : \alpha \in Ord \rangle$ of names for posets. Given any stage α, we do nothing – that is, we let $\dot{\mathbb{Q}}_\alpha$ be the trivial forcing $\{\emptyset\}$ – unless $\alpha \in Reg^*$. In that case we let $\dot{\mathbb{Q}}_\alpha$ be a $\mathcal{B}_\alpha$–name for the α^+–directed closed poset for adding a function $f : \alpha^+ \longrightarrow H(\alpha^+)$ by initial segments (note that this is essentially the forcing for adding a Cohen subset of α^+ over $V^{\mathcal{B}_\alpha}$). In $V^{\mathcal{B}_\alpha}$ let also $\dot{f}_\alpha$ be a $\dot{\mathbb{Q}}_\alpha$–name for the function being added.

Standard arguments (see [F1]) show that forcing with $\mathcal{B}$ preserves ZFC, GCH above $\aleph_0$ and cofinalities. For the reader's convenience we sketch these preservation proofs in a moment.

A simple inductive argument using GCH above $\aleph_0$ shows that each $\mathcal{B}_\alpha$ (for $\alpha \in Reg^*$) has size at most α: The result when α is an inaccessible cardinal is easy. If $\alpha = \alpha_0^+$ for α_0 a singular cardinal and $Reg^* \cap \alpha_0$ has order type $\overline{\alpha} \leq \alpha_0$, then $|\mathcal{B}_\alpha|$ is the cardinality of a collection of $2^{|\overline{\alpha}|} = |\overline{\alpha}|^+$–many sets of the form ${}^X(\bigcup_{\alpha' < \alpha_0} \mathcal{B}_{\alpha'})$ for some $X \subseteq \overline{\alpha}$. Hence, $|\mathcal{B}_\alpha| \leq \alpha_0^+ = \alpha$ since $\bigcup_{\alpha' < \alpha_0} \mathcal{B}_{\alpha'}$ has size at most $sup(Reg^* \cap \alpha_0) = \alpha_0$. And, if $\alpha = \alpha_0^+$ for α_0 regular, then $\mathcal{B}_\alpha$ is (isomorphic to) $\mathcal{B}_{\alpha_0} * \dot{\mathbb{Q}}_{\alpha_0}$, with $|\mathcal{B}_{\alpha_0}| \leq \alpha_0$ and with $\dot{\mathbb{Q}}_{\alpha_0}$ a $\mathcal{B}_{\alpha_0}$–name for a poset of size $(\alpha_0^+)^{\alpha_0} = \alpha$ by GCH above $\aleph_0$. Hence $|\mathcal{B}_\alpha| = \alpha$ again by $\alpha^{\alpha_0} = \alpha$ since we are using only nice names.

Note that $\mathcal{B}$ is ω_3–directed closed, and hence preserves all cardinals $\kappa \leq \omega_3$. Using the fact that $\mathcal{B}_\alpha$ has size at most α and that each component $\dot{\mathbb{Q}}_\beta$ on the tail is forced to be α^+–directed closed, one can prove the following result by an application of Lemma 1.2.

Lemma 3.1 *For each $\alpha \in Reg^*$, $\mathcal{B}_\alpha$ has the α^+–chain condition and $\mathcal{B}$ factors as $\mathcal{B}_\alpha * \dot{\mathcal{B}}^1$, with $\dot{\mathcal{B}}^1$ a $\mathcal{B}_\alpha$–name for an α^+–directed closed forcing.*

By an induction using this lemma it follows that $\mathcal{B}$ preserves the regularity of all $\alpha \in Reg^*$.

The preservation of ZFC is also easy: Every class–forcing preserves the ZFC axioms, with the possible exceptions of Replacement and the Power set Axiom. That $\mathcal{B}$ preserves the Power set Axiom as well follows quite directly from Lemma 3.1. For the preservation of Replacement we only need to argue, by results in [F1], that $\mathcal{B}$ is pretame (in the terminology from [F1]). This means that if $p \in \mathcal{B}$, I is a set and $\langle D_i : i \in I \rangle$ is a definable sequence of predense subclasses of $\mathcal{B}$ below p, then there is some condition q extending p and there is some ordinal α such that every $D_i \cap V_\alpha$ is predense below q. But this is true again by Lemma 3.1: Given p and $\langle D_i : i \in I \rangle$ we just have to consider some $\alpha_0 \in Reg^*$ above the cardinality of I and so that $p \in \mathcal{B}_{\alpha_0}$ and use Lemma 3.1 to find, in $V^{\mathcal{B}_{\alpha_0} \restriction p}$, a condition $\dot{q}$ in $\bigcap_{i \in I} \dot{E}_i$, where $\dot{E}_i$ denotes, in $V^{\mathcal{B}_{\alpha_0}}$, the dense subset of $\mathcal{B}/\dot{G}_{\alpha_0}$ consisting of those conditions q such that, for some $\tilde{p} \in \dot{G}_{\alpha_0}$, $\tilde{p} \cup q$ extends some member of D_i. Finally, taking $q = p \cup \dot{q}$ and taking $\alpha > \alpha_0$ such that $supp(q) \subseteq \alpha$ yields the desired conclusion.

Using once more Lemma 3.1 one can prove that $\mathcal{B}$ preserves GCH at uncountable regular cardinals. It follows then that every singular cardinal α remains strong limit after forcing with $\mathcal{B}$ and hence $(2^\alpha)^{V^\mathcal{B}} = (\alpha^{cf(\alpha)})^{V^\mathcal{B}} = (\alpha^{cf(\alpha)})^V = \alpha^+$ again by the relevant distributivity of the tail forcings. Hence, $\mathcal{B}$ preserves GCH above $\aleph_0$. And, by a density argument together with Lemma 3.1, in $V^\mathcal{B}$ it holds that each $\dot{f}_\alpha$ is a bookkeeping function for $H(\alpha^+)$.

3.2 Defining $\dot{\mathcal{C}}$

Let us move to $V_1 := V^\mathcal{B}$ now. The definition of $\dot{\mathcal{C}}$ is the following.

$\dot{\mathcal{C}}$ is again the direct limit of a reverse Easton iteration. Let $\langle \mathcal{C}_\alpha : \alpha \in Ord \rangle$ be this iteration and let $\langle \dot{\mathcal{Q}}_\alpha : \alpha \in Ord \rangle$ be the sequence of names on which it is based. Again, at any given stage α, we do nothing unless $\alpha \in Reg^*$.

Given $\alpha \in Reg^*$, suppose $\mathcal{C}_\alpha \in H(\alpha^+)$ and let $g_\alpha : \alpha^+ \longrightarrow H(\alpha^+)$ be, in $V_1^{\mathcal{C}_\alpha}$, a bookkeeping function for $H(\alpha^+)$ obtained in some canonical way from $\dot{f}_\alpha$. For example, given an ordinal $i \in \alpha^+$ we can let $g_\alpha(i)$ be $(\dot{f}_\alpha(i))_{\dot{G}}$, where $\dot{G}$ is the generic filter for $\mathcal{C}_\alpha$, if $\dot{f}_\alpha(i)$ happens to be a $\mathcal{C}_\alpha$–name for a member of $H(\alpha^+)$. This indeed gives a bookkeeping function for $H(\alpha^+)^{V_1^{\mathcal{C}_\alpha}}$ because $\dot{f}_\alpha$ is a bookkeeping function for $H(\alpha^+)^{V_1}$ and because every set in $H(\alpha^+)^{V_1^{\mathcal{C}_\alpha}}$ has a $\mathcal{C}_\alpha$–name belonging to $H(\alpha^+)^{V_1}$ (since $\mathcal{C}_\alpha \in H(\alpha^+)^{V_1}$).

This time we let $\mathcal{C}_\alpha$ force $\dot{\mathcal{Q}}_\alpha = (Code^*)_\alpha^{g_\alpha, \mathcal{W}_\alpha}$, where $\mathcal{W}_\alpha$ denotes the union of the well–orders added at all previous stages of the iteration. By $\mathcal{C}_\alpha \in H(\alpha^+)$ and by our GCH–assumption, together with results from the previous section, it follows that we may take $\mathcal{C}_{\alpha+1} = \mathcal{C}_\alpha * \dot{\mathcal{Q}}_\alpha$ to be a member of $H(\alpha^{+2})$.

For the limit stages α of our inductive definition, note that $\mathcal{C}_\alpha$ has then size at most α (for $\alpha \in Reg^*$) by considerations as for $\mathcal{B}_\alpha$.

The following result can be proved using Lemmas 2.1 and 1.2, together with the fact that each $\mathcal{C}_{\overline{\alpha}}$ (for $\overline{\alpha} \in Reg^*$) has size at most $\overline{\alpha}$.

Lemma 3.2 *$\dot{\mathcal{C}}$ is ω_2–directed closed. Also, given any $\alpha \in Reg^*$ and any $\mathcal{C}_\alpha$–generic filter G_α over V_1, the quotient forcing $\mathcal{C}/G_\alpha$ is α–directed closed in $V_1[G_\alpha]$.*

Also, for every successor cardinal α^+, $\mathcal{C}_{\alpha^+}$ is isomorphic to $\mathcal{C}_\alpha * \dot{\mathcal{Q}}_\alpha$, with $\mathcal{C}_\alpha$ a poset of size at most α and $\dot{\mathcal{Q}}_\alpha$ a name for a poset with the α^+–chain condition. In particular this means that $\mathcal{C}_{\alpha^+}$ has the α^+–chain condition.

Hence, for every $\alpha \in Reg^*$ we have again that $\dot{\mathcal{C}}$ factors as the iteration of a poset with the α^+–chain condition and an α^+–directed closed class-forcing. It follows by arguing as for $\mathcal{B}$ that $\dot{\mathcal{C}}$ preserves all cofinalities as well as ZFC and GCH above $\aleph_0$.

¿From the discussion in the previous section it follows that each $\dot{\mathcal{Q}}_\alpha$ adds over $V_1^{\mathcal{C}_\alpha}$ a well–order $\leq_\alpha$ of $H(\alpha^+)$ definable over $\langle H(\alpha^+), \in\rangle$ by a parameter–free formula and extending all the well–orders $\leq_\beta$ added at previous stages. And of course, the definition of this well–order is the same for all stages. Further, by Lemma 3.2 we also have that forcing with $\mathcal{C}/G_{\alpha+1}$ over $V_1[G_{\alpha+1}]$, for any $\mathcal{C}_{\alpha+1}$–generic $G_{\alpha+1}$, does not change $H(\alpha^+)$, so that in the end $\leq_\alpha$ remains a well–order of $H(\alpha^+)$ defined by the same formula (over $\langle H(\alpha^+), \in\rangle$) as in $V_1[G_{\alpha+1}]$.

Remark: As a bonus from the fact that $\mathcal{P}$ is ω_2–directed closed it is easy to prove that $\mathcal{P}$ preserves forcing axioms like PFA or Martin's Maximum in case they hold in the ground model. Hence, these forcing axioms are compatible with a locally definable well–order of the universe in the sense of Theorem 1.1: First we force GCH above $\aleph_0$ with an ω_2–directed closed forcing over any model of the forcing axiom, and then we force with $\mathcal{P}$.

4 Large cardinal preservation

The large cardinal preservation argument that will conclude the proof of Theorem 1.1 will be derived in part from the following general result.

Theorem 4.1 *(Lifting Theorem) Let $\kappa \leq \lambda$ be regular cardinals, suppose $\lambda^{<\lambda} = \lambda$ and $2^\lambda = \lambda^+$, and let $j : V \longrightarrow M$ be a λ–supercompact embedding derived from a normal and fine κ–complete measure on $\mathcal{P}_\kappa(\lambda)$.*

Let $\mathbb{P} = \langle \mathbb{P}_\xi : \xi \leq \lambda + 1\rangle$ be a reverse Easton iteration, based on a sequence $\mathbb{Q} = \langle \dot{\mathbb{Q}}_\xi : \xi < \lambda + 1\rangle$ of names, in which each $\dot{\mathbb{Q}}_\xi$ is forced to be trivial unless ξ is a V–regular cardinal.

Suppose $|\mathbb{P}_\lambda| = \lambda$ and $\mathbb{P}_\kappa \subseteq V_\kappa$. Suppose also that $j(\mathbb{P})_{\lambda+1} = \mathbb{P}_{\lambda+1}$.

Suppose each $\mathbb{P}_\xi$ (for $\xi < \lambda + 1$ a V–regular cardinal) forces the following statements.

(1) $\dot{\mathbb{Q}}_\xi$ is a dense suborder of the direct limit of a $< \xi$–supported forcing iteration $\langle \mathcal{Q}^\xi_i : i < \xi^+\rangle$ with the ξ^+–chain condition, and $\mathcal{Q}^\xi_i \cap \dot{\mathbb{Q}}_\xi$ has size at most ξ for every $i < \xi^+$.

(2) If $\mathcal{D}$ is a directed subset of $\mathcal{Q}^\xi_1 \cap \dot{\mathbb{Q}}_\xi$ of size less than ξ, then there is a condition $p^* \in \mathcal{Q}^\xi_1 \cap \dot{\mathbb{Q}}_\xi$ extending all conditions in $\mathcal{D}$ and such that $\mathcal{E} \cup \{p^*\}$ has a greatest lower bound whenever $\mathcal{E}$ is a directed subset of $\dot{\mathbb{Q}}_\xi$ with $|\bigcup\{supp(q) : q \in \mathcal{E}\}| < \xi$ and with $\{q \restriction 1 : q \in \mathcal{E}\} = \mathcal{D}$.

(3) ξ is a regular cardinal.

If G is a $\mathbb{P}_{\lambda+1}$–generic filter over V, then in $V[G]$ there is a $j(\mathbb{P}_{\lambda+1})$–generic filter H over M such that $j``G \subseteq H$ and whose restriction to $\mathbb{P}_{\lambda+1}$ is G.

Proof: Note that (2) poses a strong form of ξ–directed closure of $\dot{\mathbb{Q}}_\xi$, in $V^{\mathbb{P}_\xi}$, for every V–regular ξ. This property will be used in the proof of Lemma 4.3.

Let G be $\mathbb{P}_{\lambda+1}$–generic over V. As required by the conclusion, the restriction of H to $j(\mathbb{P})_{\lambda+1}$ is going to be G. This makes sense since $j(\mathbb{P})_{\lambda+1} = \mathbb{P}_{\lambda+1}$ and since every generic filter for $\mathbb{P}_{\lambda+1}$ over V is also generic for the same forcing over the submodel M.

Lemma 4.2 *In $M[G]$ there is a condition r^* in $j(\mathbb{P}_\lambda)/G$ extending $j(r) \restriction [\kappa, j(\lambda))$ for every $r \in G$.*

Proof: Let $\mathcal{X}$ be the union of all sets of the form $j(X)\backslash j(\kappa)$, with X a subset of λ belonging to V and with $X \cap \alpha$ bounded in α whenever α is a V–regular cardinal. The condition r^* will have support $\mathcal{X}$, which is also equal to $\bigcup\{j(supp(r)) : r \in G_\lambda\}\backslash j(\kappa)$ by a simple density argument.

In order to check that $\mathcal{X}$ is a legal support for a condition in $j(\mathbb{P}_\lambda)/G$ it suffices to see, for every set $X \subseteq \lambda$ as above, that $sup(\alpha \cap j(X))$ is less than α for every M–regular cardinal $\alpha \leq j(\lambda)$ above $j(\kappa)$. (This is enough since $\mathcal{X}$ is a union of $\lambda^{<\lambda}$–many sets in M, and therefore itself a member of M. It follows then that $\mathcal{X} \cap \alpha$ is bounded in α for every M–regular $\alpha > j(\kappa)$ by $\lambda^{<\lambda} < j(\kappa) < \alpha$.) But $j(X) \cap \alpha$ must be bounded in α for every such α by elementarity of j.

Let us work in $M[G]^{j(\mathbb{P}_\kappa)/G}$ for a while. We may assume $\kappa < \lambda$, as otherwise there is nothing to prove.

Note that $X := \{j(r(\kappa)) : r \in G\}$ – and in fact $j``G_\lambda$ – is a set in $M[G]$ since $|\mathbb{P}_\lambda|^V = \lambda$ and since $({}^\lambda M[G]) \cap V[G] \subseteq M[G]$. Hence, X is, in $M[G]$, a directed collection of conditions in $j(\mathbb{Q})(j(\kappa))$ of size at most $|\lambda|^{M[G]}$. Indeed, if $r_1, \dots r_n$ are finitely many conditions in G and $r \in G$ is a condition extending all of them, then $j(r \restriction \kappa) = r \restriction \kappa \in G$[23] forces in $j(\mathbb{P}_\kappa)$ (over M) that $j(r(\kappa))$ extends all of $j(r_1(\kappa)), \dots j(r_n(\kappa))$ in $j(\mathbb{Q})(j(\kappa))$.

It follows then from the fact that $j(\mathbb{Q})(j(\kappa))$ is $j(\kappa)$–directed closed in $M[G]$, together with $|\lambda|^{M[G]} < j(\kappa)$, that there is an $r^*(j(\kappa)) \in j(\mathbb{Q})(j(\kappa))$ extending $j(r(\kappa))$ for all $r \in G$.

The choice of the remaining components $r^*(j(\alpha))$ of r^* (for $\alpha \in \mathcal{X}$) goes along similar lines working in $M^{(j(\mathbb{P}_\alpha)/G)\restriction(r^* \restriction j(\alpha))}$, replacing κ with α at the appropriate places and using the fact that $r^* \restriction j(\alpha)$ extends $j(r \restriction \alpha)$ for every $r \in G$. $\square$

In $V[G]$, the number – let us call it χ – of maximal antichains of $j(\mathbb{P}_\lambda)/G$ in $M[G]$ is bounded by the cardinality of $(2^{|j(\mathbb{P}_\lambda)|})^{M[G]}$ also in $V[G]$. And $(2^{|j(\mathbb{P}_\lambda)|})^M = (2^{j(\lambda)})^M = j(\lambda^+)$ has cardinality λ^+ in V by $\lambda^{<\kappa} = \lambda$ and by $(\lambda^+)^\lambda = \lambda^+$. Hence, in V there are exactly λ^+–many nice $j(\mathbb{P}_\lambda)$–names in M for subsets of $j(\mathbb{P}_\lambda)$. It follows in particular that

$$|(2^{|\lambda|})^{M[G]}|^{V[G]} = |(2^{|j(\mathbb{P}_\lambda)|})^{M[G]}|^{V[G]} \leq |(\lambda^+)^V|^{V[G]} \leq (\lambda^+)^V,$$

and in fact $|(2^{|\lambda|})^{M[G]}|^{V[G]} = (\lambda^+)^V$ as $M[G]$ is closed under λ–sequences in $V[G]$ and $(2^{|\lambda|})^{M[G]} > |\lambda|^{M[G]} = |\lambda|^{V[G]}$. (This implies in particular that

[23] The equality follows from $\mathbb{P}_\kappa \subseteq V_\kappa$.

$(\lambda^+)^V$ is the successor, in $V[G]$, of $|\lambda|^{V[G]}$.)

We have seen that $\chi \leq (\lambda^+)^V$. Now we can build in $V[G]$ an $M[G]$–generic filter H' for $j(\mathbb{P}_\lambda)/G$ containing r^*. H' can be obtained as the upward closure, in $j(\mathbb{P}_\lambda)/G$, of a decreasing sequence $(r_\xi)_{\xi<(\lambda^+)^V} \in V[G]$ of conditions in $j(\mathbb{P}_\lambda)/G$ extending r^* and such that each $r_{2\cdot\xi+1}$ extends some condition in A_ξ, for some fixed enumeration $(A_\xi)_{\xi<(\lambda^+)^V}$ of all maximal antichains of $j(\mathbb{P}_\lambda)/G$ in $M[G]$. The construction can be continued at limit stages ξ because $(r_\zeta)_{\zeta<\xi} \in M[G]$ and $|\{r_\zeta\}_{\zeta<\xi}|^{M[G]} = |\lambda|^{V[G]}$,[24] and using the fact that $j(\mathbb{P}_\lambda)/G$ is $(|\lambda|^+)^{M[G]}$–directed closed in $M[G]$.

The final step of the construction will be to find an H'' as promised by the following lemma. Its proof is the only place where we use the full force of the condition in (2).

Lemma 4.3 *There is a $j(\dot{\mathbb{Q}}_\lambda)_{G*H'}$–generic filter H'' over $M[G*H']$ such that $j(r(\lambda))_{G*H'} \in H''$ whenever r is a condition in G.*

Proof: In $V[G_\lambda]$, $\mathbb{Q}_\lambda$ is a dense suborder of the direct limit of a $<\lambda$–supported iteration $\langle \mathcal{Q}^\lambda_i : i < \lambda^+\rangle$ with the λ^+–chain condition and in which every initial part $\mathcal{Q}^\lambda_i$ has size at most λ. Let $\langle H_i : i < \lambda^+\rangle$ be such that each H_i is the restriction to $\mathcal{Q}^\lambda_i$ of $G(\lambda)$. Since $\mathbb{P}_\lambda$ is a poset of size λ, we may assume that there are $\mathbb{P}_\lambda$–names $\dot{\mathcal{Q}}_i$ (for $i < \lambda^+$) in $H(\lambda^+)^V$ such that $(\dot{\mathcal{Q}}_i)_{G_\lambda} = \mathcal{Q}^\lambda_i$. Then, by $({}^\lambda M) \cap V \subseteq M$ we have that $j \restriction \dot{\mathcal{Q}}_i$ is a member of M for every $i < \lambda^+$.

The construction of the desired filter H'' will be somewhat more complicated than the construction of H'. The reason is that it will not be true in general that we can get a master condition for $G(\lambda)$, as finding such a condition would involve extending a certain collection of λ^+–many conditions.[25] However, this obstacle can be sorted out in the following way.

By arguing much like in the proof of Lemma 4.2 it can be seen that there is a sequence $\langle q_i : 0 < i < \lambda^+\rangle \in V[G]$ of "partial master conditions" for $G(\lambda)$. By this we mean that each $q_i \in M[G*H']$ is a master condition for H_i[26] and that $q_i = q_{i'} \restriction j(i)$ for all $i < i' < \lambda^+$:

By (2) for $\xi = j(\lambda)$ in $M[G*H']$, we may take a name q_1 for a lower bound of $\mathcal{D} = \{j(\dot{p}) : (\exists r)\ r \frown \langle \dot{p}\rangle \in G_\lambda * H_1\}$ with the property that, in $M[G*H']$,

[24] By $({}^\lambda M[G]) \cap V[G] \subseteq M[G]$.

[25] Which need not be a member of $M[G*H']$ as we are only assuming $({}^\lambda M) \cap V \subseteq M$ and not $({}^{\lambda^+} M) \cap V \subseteq M$.

[26] That is, each q_i is a $j(\dot{\mathcal{Q}}_i)_{G*H'}$–condition extending $j(\dot{q})_{G*H'}$ whenever $\dot{q}$ is a $\mathbb{P}_\lambda$–name such that $r \frown \langle \dot{q}\rangle \in G_\lambda * H_i$ for some r.

$\{q_1\} \cup \mathcal{E}$ has a greatest lower bound whenever $\mathcal{E}$ is a directed subset of $j(\dot{\mathbb{Q}}_\lambda)_{G*H'}$ with $|\bigcup\{supp(q) : q \in \mathcal{E}\}| < j(\lambda)$ and with $\{q \restriction 1 : q \in \mathcal{E}\} = \mathcal{D}$.

Each further q_i is a name for a greatest lower bound of $\mathcal{E}_i \cup \{p^*\}$ for $\mathcal{E}_i = \{j(\dot{q}) : (\exists r)\ r \frown \langle \dot{q} \rangle \in G_\lambda * H_i\}$.

$j(\dot{\mathcal{Q}}_{\lambda^+})_{G*H'}$ has the $j(\lambda^+)$–chain condition in $M[G*H']$ and $j``\lambda^+$ is cofinal in $j(\lambda^+)$,[27] so for every maximal antichain A of $j(\dot{\mathcal{Q}}_{\lambda^+})_{G*H'}$ in $M[G*H']$ there is some ordinal $i < \lambda^+$ such that $A \subseteq j(\dot{\mathcal{Q}}_i)_{G*H'}$. Also, in $M[G*H']$ there are $j(\lambda^+)$–many maximal antichains of $j(\dot{\mathcal{Q}}_{\lambda^+})_{G*H'}$.[28] As we are going to see next, after fixing in $V[G]$ an enumeration $\langle A_i : i < \lambda^+ \rangle$ of all of them[29] we can build in λ^+–many steps a filter H'' for this poset meeting all A_i and such that $H'' \cap j(\dot{\mathcal{Q}}_i)_{G*H'}$ contains q_i for every $i < \lambda^+$. This will finish the proof.

Let $(\zeta_i)_{i<\lambda^+}$ be an increasing sequence of ordinals in λ^+ such that $A_i \subseteq j(\dot{\mathcal{Q}}_{\zeta_i})_{G*H'}$ for all i. The filter H'' will be the upward closure in $j(\dot{\mathcal{Q}}_{\lambda^+})_{G*H'}$ of a decreasing sequence $\langle s_i : i < \lambda^+ \rangle$ of $j(\dot{\mathcal{Q}}_{\lambda^+})_{G*H'}$–conditions such that each s_i is a condition in $j(\dot{\mathcal{Q}}_{\zeta_i})_{G*H'}$ extending q_{ζ_i} and extending some condition in A_{i_0} if $i = i_0 + 1$. As in the construction of H', we take advantage of the closure of $M[G*H']$ under λ–sequences in $V[G]$. This ensures that all proper initial segments $\langle s_i : i < i_0 \rangle$ (for $i_0 < \lambda^+$) of $\langle s_i : i < \lambda^+ \rangle$ are in $M[G*H']$, and hence the construction can be continued at limit stages.

We can let s_0 be just q_1. At a nonzero limit stage i, s_i is just any condition in $j(\dot{\mathcal{Q}}_{\zeta_i})_{G*H'}$ extending all previous $s_{i'}$ as well as q_{ζ_i}. It can be obtained by first considering a lower bound $\overline{s}$ of $\{s_{i'}\}_{i'<i}$ with support equal to $\bigcup_{i'<i} supp(s_{i'})$ and then extending $\overline{s}$ to a condition s_i extending q_{ζ_i}. This can be accomplished since $\overline{s}$ is certainly compatible with q_{ζ_i} as it is a condition in $(j(\dot{\mathcal{Q}})_{G*H'})_{sup_{i'<i}\zeta_{i'}}$ extending $q_{\zeta_i} \restriction sup_{i'<i}\zeta_{i'}$.

At a successor stage $i = i_0 + 1$, s_i can be obtained in the following two steps: First we extend s_{i_0} to a condition s' in $j(\dot{\mathcal{Q}}_{\zeta_i})_{G*H'}$ extending q_{ζ_i}, and then we extend s' to a condition s_i in $j(\dot{\mathcal{Q}}_{\zeta_i})_{G*H'}$ extending some condition in A_{i_0}. This can be achieved as A_{i_0} is a maximal antichain of $j(\dot{\mathcal{Q}}_{\lambda^+})_{G*H'}$ and therefore of $j(\dot{\mathcal{Q}}_{\zeta_i})_{G*H'}$. Note that s_i is compatible with all further q_{i^*} since it is a condition in $j(\dot{\mathcal{Q}}_{\zeta_i})_{G*H'}$ extending q_{ζ_i} and since $q_{\zeta_i} = q_{i^*} \restriction j(\zeta_i)$ for all $i^* > i$. This finishes the construction. $\square$

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[27]As every function $g : \lambda^{<\kappa} \longrightarrow \lambda^+$ in V is bounded by some ordinal in λ^+.

[28]By $2^\lambda = \lambda^+$ and by elementarity of j.

[29]Which we can do since $j(\lambda^+)$ has size λ^+ in V.

Finally, the filter $H = (G * H') * H''$ is as desired. $\Box$

Now we are ready to finish the proof of Theorem 1.1.

Theorem 4.4 *(GCH above $\aleph_0$) For all regular cardinals $\kappa \leq \lambda$, if κ is λ–supercompact, then κ remains λ–supercompact after forcing with $\mathcal{P}$.*

Proof: Let $j : V \longrightarrow M$ be a λ–supercompact embedding derived from a normal and fine κ–complete measure on $\mathcal{P}_\kappa(\lambda)$.

Let us start by showing that j lifts to a λ–supercompact embedding $j^* : V[\tilde{G}] \longrightarrow M^*$ whenever $\tilde{G}$ is $\mathcal{B}$–generic over V. In fact we are going to see the following.

Lemma 4.5 *In $V[\tilde{G}]$ there is a $j(\mathcal{B})$–generic filter $\tilde{H}$ over M such that $j``\tilde{G} \subseteq \tilde{H}$*[30] *and such that, in addition, the restriction of $\tilde{H}$ to $j(\mathcal{B})_{\lambda+1}$ is $\tilde{G}_{\lambda+1}$.*

Proof: By the λ^+–distributivity of $\mathcal{B}/\tilde{G}_{\lambda+1}$ in $V[\tilde{G}_{\lambda+1}]$ and by $\lambda^{<\kappa} = \lambda$ it suffices to see, in $V[\tilde{G}_{\lambda+1}]$, that there is a $j(\mathcal{B}_{\lambda+1})$–generic filter $\tilde{H}$ over M such that $j``\tilde{G}_{\lambda+1} \subseteq \tilde{H}$. Indeed, we will then have that there is a fine and normal κ–complete measure $\mathcal{U}$ on $\mathcal{P}_\kappa(\lambda)$ in $V[\tilde{G}_{\lambda+1}]$ and that the function f sending a set $x \in \mathcal{P}_\kappa(\lambda)$ to $\tilde{G}_{ot(x\cap(\lambda+1))}$ – which represents the filter $(j_{\mathcal{U}}(\tilde{G}_{\lambda+1}))_{\lambda+1} \subseteq (j_{\mathcal{U}}(\mathcal{B}))_{\lambda+1}$ in the ultrapower $Ult(V[\tilde{G}_{\lambda+1}],\mathcal{U})$, with $j_{\mathcal{U}}$ being the canonical embedding – also represents $\tilde{G}_{\lambda+1}$ in $Ult(V[\tilde{G}_{\lambda+1}],\mathcal{U})$. Since $\lambda^{<\kappa} = \lambda$, forcing with $\mathcal{B}/\tilde{G}_{\lambda+1}$ over $V[\tilde{G}_{\lambda+1}]$ leaves $\mathcal{P}(\mathcal{P}_\kappa(\lambda))$ unchanged.[31] It follows that $\mathcal{U}$ remains a normal and fine ultrafilter on $\mathcal{P}_\kappa(\lambda)$ in $V[\tilde{G}]$. And the function f still represents $\tilde{G}_{\lambda+1}$ in $Ult(V[\tilde{G}],\mathcal{U})$ since the set of functions g with domain $\mathcal{P}_\kappa(\lambda)$ and $g(x) \in \tilde{G}_{ot(x\cap(\lambda+1))}$ for all $x \in \mathcal{P}_\kappa(\lambda)$ is the same in $V[\tilde{G}_{\lambda+1}]$ and in $V[\tilde{G}]$ (again by the λ^+–distributivity of $\mathcal{B}/\tilde{G}_{\lambda+1}$ in $V[\tilde{G}_{\lambda+1}]$).

This filter $\tilde{H}$ can be built using a simplified version of the construction in the proof of Lemma 4.1: As required by the lemma, the restriction of $\tilde{H}$ to $j(\mathcal{B})_{\lambda+1}$ is going to be $\tilde{G}_{\lambda+1}$. $\tilde{G}_{\lambda+1}$ is indeed $j(\mathcal{B})_{\lambda+1}$–generic over M since this poset is precisely $\mathcal{B}_{\lambda+1}$, which follows from the uniformity of the definition of the $\mathcal{B}$–iteration and from $({}^\lambda M) \cap V \subseteq M$.

[30] This suffices for the existence of the required embedding.

[31] This is true even about forcing with $\mathcal{B}/\tilde{G}_\lambda$ over $V[\tilde{G}_\lambda]$ as $\mathcal{B}/\tilde{G}_\lambda$ is λ^+–distributive in $V[\tilde{G}_\lambda]$. Here we are working with the further extension $V[\tilde{G}_{\lambda+1}]$ since we want agreement of the generic filter on the M–side with $\tilde{G}$ up to stage $\lambda + 1$ of the iteration.

The second step in the construction is to find a $j(\mathcal{B}_\lambda)/\tilde{G}_{\lambda+1}$–generic H' over $M[\tilde{G}_{\lambda+1}]$ containing $j(q) \restriction (\lambda,\, j(\lambda))$ for every $q \in \tilde{G}_{\lambda+1}$. For this we first find a master condition $r \in j(\mathcal{B}_\lambda)/\tilde{G}_{\lambda+1}$ for $\tilde{G}_\lambda$. That is, we find $r \in j(\mathcal{B}_\lambda)/\tilde{G}_{\lambda+1}$ extending $j(q) \restriction (\lambda,\, j(\lambda))$ for every $q \in \tilde{G}_\lambda$. This can be achieved since $M[\tilde{G}_{\lambda+1}]$ is closed under λ–sequences in $V[\tilde{G}_{\lambda+1}]$, $|\mathcal{B}_\lambda| = \lambda$, and since $j(\mathcal{B}_\lambda)/\tilde{G}_{\lambda+1}$ is λ^+–directed closed in $M[\tilde{G}_{\lambda+1}]$. Then we build in $V[\tilde{G}_{\lambda+1}]$, in λ^+–steps, a filter H' of $j(\mathcal{B}_\lambda)/\tilde{G}_{\lambda+1}$ containing r and meeting all members of some given enumeration of length λ^+ of all dense subsets of $j(\mathcal{B}_\lambda)/\tilde{G}_{\lambda+1}$ in $M[\tilde{G}_{\lambda+1}]$. The existence of such an enumeration follows from the fact that $|\mathcal{P}^M(j(\mathcal{B}_\lambda))|^V = |j(\lambda^+)|^V = \lambda^+$ by $\lambda^\lambda = \lambda^+$. As usual, all proper initial segments of this construction are in $M[\tilde{G}_{\lambda+1}]$, but of course the whole construction is not.

Let $\dot{\mathcal{Q}}$ be the $\mathcal{B}_\lambda$–name chosen for building $\mathcal{B}_{\lambda+1}$ (i.e., $\mathcal{B}_{\lambda+1} = \mathcal{B}_\lambda * \dot{\mathcal{Q}}$). To finish the construction it suffices to show that the filter H'' of $j(\dot{\mathcal{Q}})_{\tilde{G}_{\lambda+1}*H'}$ generated by the set of all $j(\dot{p})_{\tilde{G}_{\lambda+1}*H'}$, where $\dot{p}$ is a $\mathcal{B}_\lambda$–name such that $q \frown \langle \dot{p} \rangle \in \tilde{G}_{\lambda+1}$ for some q, is in fact generic over $M[\tilde{G}_{\lambda+1} * H']$. To see that this is the case, let D be an open and dense subset of $j(\dot{\mathcal{Q}})_{\tilde{G}_{\lambda+1}*H'}$ in $M[\tilde{G}' * H']$. D is of the form $(j(f)(j``\lambda))_{\tilde{G}_{\lambda+1}*H'}$, where f is a function in V with domain $\mathcal{P}_\kappa(\lambda)$ such that every $f(x)$ is a $\mathcal{B}_\lambda$–name for a dense and open subset of $\dot{\mathcal{Q}}$. Since $\lambda^{<\kappa} = \lambda$ and since $\dot{\mathcal{Q}}$ is λ^+-directed closed in $V^{\mathcal{B}_\lambda}$, the set E of $p \in \dot{\mathcal{Q}}$ such that $p \in f(x)$ for all $x \in \mathcal{P}_\kappa(\lambda)$ is dense in $V^{\mathcal{B}_\lambda}$. It follows that there is some $q \frown \langle \dot{p} \rangle \in \tilde{G}_{\lambda+1}$ such that q forces that $\dot{p} \in E$, and therefore $j(\dot{p})_{\dot{G}_{\lambda+1}*H'} \in (j(f)(j``\lambda))_{\tilde{G}_{\lambda+1}*H'} = D$ by elementarity of j and by the fact that $j(q) \in \tilde{G}_{\lambda+1} * H'$. $\square$

By Lemma 4.5, together with standard arguments, if $\tilde{G}$ is $\mathcal{B}$–generic over V and $V_1 = V[\tilde{G}]$, we can find in V_1 a λ–supercompact embedding $j : V_1 \longrightarrow M$ derived from a normal and fine κ–complete measure on $\mathcal{P}_\kappa(\lambda)$ and such that $(j(\tilde{G}_\alpha))_{\lambda+1} = \tilde{G}_{\lambda+1}$ for all high enough α. The proof will be finished if we can show that j lifts to a λ–supercompact embedding $j^* : V_1[G] \longrightarrow M^*$ whenever G is $\mathcal{C}$–generic over V_1.

For this it suffices to show that for every $\mathcal{C}_{\lambda+1}$–generic G over V_1 there is, in $V_1[G]$, a $j(\mathcal{C}_{\lambda+1})$–generic filter H over M such that $j``G \subseteq H$. (This is enough since $\mathcal{C}/G$ is λ^+–distributive in $V_1[G]$ by Lemma 3.2.)

We intend to apply the Lifting Theorem with $\langle \mathcal{C}_\xi \,:\, \xi \leq \lambda + 1 \rangle$ and with the sequence of names on which this iteration is built as, respectively, $\mathbb{P} = \langle \mathbb{P}_\xi \,:\, \xi \leq \lambda + 1 \rangle$ and $\langle \dot{\mathbb{Q}}_\xi \,:\, \xi < \lambda + 1 \rangle$.

Certainly, $|\mathcal{C}_\lambda| = \lambda$ and $\mathcal{C}_\kappa \subseteq V_\kappa$.

Claim 4.5.1 $\mathcal{C}_{\lambda+1} = j(\mathbb{P})_{\lambda+1}$.

Proof: It suffices to prove by induction for $\xi < \lambda+1$ that if $\mathcal{C}_\xi = j(\mathbb{P})_\xi$, then $\mathcal{C}_\xi$ forces over V that $\dot{\mathbb{Q}}_\xi = j(\mathbb{Q})_\xi$.

Since $M^{\mathcal{C}_\xi}$ is closed under λ–sequences in $V_1^{\mathcal{C}_\xi}$ and $(\xi^+)^{V_1^{\mathcal{C}_\xi}} = (\xi^+)^{V_1} \leq \lambda^+$, $M^{j(\mathbb{P})_\xi}$ computes the same $H(\xi^+)$ as $V_1^{\mathcal{C}_\xi}$.

Let $\mathcal{F} = \langle \dot{f}_\xi : \xi \leq \lambda \rangle \in V_1$.[32] We already know that $j(\mathcal{F}) \restriction \lambda + 1 = \mathcal{F}$. Also, the definition in $V_1^{\mathcal{C}_\xi}$ of $\dot{\mathbb{Q}}_\xi$ from $\dot{f}_\xi$ is uniform for all $\xi < \lambda + 1$. Hence, by elementarity of j and by $j(\mathbb{P})_\xi = \mathcal{C}_\xi$, $j(\mathbb{Q})_\xi$ is defined over the structure $\langle H(\xi^+), \in \rangle^{M^{j(\mathbb{P})_\xi}}$ $(= \langle H(\xi^+), \in \rangle^{V_1^{\mathcal{C}_\xi}})$ from $\dot{f}_\xi$ in the same way as $\dot{\mathbb{Q}}_\xi$,[33] and so they are the same object.

Now one can easily see by induction on $\xi \leq \lambda + 1$ that $\mathcal{C}_\xi = j(\mathbb{P})_\xi$ (the limit case of the induction is handled again by the closure of M in V_1, and hence the fact that M contains all possible supports for conditions in $\mathcal{C}_\xi$). $\square$

Next, let us fix an index $\xi < \lambda + 1$. Condition (1) from Theorem 4.1 holds in $V_1^{\mathcal{C}_\xi}$ for $\dot{\mathbb{Q}}_\xi$ by the discussion in Section 2, and condition (2) holds by Lemma 2.1. As for (3), we have already seen that $\mathcal{C}$ preserves all regular cardinals.

We have verified all hypotheses from Theorem 4.1 for our objects. Hence there is in $V_1[G]$ a $j(\mathcal{C}_{\lambda+1})$–generic H over M such that $j``G \subseteq H$, which is what we wanted. From this we get that j can be extended to a λ–supercompact embedding $j^* : V_1[G] \longrightarrow M^*$ again by standard lifting arguments. $\square$

We finish this section with a small remark showing, for $\kappa < \lambda$, with κ a λ–supercompact and λ singular, that at least the λ–compactness of κ is preserved by $\mathcal{P}$.

Theorem 4.6 *(GCH above $\aleph_0$) For all cardinals $\kappa < \lambda$ with λ singular, if κ is λ–supercompact, then κ remains λ–compact after forcing with $\mathcal{P}$.*

Proof: Let $\mu = cf(\lambda)$ and let $(\lambda_i)_{i<\mu}$ be a strictly increasing sequence of regular cardinals above κ converging to λ. We can assume that $\mu \geq \kappa$.

[32] That is, $\mathcal{F}$ is the restriction of $\tilde{G}$ to $\mathcal{B}_{\lambda+1}$.

[33] Note that M and V_1 compute the same regular cardinals below $\lambda + 1$.

Otherwise $\lambda^{<\kappa} = \lambda^+$ and therefore κ is λ^+–supercompact. In this case we are done by Theorem 4.4.

Let $j : V \longrightarrow M$ be a λ–supercompact embedding derived from a normal and fine κ–complete ultrafilter on $\mathcal{P}_\kappa(\lambda)$ in V and let G be $\mathcal{P}$–generic. By Theorem 4.4 we have that in $V[G]$ there is a κ–complete uniform ultrafilter $\mathcal{U}_\mu$ on μ[34] and that for every $i < \mu$ there is a normal and fine κ–complete measure $\mathcal{U}_i$ on $\mathcal{P}_\kappa(\lambda_i)$.

Let us define in $V[G]$ a filter $\mathcal{U}$ on $\mathcal{P}_\kappa(\lambda)$ by setting $X \in \mathcal{U}$ if and only if

$$\{i < \mu \, : \, \{x \cap \lambda_i \, : \, x \in X\} \in \mathcal{U}_i\} \in \mathcal{U}_\mu$$

Using the fact that $\mathcal{U}_\mu$ is a uniform κ–complete ultrafilter on μ it is easy to verify that $\mathcal{U}$ is a fine κ–complete ultrafilter on $\mathcal{P}_\kappa(\lambda)$. Hence, we have that κ is λ-compact in $V[G]$. $\square$

5 Preserving n–huge cardinals

It is possible to build, under the same cardinal arithmetic assumption, a variant $\mathcal{P}^\dagger$ of $\mathcal{P}$ satisfying conclusion (1) from Theorem 1.1 and with the additional property that $\mathcal{P}^\dagger$ preserves, for any given positive $n < \omega$, many of the n–huge cardinals there might exist in the ground model.

Let us be a little more precise. Given an elementary embedding $j : V \longrightarrow M$ with critical point κ, let $j(\kappa)_0 = \kappa$ and let $j(\kappa)_{i+1} = j(j(\kappa)_i)$ for all $i < \omega$. Given a class $\mathcal{H}$ of huge cardinals, let us call a function $F : \mathcal{H} \longrightarrow \omega \backslash \{0\}$ a *non-overlapping hugeness function* in case

(a) for every $\kappa \in \mathcal{H}$, κ is $F(\kappa)$–huge, and

(b) for all $\kappa < \kappa'$ in $\mathcal{H}$, the $F(\kappa)$–hugeness of κ is witnessed by some $j : V \longrightarrow M$ with $j(\kappa)_{F(\kappa)} < \kappa'$.[35]

Also, given an integer n, $1 \leq n < \omega$, let $\mathcal{H}_n$ denote the class of n–huge cardinals κ for which it holds that for all m and all m–huge cardinals

[34] $\mathcal{U}_\mu$ can be defined, for some μ–supercompact embedding $k : V \longrightarrow M'$ derived from a normal and fine measure on $\mathcal{P}_\kappa(\mu)$, as the collection of all $Y \subseteq \mu$ such that $sup(k``\mu) \in k(Y)$.

[35] In other words, $j : V \longrightarrow M$ is an elementary embedding with $crit(j) = \kappa$, $j(\kappa)_{F(\kappa)} < \kappa'$, and such that M is closed under sequences of length $j(\kappa)_{F(\kappa)}$.

$\kappa' < \kappa$ there is some elementary embedding $j : V \longrightarrow M$ witnessing the m–hugeness of κ' such that $j(\kappa')_m < \kappa$. Let us call an elementary embedding $j : V \longrightarrow M$ witnessing the m–hugeness of some κ' *minimal* in case any other witness $j' : V \longrightarrow M$ satisfies $j'(\kappa')_m \geq j(\kappa')_m$. Obviously we can define all $\mathcal{H}_n$ by only making reference to minimal witnesses for n–hugeness.

It is easy to see that there is a definable non-overlapping hugeness function F with the property that $dom(F) \cap \mathcal{H}_n$ is unbounded in $sup(\mathcal{H}_n)$ (if $\mathcal{H}_n \neq \emptyset$).[36] What we are going to show in this final section is that for every such F it is possible to find a partial order $\mathcal{P}^\dagger$, definable from F, which adds a locally defined well–order of the universe in the sense of Theorem 1.1 (1) and preserves the $F(\kappa)$–hugeness of all $\kappa \in dom(F)$. In particular this will show that there is a definable class–forcing $\mathcal{P}^\dagger$ adding a locally defined well–order of the universe and preserving the statement "There is a proper class of n–huge cardinals" for every $n > 0$ for which this statement is true in the ground model.

One reason why such a preservation of n–hugeness could not possibly work with $\mathcal{P}$ itself is the following: Suppose κ is a huge cardinal, as witnessed by some elementary embedding $j : V \longrightarrow M$ with critical point κ and ${}^{j(\kappa)}M \subseteq M$. Then $H(j(\kappa)^+) \subseteq M$, so if $\leq$ is to be a well–order of $H(\kappa^+)$ definable by some formula $\Phi(x, y)$ with no parameters over $\langle H(\kappa^+), \in\rangle$, then $j(\leq)$ must be a well–order of $H(j(\kappa)^+)^M = H(j(\kappa)^+)^V$ definable over $\langle H(j(\kappa)^+)^V, \in\rangle$ by the same formula $\Phi(x, y)$ (by elementarity of j). But certainly we did not take any steps in the construction of $\mathcal{P}$ for such a coherence to hold between the well–orders added at two different stages of the (second) iteration.

Our approach to proving preservation of n–hugeness will be as usual to ensure that some elementary embedding $j : V \longrightarrow M$ witnessing n–hugeness of some given κ lifts to an n–huge embedding in the extension. It is also easy to see that the success of this approach requires that the lifting arguments in the proof of Theorem 4.1 be incorporated into the very definition of $\mathcal{P}^\dagger$.

Our forcing construction can be easily seen to preserve many instances of both local and global supercompactness. We should point out, however, that we do not know how to handle all cases of local and global supercompactness at once by this forcing. The problem arises when we consider $\kappa < \kappa' < \lambda$ with κ being λ–supercompact and κ' being huge and we want to argue that in the extension both the λ–supercompactness of κ and the hugeness of κ' are preserved.

[36] Of course, $sup(\mathcal{H}_n)$ might well be Ord.

The result we are going to prove can be given the following general formulation.

Theorem 5.1 *(GCH above $\aleph_0$) Let F be a non-overlapping hugeness function. There is a formula without parameters $\varphi(x, y)$*[37] *and there is an ω_2–directed closed class–forcing $\mathcal{P}^\dagger$, definable from F, preserving ZFC and GCH above $\aleph_0$, as well as all cofinalities, and such that*

(1) $\mathcal{P}^\dagger$ forces that there is a well–order $\leq$ of the universe such that

$$\{(a, b) \in H(\kappa^+)^2 : \langle H(\kappa^+), \in\rangle \models \varphi(a, b)\}$$

is the restriction $\leq\restriction H(\kappa^+)^2$ and is a well–order of $H(\kappa^+)$ whenever $\kappa \geq \omega_2$ is a regular cardinal, and

(2) for all $\kappa \in dom(F)$, κ remains $F(\kappa)$–huge after forcing with $\mathcal{P}^\dagger$.

Note that the remark from Section 3 concerning preservation of forcing axioms applies similarly for the present result.

Restricting to non-overlapping hugeness functions as in our hypothesis will guarantee that there are no interferences in the definition of $\mathcal{P}^\dagger$ coming from overlapping embeddings corresponding to different cardinals. Brooke-Taylor ([B]) uses a similar strategy of avoiding possible overlappings of elementary embeddings for building a class-forcing extension in which there is a definable well–order of the universe, while at the same time preserving members of suitably defined sparse families of cardinals satisfying any one of various large cardinal properties (including n–hugeness). The well–order constructed in his model does not admit a local definition in our sense. The coding technique in Brooke-Taylor's construction is also different from the one we use here: He encodes a given bit of information at a suitably chosen cardinal κ by making the diamond–principle $\diamondsuit^*$ hold or fail at κ^+.

The rest of this section describes the proof of Theorem 5.1. As $\mathcal{P}$, $\mathcal{P}^\dagger$ will be a two–step iteration, which we will write $\mathcal{B}^\dagger * \dot{\mathcal{C}}^\dagger$. The first forcing $\mathcal{B}^\dagger$ will add a system of bookkeeping functions for all $H(\alpha^+)$ with $\alpha \in Reg^*$ and it will also pick a suitable ultrafilter $\mathcal{U}$ on $\mathcal{P}(\tilde{\kappa})$ for every $n > 0$ and every $\kappa \in F^{-1}(n)$ (for some cardinal $\tilde{\kappa} > \kappa$) such that the embedding $j_{\mathcal{U}} : V \longrightarrow M$ derived from $\mathcal{U}$ is a minimal witness for the n–hugeness of κ. The second forcing $\dot{\mathcal{C}}^\dagger$ will add a coherent sequence of definable well–orders of $H(\alpha^+)$

[37] In fact, the same formula that works for Theorem 1.1 works also here.

also for $\alpha \in Reg^*$. Same as before, both $\mathcal{B}^\dagger$ and $\dot{\mathcal{C}}^\dagger$ will be reverse Easton iterations of length Ord on which nontrivial things happen only at regular stages.

Both the definition of $\mathcal{B}^\dagger$ and of $\dot{\mathcal{C}}^\dagger$ in $V^{\mathcal{B}^\dagger}$ will proceed very much as in the previous $\mathcal{B}$ and $\dot{\mathcal{C}}$. The only differences will take place at stages $\alpha \in [j(\kappa)_1,\, j(\kappa)_n)$, where $\kappa \in dom(F)$ and $n = F(\kappa)$ and where $j : V \longrightarrow M$ is a minimal witness for the n–hugeness of κ. Note at this point that such a minimal witness j can always be picked in such a way that M is the ultrapower $Ult(V,\mathcal{U})$, for some fine and normal ultrafilter $\mathcal{U}$ on $\mathcal{P}(j(\kappa)_n)$ concentrating on the set of $x \subseteq j(\kappa)_n$ such that $ot(x \cap j(\kappa)_{i+1}) = j(\kappa)_i$ for all $i < n$, and that j is precisely the elementary embedding $j_{\mathcal{U}}$ derived from $\mathcal{U}$.

One first natural step in the definition of $\mathcal{P}^\dagger$ would be to give an explicit description of the $\mathcal{B}^\dagger$–iteration on the interval $[j(\kappa)_1,\, j(\kappa)_n)$ for given n and $\kappa \in F^{-1}(n)$ (where j is a minimal witness for the n–hugeness of κ). Rather than doing that, we are going to assume right away that $\mathcal{B}^\dagger$ has already been defined. We are going to assume as well that $\mathcal{B}^\dagger$ is ω_3–directed closed and preserves ZFC, GCH (above $\aleph_0$) and cofinalities and that it adds a bookkeeping function for $H(\alpha^+)$ for all $\alpha \in Reg^*$. Finally, we will also assume, for every n and every $\kappa \in F^{-1}(n)$, that the generic filter $\tilde{G}$ for $\mathcal{B}^\dagger$ picks a normal and fine ultrafilter $\mathcal{U}_\kappa$ on $[\kappa_n]^{\kappa_{n-1}}$ in V (for cardinals $(\kappa_i)_{0<i\leq n}$ above $\kappa_0 := \kappa$) concentrating on the set of x such that $ot(x \cap \kappa_{i+1}) = \kappa_i$ for all $i < n$, and such that the embedding $j_{\mathcal{U}_\kappa} : V \longrightarrow M$ derived from $\mathcal{U}_\kappa$ is a minimal witness for the n–hugeness of κ in V, and that $j_{\mathcal{U}_\kappa}$ can be lifted to $j^*_{\mathcal{U}^*_\kappa} : V[\tilde{G}] \longrightarrow Ult(V[\tilde{G}],\mathcal{U}^*_\kappa)$ for some $\mathcal{U}^*_\kappa \in V[\tilde{G}]$ which, in $V[\tilde{G}]$, is a normal and fine ultrafilter on $\mathcal{P}(\kappa_n)$ concentrating on the relevant set and definable from $\tilde{G}$.

Under these assumptions we can define $\dot{\mathcal{C}}^\dagger$ in $V^{\mathcal{B}^\dagger}$ and we can prove the desired conclusions for this forcing. It should then be clear how to step back and implement the right definition of $\mathcal{B}^\dagger$ and how to prove the facts about it that we have assumed (the work for $\mathcal{B}^\dagger$ is in fact easier).

5.1 $\dot{\mathcal{C}}^\dagger$

Let us work in $V_1 := V^{\mathcal{B}^\dagger}$ and let us write $\mathcal{C}^\dagger$ for $\dot{\mathcal{C}}^\dagger$. As we said, the $\mathcal{C}^\dagger$–iteration $\langle \mathcal{C}^\dagger_\alpha \,:\, \alpha \in Ord\rangle$ is going to be defined in the same way as the $\dot{\mathcal{C}}$–iteration except on those stages α belonging to the interval $[\kappa_1,\, \kappa_n)$ for some n and some $\kappa \in F^{-1}(n)$.[38] Let us fix such a κ and let us assume for concreteness that $n = 4$.[39] Let $\mathcal{U}$ be the ultrafilter $\mathcal{U}^*_\kappa$ corresponding to the lifting of the elementary embedding $j_{\mathcal{U}_\kappa} : V \longrightarrow M$[40] and let $j : V_1 \longrightarrow M$ be the corresponding elementary embedding. Let also $\kappa_5 = j(\kappa_4)$. Finally, let $\langle \dot{\mathbb{Q}}_\alpha \,:\, \alpha \in Ord \rangle$ denote the sequence of names for posets on which the iteration is being built.

First of all, note that every filter generic for $\mathcal{C}^\dagger_{\kappa_1}$ over V_1 is also generic for the same forcing over M. In $V_1^{\mathcal{C}^\dagger_{\kappa_1}}$, we let $\dot{\mathbb{Q}}_{\kappa_1}$ be the restriction of $(Code^*)^{g_{\kappa_1},\, \mathcal{W}_{\kappa_1}}_{\kappa_1}$ – where, as would be expected, g_{κ_1} denotes a name for a bookkeeping function for $H(\kappa_1^+)^{V_1^{\mathcal{C}^\dagger_{\kappa_1}}}$ canonically obtained from the generic object for $\mathcal{B}^\dagger$ and $\mathcal{W}_{\kappa_1}$ is the union of all well–orders added previously (and similarly for other indices) – to $r^1(\kappa_1)$ for a certain master condition r^1 for $\dot{G}_{\kappa_1}$, which we are going to call the 'canonical' master condition for $\dot{G}_{\kappa_1}$ starting at stage κ_1.

To be more specific, r^1 is, in $V_1^{\mathcal{C}^\dagger_{\kappa_1}}$, the unique condition in $j(\mathcal{C}^\dagger_{\kappa_1})/\dot{G}_{\kappa_1}$ (this is what we mean by 'starting at stage κ_1') extending $j(r) \restriction [\kappa_1,\, \kappa_2)$ for every $r \in \dot{G}_{\kappa_1}$, obtained as in the proof of Lemma 4.2 in the previous section, such that, for all $\alpha \in supp(r^1) = \bigcup\{supp(j(r)) \,:\, r \in \dot{G}_{\kappa_1}\} \backslash \kappa_1$, $r^1(\alpha)$ is forced to be the unique condition in $(Code^*)^{g_\alpha,\, \mathcal{W}_\alpha}_\alpha$ extending all $j(r)(\alpha)$ ($r \in \dot{G}_{\kappa_1}$) with $D(r^1(\alpha))$ being equal to $(sup\{D(j(r)(\alpha)) \,:\, r \in \dot{G}_{\kappa_1}\})+1$ and putting the top element of $D(r^1(\alpha))$ outside the domains of all club–sequences $\vec{C}_i$ and outside all stationary sets S_i ($i < \alpha$), where the $\vec{C}_i$'s and S_i's are the objects being added at the first stage of the iteration corresponding to $(Code^*)^{g_\alpha,\, \mathcal{W}_\alpha}_\alpha$. It is not difficult to see, using the closure of M, that this r^1 is in fact a condition in M.

Similarly, given any ordinal $\alpha \in [\kappa_1,\, \kappa_2)$, we define $\dot{\mathbb{Q}}_\alpha$ in $V_1^{\mathcal{C}^\dagger_\alpha}$ as the restriction of $(Code^*)^{g_\alpha,\, \mathcal{W}_\alpha}_\alpha$ to $r^1(\alpha)$. (This makes sense since, by definition, $V_1^{\mathcal{C}^\dagger_{\kappa_1}}$ forces that every condition in $\mathcal{C}^\dagger_\alpha/\dot{G}_{\kappa_1}$ extends $r^1 \restriction \alpha$.)

Now, by elementarity of j we have, for every ordinal α in the interval $[\kappa_2,\, \kappa_3)$, that $j(\dot{\mathbb{Q}})_\alpha$ is forced over $M^{j(\mathcal{C}^\dagger)_\alpha}$ to be $(Code^*)^{j(g)_\alpha,\, \mathcal{W}_\alpha}_\alpha \restriction j(r^1)(\alpha)$.

The iteration on the interval $[\kappa_2,\, \kappa_3)$ can be roughly described as the

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[38] Note that the values of κ_i $(0 < 1 < n)$ depend in principle on the ultrafilter $\mathcal{U}_\kappa$ picked by the generic object for $\mathcal{B}^\dagger$.

[39] The general case is just the same.

[40] $\mathcal{U}$ is, in V_1, a normal and fine ultrafilter on $[\kappa_4]^{\kappa_3}$ concentrating on the set of x such that $ot(x \cap \kappa_{i+1}) = \kappa_i$ for all $i < 4$.

restriction of $j(\mathcal{C}_{\kappa_2})$ to the 'canonical' master condition for $\dot{G}_{\kappa_2}$ starting at stage κ_2. More precisely, for any given $\alpha \in [\kappa_2, \kappa_3)$, $\dot{\mathbb{Q}}_\alpha$ is defined in $V_1^{\mathcal{C}^\dagger_\alpha}$ to be the restriction of $j(\dot{\mathbb{Q}})_\alpha$ to $r^2(\alpha)$, where r^2 is the 'canonical' master condition for $\dot{G}_{\kappa_2}$ starting at stage κ_2 (and where the property of being this kind of 'canonical' master condition is naturally defined).

Finally, the description of the iteration on the last interval $[\kappa_3, \kappa_4)$ is as one would expect: Given $\alpha \in [\kappa_3, \kappa_4)$, $\dot{\mathbb{Q}}_\alpha$ is forced over $V_1^{\mathcal{C}^\dagger_\alpha}$ to be the restriction of $j(\dot{\mathbb{Q}})_\alpha$[41] to $r^3(\alpha)$, where r^3 is the 'canonical' master condition for $\dot{G}_{\kappa_3}$ starting at stage κ_3, and where again this is defined in the natural way.

The definition of the iteration from, and including, stage κ_4 is the same as for the $\dot{\mathcal{C}}$–iteration until the next member (if any) of $dom(F)$ is encountered.

By arguments as those in Section 3 one can see that $\mathcal{C}^\dagger$ is an ω_2–directed closed class–forcing preserving ZFC, GCH (above $\aleph_0$) and cofinalities, that its tails have the expected degree of directed closure, and that it adds the required coherent system of definable well–orders of $H(\alpha^+)$ for $\alpha \in Reg^*$.

As to the relevant large cardinal preservation, if we assume that $\mathcal{C}^\dagger_\kappa$ preserves the n–hugeness of each cardinal in $F^{-1}(n) \cap \kappa$, for any $n \geq 1$, then it will suffice to see that $\mathcal{C}^\dagger_{\kappa_4+1}$ preserves the 4–hugeness of κ for our fixed 4–huge cardinal κ. Indeed, given a $\mathcal{C}^\dagger_\kappa$–generic G_0 over V_1, the κ–distributivity of $\mathcal{C}^\dagger/G_0$ in $V_1[G_0]$ will entail, for all nonzero $n < \omega$, that all cardinals in $F^{-1}(n) \cap \kappa$ remain n–huge in $V_1[G_0]^{\mathcal{C}^\dagger/G_0}$. Also, since the 4–hugeness of κ is equivalent to the existence of a certain ultrafilter on $[\kappa_4]^{\kappa_3}$ and since $\kappa_4^{\kappa_3} = \kappa_4$ (by the inaccessibility of κ_4), the fact that the tail of the forcing starting at stage $\kappa_4 + 1$ is κ_4^+–distributive in $V_1[G_0]^{\mathcal{C}^\dagger_{\kappa_4+1}/G_0}$ will imply that κ remains 4–huge in the end.

Let thus G be $\mathcal{C}^\dagger_{\kappa_4+1}$–generic over V_1. By the standard arguments it is enough to find in $V_1[G]$ a $j(\mathcal{C}^\dagger_{\kappa_4+1})$–generic filter H over M with $j``G \subseteq H$. The restriction of H to $j(\mathcal{C}^\dagger)_{\kappa_4}$ is going to be just G_{κ_4}. By the definition of the iteration on the stages $\alpha \in [\kappa_1, \kappa_4)$ it is not difficult to verify that G_{κ_4} is indeed generic for $j(\mathcal{C}^\dagger)_{\kappa_4}$ over M and that it contains $j(r)$ for every $r \in G_{\kappa_3}$. Actually, the changes in the definition of $\mathcal{C}^\dagger$ (with respect to the definition of the $\dot{\mathcal{C}}$–iteration) on the interval $[\kappa_1, \kappa_4)$ have been introduced precisely to make this claim true.

[41] Note that this is the poset $((Code^*)_\alpha^{j(g)_\alpha, \mathcal{W}_\alpha} \restriction j(r^1)(\alpha)) \restriction j(r^2)(\alpha)$ as computed in $M^{j(\mathcal{C}^\dagger)_\alpha}$.

It remains to find a $j(\mathcal{C}^{\dagger}_{\kappa_4+1})/G_{\kappa_4}$–generic filter $\tilde{H}$ over $M[G_{\kappa_4}]$ containing $j(r) \restriction [\kappa_4, \kappa_5]$ for all $r \in G$. This can be achieved in two steps very much as in the proof of the Lifting Theorem in the previous section.

The first step is to find a $j(\mathcal{C}^{\dagger}_{\kappa_4})$–generic filter H' over $M[G_{\kappa_4}]$ containing $j(r) \restriction [\kappa_4, \kappa_5)$ for all $r \in G$. For this, we first build a 'canonical' master condition r^* for G_{κ_4} starting at stage κ_4 exactly as in the proof of Lemma 4.2. This condition is in $M[G_{\kappa_4}]$ since ${}^{\kappa_4}M[G_{\kappa_4}] \subseteq M[G_{\kappa_4}]$ holds in $V_1[G_{\kappa_4}]$ and since $\mathcal{C}^{\dagger}_{\kappa_4}$ has size κ_4. Then we note that $\mathcal{P}^M(j(\mathcal{C}^{\dagger}_{\kappa_4}))$ has size, in V_1, equal to $|(\kappa_4^+)^{[\kappa_4]^{\kappa_3}}|^{V_1} = \kappa_4^+$ (by our GCH–assumption) and that therefore, similarly as in the proof of the Lifting Theorem, we can build in $V_1[G]$, in κ_4^+ steps, a filter H' of $j(\mathcal{C}^{\dagger}_{\kappa_4})/G_{\kappa_4}$ containing r^* and meeting all maximal antichains belonging to $M[G_{\kappa_4}]$. The construction certainly does not belong to $M[G_{\kappa_4}]$, but each of its proper initial segments does.

The second and final step is to find a $j(\dot{\mathbb{Q}}_{\kappa_4})_{G_{\kappa_4} * H'}$–generic filter H'' over $M[G_{\kappa_4} * H']$ containing $j(r(\kappa_4))_{G_{\kappa_4} * H'}$ whenever r is a condition in G, and we can do that basically as in the proof of Lemma 4.3: Again we can find a sequence $\langle q_i : 0 < i < \kappa_4^+ \rangle \in V_1[G]$ of "partial master conditions" for $G(\kappa_4)$, where again this means that each $q_i \in M[G_{\kappa_4} * H']$ is a master condition for H_i – where H_i is the generic filter for $\mathcal{Q}_i$ for each i, and where $\langle \mathcal{Q}_i : i \leq \kappa_4^+ \rangle$ is the $< \kappa_4$–supported iteration leading to $Code^{g_{\kappa_4}, \mathcal{W}_{\kappa_4}}_{\kappa_4}$ – and that $q_i = q_{i'} \restriction j(i)$ for all $i < i' < \kappa_4^+$.

Again we use the $j(\kappa_4^+)$–chain condition of $j(\dot{\mathbb{Q}}_{\kappa_4})_{G * H'}$ in $M[G_{\kappa_4} * H']$ and the fact that $j``\kappa_4^+$ is cofinal in $j(\kappa_4^+) = (\kappa_5^+)^M$[42] to build the required filter H'' in κ_4^+ steps making sure to always stay compatible with all members of our sequence of partial master conditions and making sure that we eventually hit them all. This finishes the proof of Theorem 5.1.

The results in this article leave several questions unanswered. We will finish with a list of some of them.

Question 5.1 *Does the forcing $\mathcal{P}$ from Theorem 1.1 preserves all instances of singular supercompactness? That is, does $\mathcal{P}$ preserve the λ–supercompactness of κ whenever $\lambda > \kappa$ is singular and κ is a λ–supercompact cardinal in the ground model?*

Question 5.2 *Assume GCH. Is there any poset preserving GCH and co-*

[42]Which follows from the fact that every function from $[\kappa_4]^{\kappa_3}$ into κ_4^+ is bounded by some ordinal in κ_4^+.

finalities and adding a well–order of $H(\omega_2)$ definable over $\langle H(\omega_2), \in\rangle$ by a parameter–free formula?

Question 5.3 *Is it possible to force a locally defined well–order of the universe, in the sense of Theorem 1.1, while at the same time preserving all (or many) members of some interesting class of large cardinals, and while also forcing combinatorial principles holding in L, like morasses or (versions of) $\square_\kappa$?*

Question 5.4 *Is it possible to force a locally defined well–order of the universe while at the same time preserving all huge cardinals?*

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