Ordinal Recursion Theory

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Introduction

In a fundamental paper, Kreisel and Sacks [1965] initiated the study of "metarecursion theory", an analog of classical recursion theory where ω is replaced by Church-Kleene ω_1, the least non-recursive ordinal. Subsequently, Sacks and his school developed recursion theory on arbitrary Σ_1-admissible ordinals, now known as "α-recursion theory".

In Section I of the present article, we present the basic concepts and techniques of this theory, putting particular emphasis on the main new ideas that have been introduced to study recursion-theoretic problems assuming only Σ_1-admissibility on a domain greater than ω. As Σ_1-admissibility is easily lost under relativization, we turn to "β-recursion theory" (Section II) which attempts to develop recursion theory on arbitrary limit ordinals. In Section III, the final part of this article, we take up the topic of "admissibility spectra", where instead of studying the definability of subsets of a fixed Σ_1-admissible ordinal, we ask: given a set X, which are the ordinals Σ_1-admissible relative to X ?

The reader will notice that Jensen's work on the fine structure theory of Gödel's L features prominently throughout. Indeed a major development of ordinal recursion theory is the infusion of set-theoretic ideas in studying recursion-theoretic problems. The unmistakeable presence of a strong set-theoretic flavor in the subject of admissibility spectra is especially pronounced. We thus view the appearance of Jensen's paper (Jensen [1972], preliminary copies of which had been circulated earlier), at a time when ordinal recursion theory was being developed, to be a fortuitous happening.

Some of the techniques and ideas which were invented in ordinal recursion theory have recently found applications in "recursion theory on fragments of Peano arithmetic". This is an unexpected turn of events which signal a basic unity among various fields in recursion theory and fine structure theory. We touch briefly on this work at the end of Section I.

I. α-Recursion Theory

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We begin with some basic notions. Recall Gödel's constructible universe L, defined as $\cup\{L_\alpha | \alpha \text{ an ordinal}\}$. A limit ordinal α is Σ_n-*admissible* if L_α satisfies the replacement axiom for Σ_n formulas (with parameters in L_α) in ZF set theory. If α is Σ_n-admissible for some $n \geq 1$, there is a $\Sigma_1(L_\alpha)$ bijection between α and L_α, allowing one to identify these two objects if and when necessary. Σ_1-admissible ordinals are sometimes referred to simply as admissible ordinals. Unless otherwise specified, we fix α to be an admissible ordinal henceforth.

A set $K \subset \alpha$ is α-*finite* if $K \in L_\alpha$. A function is partial α-*recursive* if its graph is $\Sigma_1(L_\alpha)$. A set is α-*recursively enumerable* (α-RE) if it is the domain of a partial α-recursive function. $A \subset \alpha$ is α-recursive if both A and $\alpha \setminus A$ are α-RE. In terms of definability, a set is α-recursive if and only if it is $\Delta_1(L_\alpha)$. It is α-finite if and only if it is α-recursive and bounded in α.

All the basic results in classical recursion theory, for example those covered in the first seven chapters of Rogers [1967], hold for all Σ_1 admissible ordinals. Thus a set $K \subset \alpha$ is RE if and only if it is the range of a total α-recursive function; there is an effective (i.e. $\Sigma_1(L_\alpha)$ definable) enumeration of all α-finite sets and all α-RE sets; Kleene's Recursion Theorem is true for each α. We denote by W_e the eth α-RE set and by K_e the eth α-finite set under the respective effective enumerations.

Reducibility The notion of reducibility provides a means of comparing the relative complexity of subsets of α. Given $A \subset \alpha$, define by the collection of neighborhood conditions of A the set

$$N(A) = \{(c, d) | K_c \subset A \ \& \ K_d \subset \alpha \setminus A\}.$$

We say that A is α-RE in $B \subset \alpha$ if there is an e such that for all $x < \alpha$,

$$x \in A \leftrightarrow (\exists c)(\exists d)[(x, c, d) \in W_e \ \& \ (c, d) \in N(B)\}.$$

A is *weakly* α-*recursive in* B, written $A \leq_{w\alpha} B$, if A and $\bar{A}$ are α-RE in B. Define by A^* the set $\{u | K_u \subset A\}$. Then A is α-*recursive in* B, written $A \leq_\alpha B$, if A^* and $\bar{A}^*$ are α-RE in B.

Thus A is α-recursive in B provided there is an algorithm such that for any given α-finite set K, it is possible to use the algorithm, with B as an oracle, to conclude within α-finite time whether K is a subset of A or disjoint from A. It is not difficult to verify that $\leq_\alpha$ is reflexive and transitive. A and B are said to have the same α-degree, written $A \equiv_\alpha B$, if $A \leq_\alpha B$ and $B \leq_\alpha A$. Although $\leq_{w\omega}$ is equivalent to $\leq_\omega$ and therefore transitive, $\leq_{w\alpha}$ is not transitive in general.

The least complicated α-degree, which we denote by $\mathbf{0}$, is the α-recursive degree, which consists of α-recursive sets. A degree is an α-RE degree if it contains an α-RE set. There is a *greatest* α-RE degree $\mathbf{0}'$ which contains the α-RE set $\emptyset' = \{(x, e) | x \in W_e\}$, in which every α-RE set is α-recursive.

There is an analog of Church's thesis for α-recursion theory which we shall appeal to in this article. This thesis allows a more informal presentation of the topics to be covered, emphasizing intuition over formalism.

The key motivation of ordinal recursion lies in the search for a "generalized" recursion theory. It is evident that the notion of effective computation applies to a wider class of mathematical structures, as exemplified in Kleene's work on ordinal notations (Kleene [1938]). Kreisel and Sacks [1965] initiated the study of recursion theory on Church-Kleene ω_1, and this led to the subsequent introduction of the theory of Σ_1 admissible ordinals by Sacks and his school.

A closer examination reveals that if the full replacement axiom is assumed in L_α, then many difficult proofs in classical recursion theory go through almost routinely, without major modifications of the classical construction (there are exceptions: cf. the section on maximal sets). ¿From the point of view of effective computation, where "Σ_1"-ness is identified with "effectively enumerable", it should be sufficient to assume only Σ_1-replacement axiom to arrive at a satisfactory recursion theory (though all is not lost even when this crucial assumption is removed in β-recursion theory, see Section II). This view is supported by the successful solution of Post's problem for all admissible ordinals (Theorem 2 below).

We give here some examples of admissible ordinals:

(a) $\alpha = \omega$, the classical case. Then α is Σ_n-admissible for all $n < \omega$. The same conclusion holds for any regular constructible cardinal;

(b) $\alpha = \omega_1^{CK}$, Church-Kleene ω_1. Here α is Σ_1 but not Σ_2-admissible. There is also a $\Sigma_1(L_\alpha)$ map from α into ω (ω is called the Σ_1-projectum of α). A subset of ω is α-RE if and only if it is Π_1^1 definable;

(c) $\alpha = \delta_2^1$, the least ordinal which is not the order type of a Δ_2^1 set of natural numbers. In this case a set of natural numbers is α-RE if and only if it is Σ_2^1 definable.

(d) $\alpha = \aleph_\omega^L$, the ωth constructible cardinal of L. There is a $\Sigma_2(L_\alpha)$ cofinal map from ω into α. Hence α is not Σ_2-admissble. On the other hand, every infinite cardinal in a well-founded model of ZF is Σ_1-admissible.

Regularity

A set $A \subset \alpha$ is *regular* (in α) if its restriction to every $\gamma < \alpha$ is α-finite. It follows that every set of natural numbers is regular. On the other hand, in ω_1^{CK}, Kleene's $\mathcal{O}$, a complete Π_1^1-set of natural numbers, is not regular (in ω_1^{CK}), even though it is bounded and ω_1^{CK}-RE. Non-regularity is a major feature which distinguishes ordinal recursion theory from the classical theory. Non-regular α-RE sets are sets with bounded parts which cannot be enumerated in α-finite time. Their existence renders some of the standard techniques ineffective. Nevertheless, at least for α-RE sets and for the study of α-degrees, this difficulty can be circumvented:

Theorem 1 (Sacks [1966]) *Let α be admissible. Then every α-RE degree contains a regular α-RE set.*

Maass [1978] showed that there is a parameter free Σ_1 function f such that for any α and $e < \alpha$, W_e and $W_{f(e)}$ have the same α-degree and $W_{f(e)}$ is regular.

Definability

Jensen's work on the fine structure of L [1972] turns out to be a key component in the development of ordinal recursion theory, a development which arguably exemplifies the successful integration of set-theoretic and recursion-theoretic ideas. In retrospect, the secret to the solutions of such basic problems as Post's problem, Sacks Splitting Theorem, and the Density Theorem for all admissible α, rests on the insight that the complexities of the classical constructions, with the intervention of fine structure theory, may be refined to achieve the goals, provided that one chooses the appropriate definable objects within α to carry out the necessary priority arguments. On the other hand, in certain cases where such an approach fails, it is shown that the problems being considered have negative solutions. Problems such as the existence of maximal sets, and ordering of α-degrees above $\mathbf{0}'$, are examples.

We list here several important objects in fine structure theory that play pivotal roles in ordinal recursion. Let $B \subset \alpha$.

The Σ_n-cofinality of (L_α, B) is defined to be the least γ for which there is a $\Sigma_n(L_\alpha, B)$ function from γ cofinally into α. We denote this ordinal by $\kappa_n(B)$, or simply write it as Σ_n-cofinality (α, B). Clearly $\kappa_n(\emptyset) = \alpha$ if and only if α is Σ_n-admissible.

The Σ_n^B-projectum of α, denoted $\alpha_n^*(B)$ (or sometimes Σ_n-projectum (α, B)), is the least ordinal $\gamma \leq \alpha$ for which there is a $\Sigma_n(L_\alpha, B)$ map from α into γ. If $B = \emptyset$, Jensen's theory provides several characterizations of this ordinal: (a) it is the largest limit ordinal less than or equal to α in which every bounded $\Sigma_n(L_\alpha)$ set is α-finite; (b) it is the least ordinal γ for which there is a $\Sigma_n(L_\alpha)$ map from a subset of γ onto α. When B is α-RE and regular, the above characterization continues to hold with $\Sigma_n(L_\alpha)$ replaced by $\Sigma_n(L_\alpha, B)$, given and used in Shore's proof of the Density Theorem [1976].

We use the notations α_n^* and κ_n when B is empty. When $n = 1$, we omit the subscript 1 and simply write α^* and $\alpha^*(B)$ instead. In α-recursion theory, it is important to present the set of requirements with as short a list as possible. The ordinal α^* or $\alpha^*(B)$ are often used for this purpose.

We say that $\lambda < \alpha$ is an α-cardinal if there is no α-finite injection of λ into a smaller ordinal. If $\alpha_n^* < \alpha$ (or if $\kappa_n(B) < \alpha$), it is not difficult to prove that it is an α-cardinal. And $\alpha^* < \alpha$ implies that it is the greatest α-cardinal.

A set B is *hyperregular* if $\kappa_1(B) = \alpha$. In other words, B is hyperregular if (L_α, B) is a Σ_1-admissible structure. It is not difficult to verify that every α-recursive set is hyperregular. However, there are α-RE sets which do not satisfy hyperregularity. As an example, consider the set B of non-cardinals in $\alpha = \aleph_\omega^L$. This is an α-RE set whose complement is of order type ω. Then $f(n) = n$th member of $\alpha \setminus B$ is a function weakly α-recursive in B, mapping ω unboundedly into α. It turns out that for $\alpha = \aleph_\omega^L$, the only nonhyperregular α-RE set is of complete degree $\mathbf{0}'$, and every set of α-degree above this is non-hyperregular, while any set which does not compute B defined above is hyperregular (under the axiom of constructibility). In particular, every incomplete α-RE set is hyperregular.

Hyperregularity is a strong condition which ensures that computations carried out on α-finite sets using oracles are completed in α-finite steps. Its recursion-theoretic consequences are significant: For example, for the α considered above, there is no incomplete

α-RE set whose "jump" (an analog of the classical notion) is strictly above $\emptyset'$ (hence no incomplete "high set") (Shore [1976a]). On the other hand, many tools, such as priority arguments, are not relatiivizable to non-hyperregular sets. This introduces additional complications to the study of α-recursion theory. Different techniques are needed in many cases and, in the most extreme case, non-hyperregularity leads to radically different degree-theoretic results (see Section 5 below).

The α-Finite Injury Priority Method

The method of finite injury priority argument introduced by Friedberg and Muchnik to solve Post's problem marked the advent of modern recursion theory. This method has since been joined by a variety of highly complex and ingenious techniques invented to handle problems about RE sets and their degrees, of which the Friedberg-Muchnik proof is now seen to be the simplest. Solving Post's problem may indeed be considered the first important test for any reasonable ordinal recursion theory.

Theorem 2 (Friedberg-Muchnik Theorem) *Let α be an admissible ordinal. There exist α-RE sets A and B with incomparable α-degrees.*

Corollary (Solution of Post's Problem) *There exists an incomplete, α-RE and non-α-recursive degree.*

We sketch the solution by Sacks and Simpson [1972]. This is a proof which has a strong model-theoretic and set-theoretic flavor, in contrast to that of Lerman [1972] which has a stronger recursion-theoretic tilt (it is worth noting that Lerman's approach may be refined to provide a parameter free construction of the sets A and B, yielding a uniform solution, for all admissible ordinals, to Post's problem (Lerman, unpublished)).

Proof of Theorem 1: Consider requirements of the type

$$R_e : \{e\}^A \neq B$$

and those with the roles of A and B reversed. The basic strategy is to diagonalize against equalities whenever possible while preserving computations, respectingg requirements of higher priority if and when necessary. The strategy succeeds in the classical theory because (a) for any e_0, there is a stage after which no requirement of higher priority than e_0 gets injured, and (b) it can be established that each requirement R_e gets imjured at most finitely many (indeed 2^e) times. Closer inspection shows that (a) is essentially a Σ_2 condition, and when satisfied, is sufficient to derive (b). What the construction demands then is for a Σ_1-admissible ordinal α to perform a Σ_2 task. (We will see later that with the Density Theorem, the required task is even more onerous—at almost the Σ_3 level).

Since α is in general not Σ_2-admissible, a straighforward adaptation of the original approach will clearly fail. Instead, the following two lemmas provide an insight into how the difficulties may be overcome:

Sacks-Simpson Lemma *Let κ be a regular α-cardinal. Suppose that K is an α-finite set of α-cardnality less than κ such that $\{I_d | d \in K\}$ is a simultaneous α-RE sequence of α-finite sets each of which has α-cardinallity less than κ. Then $\cup_{d \in K} I_d$ is α-finite and of α-cardinality less than κ.*

An ordinal $\sigma < \alpha$ is said to be α-stable if L_σ is a Σ_1-elementary substructure of L_α.

α-Stability Lemma *If $\omega < \alpha = \alpha^*$, then it is a limit of α-stable ordinals.*

The first step to the solution is to provide a short indexing of requirements with its associated list of priorities. There are several cases to consider. First suppose that α has a greatest α-cardinal κ.

(a) $\kappa = \alpha^* < \alpha$. In this case we use the Σ_1-projectum α^* of α to provide a list of requirements. Let p be a one-one α-recursive map from α into α^*. Requirement R_d is said to have higher priority than requirement R_e if $p(d) < p(e)$. This shorter list of indices ensures that every α-RE set bounded in α^* is α-finite, an essential feature that is needed during the inductive stage to verify that every requirement is satisfied.

Now commence with the construction using the revised indexing of requirements. For each $e < \alpha$, let I_e denote the injury set (defined in the usual sense) associated with R_e. The main observation here is that if there is a regular α-cardinal $\rho \leq \alpha^*$ such that $p(e) < \rho$ and $p(d) < p(e)$ implies that I_d has α-cardinality less than ρ, then the Sacks-Simpson Lemma ensures that I_e has α-cardinality less than ρ as well. This is sufficient to show that the requirement with the highest priority after that R_e is injured less than ρ times. Induction hypothesis then allows one to conclude that every requirement is eventually satisfied.

By the α-Stabiulity Lemma 2.4, let β be the order type of α-stable ordinals above κ.

(b) $\alpha^* = \alpha$. If $\beta = \alpha$ then we use the identity function for priority listing, and modify the classical construction slightly. Lemma 2.4 and the construction provides the necessary tool to argue that every requirement settles down before the next α-stable ordinal. If $\beta < \alpha$, then there is a $\Sigma_2(L_\alpha)$ bijection p between α and $\kappa \cdot \beta$ (since the property of "being α-stable" is $\Pi_1(L_\alpha)$). We use $\kappa \cdot \beta$ to index the requirements and say that R_d has higher priority than R_e if $p(d) < p(e)$. The positions of the priorities are given by an α-recursive approximation p' of p. This gives meaning to "the priority of R_e at stage σ is ν". The "final priority" of R_e is then $p(e)$, which is the limit of $p'(\sigma, e)$ as σ tends to α.

Construction proceeds as before, using p' to guide the priority ordering at each stage. The rules governing the injury of requirements in order of priority at each stage are observed. Exploiting the property of Σ_1-stability, coupled with Lemma 2.4, ensures that all requirements of priority at least $\kappa \cdot \nu$ settle down by the $\nu + 1$-th α-stable ordinal.

Finally, if α is a limit of α-cardinals (analogous to ω), one uses an indexing provided by the identity function on α. The argument then proceeds as in Case (a).

The Density Theorem

Theorem 3 (Shore [1976]). *Let* $\mathbf{b} < \mathbf{c}$ *be* α*-RE degrees. Then there is an* α*-RE degree* $\mathbf{a}$ *such that* $\mathbf{b} < \mathbf{a} < \mathbf{c}$.

This theorem is one of the first successful liftings of infinite injury priority argument to ordinal recursion theory. We sketch the key ideas here. Fix $B <_\alpha C$ to be regular α-RE sets (Theorem 1). An α-RE set A of intermediate α-degree is to be constructed.

(Shore Incompleteness Lemma) *Supposen* B *is an incomplete* α*-RE set. Then* $\kappa_1(B) \geq \alpha^*(B)$*. Furthermore there is a* $\Sigma_1(L_\alpha, B)$ *map from* $\kappa_1^*(B)$ *onto* α.

Proof. We sketch the proof of the first half of the lemma. Assume $\kappa_1^*(B) < \alpha^*(B)$. Let D be a regular α-RE set. We show that $D \leq_\alpha B$. Fix $g : \kappa_1(B) \to \alpha$ to be cofinal. Consider

$$K = \{(\gamma, \delta) | D \cap g(\gamma) \subset D^{g(\delta)}\}.$$

Now K is a $\Pi_1(B)$ set bounded below $\alpha^*(B)$, and so is α-finite. Using it as a parameter set, we see that $D \leq_\alpha B$. Hence B is complete.

The Lemma says essentially that if B is incomplete, then (L_α, B) is a *weakly admissible* structure. Weak admissibility allows many $\Sigma_2(B)$ constructions, with suitable modifications, to go through (for example Post's problem in β-recursion theory, cf. Section II).

There are essentially three key ingredients used in the proof of Theorem 3: The use of $\alpha^*(B)$ for a sufficiently short listing of the set of requirements; the exploitation of the blocking technique, in which requirements are grouped into $\kappa_2(B)$ many blocks of the same priority; and the use of $\kappa_1(B)$ and its associated cofinal function to measure lengths of agreements between computations in the course of the construction. We elaborate the points below.

(a) Let $p \leq_{w\alpha} B$ be an injection from α into $\alpha^*(B)$. There is a simultaneous α-recursive approximation $\{p_\sigma\}$ of p such that for all x, $p_\sigma(x) = p(x)$ for all sufficiently large σ. Requirements are given a short list of length $\alpha^*(B)$ using p. The principal feature of this ordinal exploited in the proof of the Density Theorem is that every set α-RE in B and bounded below $\alpha^*(B)$ is α-finite.

(b) In the construction there are altogether $\kappa_2(B)$-many blocks of requirements. Requirements in the same block are accorded the same priority. This reduces at once the number of injury sets to a manageable level. During verification step, one does induction on $z < \kappa_2(B)$, and argues first of all that the set of permanent injuries inflicted on the computations is bounded within each block, and secondly that such a bound may be found in a $\Sigma_2(L_\alpha, B)$ manner as a function of z. The fact that $z < \kappa_2(B)$ then ensures that a uniform bound exists for all blocks $z' \leq z$.

(c) $\kappa_1(B)$ is also known as the *recursive cofinality of* B. Let $k : \kappa_1(B) \to \alpha$ be a cofinal map weakly α-recursive in B. There is a simultaneous α-recursive sequence of α-recursive functions $\{k_\sigma\}$ such that for each $y < \kappa_1(B)$, $k_\sigma|y = k|y$ for all sufficiently large σ. By the Shore Incompleteness Lemma we may choose k to be a surjective map. The calculations of lengths of agreement between two computations will be based on $k|y$, for $y < \kappa_1(B)$. Furthermore, the surjectivity of k allows the construction to pick up every α-finite set contained in C. During the construction, such sets are coded into A (which in turn causes complications). Each of these strategies is designed to ensure that should B be able to compute A, or A compute C, then in fact $C \leq_\alpha B$, a contradiction.

There are two types of requirements. The positive requirements $\{e\}^B \neq A$ for each e, which attempts to ensure that the set A to be constructed is not α-recursive in B, and negative requirements $\{e\}^A \neq C$ for each e, which arranges that C is not α-recursive in A. These requirements are grouped into blocks indexed by $\kappa_2(B)$ with the aid of the following lemma. Denote $B^{<\sigma}$ to be the set of ordinals enumerated in B before stage σ. Assume $\kappa_1(B) > \omega$. Then $B^{<\sigma} = B \cap \sigma$ (i.e. σ is B-correct) for unboundedly many σ.

Blocking Lemma *There is a function $g : \kappa_2(B) \to \alpha^*(B)$ which is $\Sigma_2(L_\alpha, B)$, together with a simultaneous α-recursive sequence $\{g_\sigma\}$ of g such that*

(a) $g_\sigma(z) \geq g(z)$ for all sufficiently large σ;

(b) For all $z < \kappa_2(B)$, and for all sufficiently large B-correct σ, $g_\sigma|z = g|z$.

We say that $\{g_\sigma\}$ is a *tame approximation* of g in view of the Blocking Lemma. We shall only consider $\kappa_1(B) > \omega$ here. The case when $\kappa_1(B) = \omega$ is considerably simpler. We say that a requirement with index e is in block z if $p(e) < g(z)$. With the blocking lemma, it makes sense using $\{g_\sigma\}$ to say that a requirement is "in block z at stage σ". Indeed by the Blocking Lemma, if a requirement is in block z, then it is in block z for all sufficiently large B-correct σ. Furthermore, by tameness property, this occurs uniformly in z,

As in the classical construction, we code the set B into the even part of A, and think of the odd part of A as consisting of triples (z, x, σ). The three ordinals are related via a length of agreement function: Suppose at stage σ there is an e in block z, with $A^{<\sigma}|k_\sigma(x)$ agreeing with $\{e\}_\sigma^{B^{<\sigma}}|k_\sigma(x)$. Then ordinal $k_\sigma(x)$ is said to be the length of agreement of computation for e at stage σ.

This length of agreement is destroyed (i.e. computation restarts) at a later stage if new elements below σ enter either A or B, since such occurances are likely to invalidate any computations reached so far. Those agreements which are never destroyed are called permanent. They turn out to be α-recursively identifiable by B. Precaution is taken so that if (z, x, σ) enters A, then $K_{k_\sigma(x)}$, the $k_\sigma(x)$th α-finite set, is contained in C. Since C is regular, this can be verified at some stage. And since k is a surjective map, all the relevant α-finite sets will be considered at some stage. The objective here is to code enough of C

into A so as to obtain the following lemma:

Lemma

(i) *For each* e *in block* z, $\{e\}^B \neq A$;

(ii) *Within each block, the permanent lengths of agreemnt are bounded below* α.

Proof: The idea is that if (i) fails, then using B to identify permanent lengths of agreement, one is able to compute C (which are coded in A) from B, a contradiction.

A special feature of the blocking technique is that requirements within the same block work together to achieve collectively a longer length of agreement. To prove (ii), one uses the fact that the set K of e's in block z for which $\{e\}^B$ is total on $k(x)$ and not equal to $A|k(x)$ after an agreement had been reached earlier, is a set Σ_1 in B and bounded below $\alpha^*(B)$, hence α-finite. It is then sufficient to consider only $e \in z \setminus K$. Repeating an argument similar to that for (i) above on the set $z \setminus K$, but this time collectively on all the computations that provide permanent lengths of agreement, shows that if (ii) is false, then again $C \leq_\alpha B$.

Consider requirements e in block z. A negative requirement is intended to preserve computations of the form $\{e\}^{A^{<\sigma}}|k_\sigma(x)$ to make it different from $C|k_\sigma(x)$. At stage σ, each requirement e is assigned a marker which is placed at the least ordinal $\nu_{e,\sigma}$ greater than the negative facts used about $A^{<\sigma}$ in the computation above. The idea is that for as long as markers stay, then no new ordinal below their positions is allowed to enter A. On the other hand, should a new element below σ enter B at a later stage, then all markers assigned at stage σ are removed, clearing the way for ordinals below $\nu_{e,\sigma}$ to be added to A if and when necessary. These markers may reappear subsequently (say at $\zeta > \sigma$) occupying different positions provided that, roughly speaking, there is a *collective* length of agreement between $C^{<\zeta}$ and $\{\{e\}_\zeta^{A^{<\zeta}}\}$, e in block z, longer than those achieved before.

The construction of the set A involves the coding of B into the even part of A (to ensure $B \leq_\alpha A$), and the manipulation of positive and negative requirements. A negative requirement (marker) is permanent if it is never removed. The set of permanent negative requirements within a block has to be bounded else one argues that C is α-recursive in B. Furthermore, it can be arranged that within a block, the limit inferior of the positions of the markers that stay behind at the end of each stage of construction is bounded below α, and may be computed from C. This allows unboundedly many opportunities for ordinals above certain level to enter A, and is crucial to the success of the construction. With this it is also possible to show that C is not α-recursive in A.

The final thread is to establish $A \leq_\alpha C$. This is achieved through arranging the construction so that the set of permanent negative requirements is α-recursive in C. We omit the details.

Non-existence of Maximal Sets

In this and the next section, we give two examples of problems which have negative solutions in ordinal recursion. The first, due to Lerman [1974], states roughly that there is a lattice-theoretic property of α-RE sets which is inherently definably countable. More precisely,

Theorem 4 *There is a maximal α-RE set if and only if there is a function f which is S_3-definable mapping α onto ω.*

The notion of maximality is derived from the classical one: M is maximal if and only if its complement $\bar{M}$ is unbounded, and there is no α-RE set which splits $\bar{M}$ into two non-α-finite parts. We say that f is S_3-definable if there is an α-recursive function f' such that for all $x < \alpha$,

$$\lim_{\tau}\lim_{\sigma} f'(\tau, \sigma, x) = f(x).$$

Thus in our terminology, we may say that there is a maximal α-RE set if and only if the S_3-projectum of α is ω. This complete characterization of the existence of maximal sets raises a very interesting but apparetly quite difficult question: is there a classification of recursion-theoretic problems which are inherently linked to the cardinality of the universe ?

The following weak form of Theorem 4 shows how the size of α has a bearing on the existence of maximal sets:

Theorem 5 *If there is a maximal α-RE set, then α is countable.*

To prove this theorem, we consider κ_2 which is Σ_2-cofinality (α), and α_2^*, the Σ_2-projectum of α.

Proof of Theorem 5: Let M be a maximal set. We first claim that $\kappa_2 \geq \alpha_2^*$. To do this, build a simultaneous α-recursive sequence of pairwise disjoint α-finite sets $\{H_\nu\}_{\nu<\kappa_2}$ such that $\bar{M} \cap (\cup_{\nu<\kappa_2} H_\nu)$ is not α-finite, and each H_ν contains at most one member of $\bar{M}$. Now the set

$$K = \{\nu | H_\nu \cap \bar{M} \neq \emptyset\}$$

is a Σ_2 definable subset of κ_2. If $\kappa_2 < \alpha_2^*$, then K is α-finite, in which case it is possible to split K into two non-empty parts K_1 and K_2 so that $\cup_{\nu\in K_1} H_\nu$ and $\cup_{\nu\in K_2} H_\nu$ each contains a non-α-finite unbounded subset of $\bar{M}$, contradicting maximality of M. Thus $\kappa_2 \geq \alpha_2^*$.

Next we argue that κ_2 is in fact countable. To do this, let β be the order type of $\bar{M}$. Partition α into an α-recursive sequence of pairwise disjoint α-RE sets $\{A_\nu\}_{\nu<\kappa_2}$. Define

$$B_\nu = \{\gamma | \exists\sigma[\text{order type of } \gamma \setminus M^\sigma] \in A_\nu\}.$$

It can be shown that for each ν, unboundedly many members of $\bar{M}$ belongs to B_ν. By maximality, $\bar{M} \setminus B_\nu$ is α-finite for all $\nu < \kappa_2$. Let $h(\nu)$ be the supremum of this α-finite set. We claim:

For each $y \in \bar{M}$, there are only finitely many ν's such that $h(\nu) < y$.

Fix a $y \in \bar{M}$. Suppose there are infinitely many ν's such that $h(\nu) < y$. This means that $y \in B_\nu$ for each of these ν's. Since the A_ν's are pairwise disjoint, y must have entered the B_ν's at different stages σ exhibiting infinitely many different order types for $y \setminus M^\sigma$. But this contradicts the well-ordering of ordinals. This proves the claim.

It follows from the claim that κ_2 and hence α_2^* is countable. We conclude that α is countable.

Post's Problem Above $\emptyset'$ And Set-theoretic Methods

The second example in the negative direction concerns α-degrees above $\mathbf{0}'$. We discuss how Silver's work on singular cardinals of uncountable cofinality when merged with Jensen's theory is exploited to derive a strong structural difference in degree theory for a class of admissible ordinals. Further applications are discussed in Section II.

The problem to consider is simple: Does Post's problem hold above any α-degree ? In other words, for any set A, do there exist sets B and C RE in A such that $A <_\alpha B$ and $A <_\alpha C$, and B, C have incomparable α-degrees ? A related, and more general, question asks if there exist incomparable α-degrees above any given degree. A basic theorem of Kleene-Post states that this holds when $\alpha = \omega$. For $\alpha = \aleph_{\omega_1}$, the answer turns out to be negative in a very strong way:

Theorem 6 (Friedman [1981]) *Assume V=L. If $\alpha = \aleph_{\omega_1}$, then the α-degrees above* **0'** *are well-ordered, with successor provided by the jump operator.*

Proof: A complete proof requires a heavy dose of Jensen's fine structure theory. We give a sketch here of the proof of the easy half. Given $A, B \geq_\alpha \emptyset'$, define the growth function g_A of A so that $g_A(\delta)$ is the least ordinal u such that $A \cap \aleph_\delta \in L_u$. Define g_B similarly. Then either $g_A(\delta) \geq g_B(\delta)$ for stationarily many δ, or $g_A(\delta) < g_B(\delta)$ for closed and unboundedly many δ. Silver's analysis of growth functions [1974] shows that in the former case $A \leq_\alpha B$, while in the latter case $A >_\alpha B$. As a consequence, if $A <_\alpha B$, then $g_A(\delta) < g_B(\delta)$ for a closed and unbounded set of δ's. Using this, the well-ordering property follows from the observation that a countable intersection of closed and unbounded sets is closed and unbounded. Hence a countable descending chain of α-degrees above $\emptyset'$ has a least element.

In Friedman [1981] it is shown that the well-ordering of α-degrees above $\mathbf{0}'$ is actually achieved through the jump operator, and these α-degrees are represented by "master codes" in Jensen's sense.

The situation for countable cofinality turns out to be radically different. Harrington and Solovay have independently shown that incomparable α-degrees exist above $\mathbf{0}'$ for $\alpha = \aleph_\omega^L$. The following result (Chong and Mourad [in preparation]) solves Post's problem above $\mathbf{0}'$:

Theorem 7 *Let $\alpha = \aleph_\omega^L$. Then there exist sets A and B, α-RE in and above $\emptyset'$, which are of incomparable α-degree.*

Since such sets A and B are necessarily non-hyperregular, the classical approach of finite injury argument no longer applies. Instead, a refinement of the method first used in establishing the Friedberg-Muchnik Theorem for $B\Sigma_1$-models of arithmetic (Chong

Mourad [1992]), called unions of intervals, is exploited to ensure that all requirements are met within ω-steps. Since A and B lie above $\emptyset'$ and are therefore able to "climb up" α in ω-many steps, the construction succeeds.

Applications to Fragments of Peano Arithmetic

One of the most interesting applications of techniques of α-recursion theory in recent years has been in the area of *reverse recursion theory*. Starting with the basic axioms of Peano arithmetic without the induction scheme, one asks:

> What is the proof-theoretic strength of a given theorem in recursion theory ? In particular, how much of the induction scheme is required to prove the theorem ?

Kirby and Paris [1978] have provided a hierarchy of theories of increasing proof-theoretic strength, and this hierarchy forms the basis for the study of subrecursive recursion theory. Let P^- be axioms of Peano arithmetic with exponentiation but without the induction scheme. Let $I\Sigma_n$ denote the induction scheme for all Σ_n formulas, and $B\Sigma_n$ to be replacement (collection) axiom for Σ_n formulas: every Σ_n-function maps a "finite set" (in the sense of the given model) onto a "finite set". Then with P^- as the underlying theory, one has $(n \geq 0)$ $B\Sigma_{n+1}$ to be strictly stronger than $I\Sigma_n$, which is in turn strictly stronger than $B\Sigma_n$.

Slaman and Woodin [1989] initiated the study of recursion theory on fragments of Peano arithmetic. We illustrate here how techniques of ordinal recursion theory are adapted to investigate problems in this area.

Theorem 8 (Chong and Mourad [1992]) *$P^- + B\Sigma_1$ proves the Friedberg-Mucknik theorem.*
Proof: Simpson (unpublished) observed that $I\Sigma_1$ was sufficient to verify that the standard construction works. Thus let $\mathcal{M}$ be a model of $P^- + B\Sigma_1$ in which Σ_1-induction fails. There is then a cofinal $\Sigma_1(\mathcal{M})$ map f defined on a $\Sigma_1(\mathcal{M})$-definable "cut" X. This map f on $\mathcal{M}$ acts very much like a Σ_2-cofinal function of $\aleph_\omega^L$ (with domain ω), or indeed a Σ_1-cofinal map on a rudimentarily closed β which is not admissible (β-recursion theory in Section II). The idea now is to treat $\mathcal{M}$ as having "cofinality X" (so that $\mathcal{M} = \cup_{t \in X} M_t$, and $M_t \subset M_{t+1}$), and construct a "Friedberg-Muchnik pair" by satisfying the requirements successively within each M_x.

The following example shows how the methods of Shore [1976a] is applied.

Theorem 9 (Mytilinaios and Slaman [1988]) *$P^- + B\Sigma_2$ does not prove the existence of an incomplete high RE set.*
Proof: There is a model $\mathcal{M}$ of $P^- + B\Sigma_2$ in which Σ_2-induction fails (with ω as the

domain of a $\Sigma_2(\mathcal{M})$-cofinal function f), and in which every real is "coded" (meaning it is the initial segment of a "finite" set). The function f is recursive in $\emptyset'$. If A is an incomplete RE set in $\mathcal{M}$, then for each $n \in \omega$, there is a least $g(n)$ such that $e \in A'|f(n)$ if and only if $\{e\}^{A^{g(n)}}_{g(n)}(e)\downarrow$, else A will be complete. Now $n \mapsto g(n)$ is coded, and so one may use it to compute A' from $\emptyset'$.

Chong and Yang [to appear] have recently shown that the existence of a maxmal set, as well as that of an incomplete high set, is equivalent to $P^- + I\Sigma_2$. In general, just as for α-recursion theory, infinite injury priority method is less well understood. Groszek, Mytilinaios and Slaman [to appear] have recently shown that $P^- + B\Sigma_2$ proves the Density Theorem. The proof-theoretic classification of this theorem is not known.

We refer the reader to Chong [1984] and Sacks [1990] for more complete treatments on α-recursion theory.

II. β-Recursion Theory

Studying the global structure of the α-degrees clearly exposes the need to deal with failures of admissibility: even though an ordinal is admissible it may fail to be relative to a set whose degree we wish to analyze. Indeed, the main thrust of the work in α-recursion theory has been to demonstrate that recursion-theoretic constructions from classical recursion theory which seem to require a large amount of admissibility, say Σ_2 or even Σ_3, can actually be refined so as to succeed with only the assumption of Σ_1-admissibility. In view of this it is natural to ask: Can the assumption of Σ_1-admissibility be eliminated ?

However on hindsight it is fair to say that a stronger motivation for the development of β-recursion theory was to find new applications of the beautiful work of Jensen [1972] on the fine structure of L, to ordinal recursion theory. Jensen's work ignores admissibility distinctions but concentrates only on iterations of the jump operator ("master codes"); β-recursion theory extends his idea to degree theory in general.

The basic notions in β-recursion theory are defined using Jensen's hierarchy for L, the J_α-hierarchy, which enjoys the following properties :

(a) $J_0 = \emptyset$, $J_{\alpha+1} \cap P(J_\alpha) =$ Definable subsets of J_α (with parameters), $J_\lambda = \cup\{J_\alpha \mid \alpha < \lambda\}$ for limit λ.

(b) J_α obeys $\Sigma_0(J_\alpha)$-comprehension and is closed under pairing.

Of course the improvement over the L_α-hierarchy is closure under pairing. Unfortunately $J_\alpha \cap \mathrm{ORD}$ is $\omega\alpha$ and not α. So we define, for limit β : $S_\beta = J_\alpha$ where $\beta = \omega\alpha$. β-recursion theory takes place on the set S_β.

The notions Σ_n-cofinality and Σ_n-projection apply to β as they do in the admissible case: Σ_n-cofinality (β) = leastγ such that there is a cofinal $f : \gamma \to \beta$ which is $\Sigma_n(S_\beta)$; Σ_n-projectum (β) = least γ such that there is a one-one $f : \beta \to \gamma$ which is $\Sigma_n(S_\beta)$. These are either equal to β or are β-cardinals (cardinals in the sense of S_β). An important result of Jensen [1972] states that Σ_n projectum (β) is also the least γ such that some $\Sigma_n(S_\beta)$ subset of γ is not an element of S_β.

We are ready to define the basic notions of β-recursion theory. As in α-recursion theory, $A \subseteq S_\beta$ is β-recursively enumerable, β-recursive, β-finite if and only if A is $\Sigma_1(S_\beta)$, $\Delta_1(S_\beta)$, an element of S_β, respectively. However when β is inadmissible (i.e., Σ_1 cofinality $(\beta) < \beta$), a new and stronger notion of β-RE (β-recurrsively enumerable) arises: A is *tamely* β-RE if $A^* = \{x \in S_\beta \mid x \subseteq A\}$ is β-RE. This is equivalent to saying that A is the union of a β-recursive sequence $\langle A^\sigma \mid \sigma < \beta\rangle$ with the property that if $x \subseteq A$ and β-finite then $x \subseteq A^\sigma$ for some $\sigma < \beta$.

The weak and strong reducibilities $\leq_{w\beta}$, $\leq_\beta$ are defined as they are in α-recursion theory: One way of achieving these definitions is through the use of "neighborhood conditions": define $N(A) = \{\langle x, y\rangle \mid x, y \text{ are } \beta\text{−finite}, x \subseteq A, y \subseteq \bar{A}\}$. B is β-RE in A if for some β-RE W, $x \in B$ if and only if $\exists z \in N(A)\ [(x, z) \in W]$. Then $B \leq_{w\beta} A$ if and only if B, $\bar{B}$ are both β-RE in A, and $B \leq_\beta A$ if and only if B^* and $\bar{B}^*$ are both β-RE in A (if and only if B, $\bar{B}$ are both "tamely" β-RE in A). The strong reducibility $\leq_\beta$ is transitive.

Now some genuinely new phenomena arise in the inadmissible case, with regard to β-reducibility. These are summarized in the following result.

Theorem 1 (Friedman [1979]) *Assume that β is inadmissible. Then there is a β-recursive set A such that:*

(i) $\emptyset <_\beta A <_\beta C$ where C is a complete β-RE set.

(ii) Any tamely β-RE set and any β-recursive set is β-reducible to A.

(iii) $C \leq_{w\beta} A$.

Thus β-recursiveness does not imply β-reducibility to $\emptyset$, and the complete β-RE set is weakly β-reducible to a β-recursive set !

It is easy to define A (in fact A can be taken to be a Δ_1 master code in the sense of Jensen [1972]). Let $f : \Sigma_1 - \text{cofinality}(\beta) \to \beta$ be $\Sigma_1(S_\beta)$ and cofinal, and take $A = \{(e, x, \gamma) \mid x \in W_e \text{ by stage } f(\gamma), \gamma < \Sigma_1\text{-cofinality } (\beta)\}$, where W_e is the eth β-RE set. Then A is β-recursive and since $x \notin W_e$ if and only if $\{e\} \times \{x\} \times \gamma \subseteq \bar{A}$ we get $C \leq_{w\beta} A$. The other properties are not difficult to verify.

The β-degree of A is referred to as $\mathbf{0}^{1/2}$ and serves as a new type of jump operator in β-recursion theory. Of course $\mathbf{0}^{1/2}$ provides an easy solution to a version of Post's Problem in the inadmissible case; however it does not answer the following question, which has come to be adopted as the official version of Post's Problem in β-recursion theory.

Post's Problem Do there exist β-RE sets A, B such that $A \not\leq_{w\beta} B$, $B \not\leq_{w\beta} A$?

As in α-recursion theory, Post's Problem has served as a driving force behind much of the work in β-recursion theory.

Early on it became apparent that with regard to questions such as Post's Problem the inadmissible ordinals divide into two very different classes. (This distinction occurred earlier in Jensen's proof of Σ_2 uniformization.) β is *weakly admissible* if Σ_1-cofinality $(\beta) \geq \Sigma_1$-projectum (β). Otherwise β is *strongly inadmissible.* In the former case many arguments from α-recursion theory can be adapted, for the following reason : if β is weakly admissible (but inadmissible) then there is a β-recursive bijection between S_β and Σ_1-cofinality (β). Moreover there is a β-recursive $A \subseteq \Sigma_1$-cofinality $(\beta) = \kappa$ which is a Δ_1 master code for S_β in Jensen's sense: $B \subseteq K$ is β-RE iff B is $\Sigma_1(L_\kappa, A)$. Thus β-recursion theory is closely related to κ-recursion theory, relativized to A and the structure (L_κ, A) is admissible. This is sufficient to reduce the solution to Post's Problem for β to the previosuly known (positive) solution for (L_κ, A). Exactly how much can be reduced from β to (L_κ, A) is analyzed in Maass [1978a].

The greater challenges in β-recursion theory arise in the strongly inadmissible case. Techniques from admissibility theory no longer apply; instead methods from combinatorial set theory are needed. The first attack on Post's Problem in the strongly inadmissible case appears in Friedman [1980].

Theorem 2 (Friedman [1980]) Suppose β has regular projectum: Σ_1-projectum (β) is regular with respect to β-recursive functions. Then Post's Problem has a positive solution.

The proof uses an effective analog of Jensen's $\diamondsuit$-principle. We provide here a sketch of the proof, in the special case where Σ_1-projectum $(\beta) = \aleph_1^L$. We may assume that Σ_1-confinality $(\beta) = \omega$ (else β is weakly admissible) but actually the proof makes no use of this.

We build β-RE $A, B \subseteq \aleph_1^L$ so as to meet the requirements $R_e^A : \bar{B} \neq W_e^A$ and $R_e^B : \bar{A} \neq W_e^B$, where of course W_e^A is the eth set β-RE in A. To achieve R_e^B we want an $x \notin A$ and a neighborhood condition $y \subseteq B,\ z \subseteq \bar{B}$ so that $(x, (y, t)) \in W_e$. One difference from the admissible case is that we may in fact have to actively guarantee $y \subseteq B$ as otherwise there may be no stage $\sigma < \beta$ where $y \subseteq B^\sigma$, due to the lack of tameness. It is possible however to arrange a weak form of tameness (through use of additional requirements) to insure that in fact $y - B^\sigma$ is countable at some stage, so we need only act to put a countable set into A or B for the sake of each requirement.

The second and most striking difference from the admissible case is that we act on each requirement *at most once.* What enables us to make this restriction is the following. Requirements can be listed in a sequence $\langle R_\delta \mid \delta < \aleph_1^L \rangle$ and as we are only putting countable sets into A or B to satisfy requirements there will be a closed unbounded set of requirements R_δ such that all action taken by R'_δ, $\delta' < \delta$ takes place below δ. Moreover R_δ will only seek to protect ordinals $\geq \delta$ from entering A or B so will never be injured. If we arrange that each requirement appears as R_δ for a stationary set of δ's then each requirement will have the opportunity to act without injury. (So in fact this not really an injury argument at all.)

Finally notice however that we have prohibited requirement R_δ from taking any action below δ; this requires that R_δ has a way of "guessing" at $A \cap \delta$, $B \cap \delta$. The necessary guesses are provided by Jensen's $\diamondsuit$-principle. We end this sketch with no more than a statement of $\diamondsuit$.

(**$\diamondsuit$-principle**) Suppose $E \subseteq \aleph_1^L$ is stationary. Then there exists $\langle G_\alpha \mid \alpha \in E \rangle$ such that :

(a) $G_\alpha \subseteq \alpha$ for $\alpha \in E$.
(b) If $A \subseteq \aleph_1^L$ then $\{\alpha \in E \mid A \cap \alpha = G_\alpha\}$ is stationary.

(In the general case of Theorem 2 we must weaken this somewhat but the general idea is the same.)

The final case, where β is strongly inadmissible with *singular* projectum is entirely different. In fact Post's Problem may have a negative solution! We illustrate the result with a typical example : $\beta = \alpha \cdot \omega$ where $\alpha = \aleph_{w_1}^L$.

Theorem 3 (Friedman [1978]) *Let C be the complete β-RE set. If A is β-RE then either $A \leq_\beta \emptyset$ or $C \leq_{w\beta} A$.*

The proof makes use of the work in Silver [1974] on the singular cardinal problem in set theory (as was for Theorem 5.1 in Section I). We confine ourselves here to only a very rough sketch of the proof. The main idea is to look at *growth rates* for subsets of α. Specifically, suppose $A \subseteq \alpha$ is constructible and define $f_A(\alpha) =$ least δ such that $A \cap \aleph_\gamma^L$ belongs to L_δ. Then it can be shown that if $f_A(\gamma) \leq f_B(\gamma)$ for unboundedly many γ then in fact A is weakly β-reducible to B. This can be extended to $\beta = \alpha \cdot \omega$ to show that in fact any two subsets of β are $\leq_{w\beta}$-comparable. If C is the complete β-RE set then associated to C is a growth rate f which is the limit of β-finite growth rates f_n, $n \in \omega$. Thus either f_A is dominated by some f_n and is hence β-finite or f_A dominates f in which case $C \leq_{w\beta} A$. The uncountable cofinality of α is used both to apply Silver's work and to simultaneously bound the f_n's in this last argument.

III. The Admissibility Spectrum

Until now we have fixed an ordinal α (admissible or not) and studied definability for subsets of α. In this section we invert the process: fix a subset x of some cardinal κ, a theory T and consider the *T-spectrum* of $x = \Lambda_T(x) = \{\alpha \mid L_\alpha[x] \models T\}$. Thus natural classes of ordinals can be defined from sets x and we can ask for a characterization of which classes arise in this way.

Most of the work in this area has concentrated on the case $\kappa = \omega$, $T = KP=$ Admissible Set Theory. However there is a good understanding of $\alpha_T(x) = \min \Lambda_T(x)$ for arbitrary κ and other theories such as $KP_n = \Sigma_n$-Admissibility, ZF. We will mention some of the latter work as well.

The first result in this area is due to Sacks.

Theorem 1 (Sacks [1976]) *If $\alpha > \omega$ is admissible and countable then $\alpha = \omega_1^R = \alpha_{KP}(R)$ for some real R.*

There are many proofs of Theorem 1, but the most adaptable (see Friedman [1986]) is via the method of almost disjoint forcing. As a first step we can add $A_0 \subseteq \alpha$ so that α is A_0-admissible and $L_\alpha[A_0] \models$ every set is countable. This is done by (Levy) forcing with finite conditions $\mathcal{P}$ from $\alpha \times \omega$ into α such that $\mathcal{P}(\beta, n) < \beta$. Second, we can add $A_1 \subseteq \alpha$ so that α is (A_0, A_1)-admissible and $\beta < \alpha$ implies β is not $(A_0 \cap \beta, A_1 \cap \beta)$-admissible. This is done with conditions $\mathcal{P} : \beta \to 2$ such that $\beta' \leq \beta$ implies β' is not $(A_0 \cap \beta', \mathcal{P} \cap \beta')$-admissible.

Now we can canonically assign a real R_β to each $\beta < \alpha$ so that if $\beta_1 \neq \beta_2$ then $R_{\beta_1} \cap R_{\beta_2}$ is finite. By "canonical" we mean that R_β is defined in $L_{\beta+1}[A_0 \cap \beta, A_1 \cap \beta]$, uniformly. Then we code $A = A_0 \vee A_1$ by a real R using conditions $(r, \overline{r})$ where r is a finite subset of $\omega, \overline{r}$ a finite subset of $\{R_\beta \mid \beta \in A\}$ and $(r_0, \overline{r}_0) \leq (r_1, \overline{r}_1)$ if $r_0 \supseteq r_1$, $\overline{r}_0 \supseteq \overline{r}_1$ and $n \in r_0 - r_1$ implies $n \notin R_\beta$ for each $R_\beta \in \overline{r}_1$. The result is that $\beta \in A$ if and only if $R \cap R_\beta$ is finite and thus $A \cap B$ is Δ_1 over $L_\beta[R]$ for each $\beta < \alpha$. So β is not R-admissible for $\beta < \alpha$. Preserving the admissibility of α requires a bit of care, but is based on the simple fact that almost disjoint forcing satisfies the countable chain condition.

Jensen extended Sacks' result to countable sequences of countable admissibles. For a proof of the following result see Friedman [1986].

Theorem 2 (Jensen) *Supose X is a countable set of countable admissibles greater than ω and $\alpha \in X \to \alpha$ is $X \cap \alpha$-admissible. Then for some real R, X is an initial segment of $\Lambda_{KP}(R)$.*

The proof strategy for Theorem 2 is similar to that used in Theorem 1: first add $A \subseteq \alpha$ preserving admissibility so that $\beta < \alpha$ is $A \cap \beta$-admissible if and only if $\beta \in X$, and then code A by a real using almost disjoint forcing. But as we must preserve the admissibility of ordinals in X (while destroying admissibility for ordinals not in X) the argument is more delicate and has the interesting feature that extendibility of conditions for the desired forcing is established using forcing.

There are severe limitations on how much more can be done concerning admissibility spectra in *ZFC* alone. This is illustrated by the next result. A class $X \subseteq$ ORD is Σ_1-complete if Y is $\Delta_1([X], X)$ whenever $Y \subseteq$ ORD is $\Sigma_1(L)$.

Theorem 3 *Let $\Lambda(R)$ abbreviate $\Lambda_{KP}(R) = \{\alpha \mid \alpha$ is R-admissible$\}$.*
(a) $R \in L \to \Lambda(R) \supseteq \Lambda(0) - \beta$ for some $\beta < \aleph_1^L$.
(b) If $R \in L[G]$, G is $\mathcal{P}$-generic over L, and $\mathcal{P} \in L$, then $\Lambda(R) \supseteq \Lambda(0) - \beta$ for some β.
(c) Suppose that $R \in L[G]$ and $G \subseteq \mathcal{P}$. If G is $\mathcal{P}$-generic over the amenable structure $(L, \mathcal{P})$, then $\Lambda(R)$ is not Σ_1-complete.

Proof: (a) Let β be large enough so that $R \in L_\beta$. (b) Let β be large enough so that $\mathcal{P} \in L_\beta$. (c) If $\Lambda(R)$ is Σ_1-complete then L-Card $= \{\kappa \mid L \models \kappa$ is a cardinal $\}$ is $\Delta_1(L[R])$

and hence by reflection, $(\kappa^+)^L < \kappa^+$ for large enough cardinals κ. By Jensen's Covering Theorem, $0^\# \in L[R]$. But $0^\#$ does not satisfy the hypothesis of (c) (see Beller, Jensen and Welch [1982]).

By (a), (b) of this result we see that class-forcing is required to get a nontrivial admissibility spectrum (without assuming $0^\#$) and we should not expect such a spectrum to be Σ_1-complete.

Using a variant of Jensen coding, R. David and S. Friedman independently obtained a class-generic real R such that $\Lambda_{KP}(R) \subseteq$ Admissible Limits of Admissibles. This is a special case of the following result which appeared in David [1989].

Theorem 4 (David and Friedman) *Suppose $\varphi(\alpha)$ is Σ_1 and $\alpha \in L$-Card $\to L \models \varphi(\alpha)$. Then there is a real R class-generic over L such that $\Lambda_{KP}(R) \subseteq \{\alpha \mid L \models \varphi(\alpha)\}$.*

This result is optimal in the sense that if $\varphi(\alpha)$ is the Π_1 formula "α is a cardinal" then the conclusion must fail by Theorem 3(c).

We give some idea of the proof of Theorem 4. The desired forcing is made up of certain "building blocks" that are not difficult to describe. Jensen coding methods are used to put these building blocks together.

We wish to arrange that if α is R-admissible then α is a limit of admissibles. Suppose that we have $D \subseteq \aleph_1^L$ so that if α is D-admissible then α is a limit of admissibles. Then we could hope to choose R so as to code D and satisfy the desired property.

The problem is that if we code D by R in the usual way (with almost disjoint forcing) we only get: for all α, $D \cap (\aleph_1)^{L_\alpha}$ is $\Delta_1(L_\alpha[R])$. So in fact what we need about D is: $L_\alpha[D \cap \xi] \models KP + \xi = \aleph_1$ implies α is a limit of admissibles. For then we need only recover $D \cap (\aleph_1)^{L_\alpha}$ inside $L_\alpha[R]$ to guarantee that α is a limit of admissibles.

How do we obtain D ? The natural thing is to force with conditions d which are initial segments of a potential D. Now we come to the main points in the proof.

(1) Extendibility is easy for this forcing because given d and $\gamma < \aleph_1^L$ we are free to extend d to length γ by killing the admissibility of all ordinals between $\sup(d)$ and γ. It is crucial for this argument that we are only concerned with killing admissibility, not with preserving it.

(2) Cardinal-preservation for this forcing is easy to prove assuming there is $D_2 \subseteq (\aleph_2)^L$ such that: $L_\alpha[D \cap \xi] \models KP + \xi = \aleph_2 \to \alpha$ a limit of admissibles.

Thus we are faced with the original problem, one cardinal higher ! The solution (due to Jensen in the proof of his Coding Theorem) is to build $R, D_1, D_2, \cdots$ simultaneously.

Finally we introduce the requirement of admissibility preservation into the above. Note that in the conclusion of Theorem 4 we have $\subseteq$ and not equality; indeed the freedom to kill admissibility is crucial to the extendibility argument in (1) above.

Nonetheless we can ask for a real R for which we can control its (nontrivial) admissibility spectrum. This requires the method of strong coding.

Theorem 5 (Friedman [1987]) There is a real R, class-generic over L, such that $\Lambda_{KP}(R) =$ Admissible limits of admissibles.

To prove Theorem 5 we can approach the problem much as in the proof of Theorem 4, however extendibility of conditions is much more difficult. The desired extension of d to length γ must be made generically, so as to preserve the admissibility of admissible limits of admissibles. (Note that this idea was foreshadowed by Jensen's proof of Theorem 2.) Thus conditions must be constructed out of generic sets for "local" versions of the very same forcing. So in fact we construct a strong coding $\mathcal{P}^\beta \subseteq L_\beta$ at each admissible β and then inductively build $\mathcal{P}^\beta$ out of generic sets for various $\mathcal{P}^{\beta'}$, $\beta' < \beta$.

A complete characterization of admissibility spectra is not known. A related question, which may indeed be a prerequisite for such a characterization, is the following : which $A \subseteq$ ORD can be Δ_1-definable in a real class-generic over L ? On this latter problem there has been some significant progress. The following will appear in Friedman-Velickovic [1995].

Theorem 6 (Friedman) Suppose $V = L$ and that $A \subseteq$ ORD obeys the Condensation Condition. Then A is Δ_1 in a real class-generic over L, preserving cardinals.

We refer the reader to Friedman-Velickovic [1995] for a definition of the Condensation Condition and a proof of Theorem 6.

Other Work Much is known about $\alpha_T(x) = \min \Lambda_T(x)$, $x \subseteq \kappa$, for $T = KP_n$, ZF and arbitrary infinite cardinals κ, assuming $V = L$. We confine ourselves here to only a brief account.

First we consider the (remaining) cases when $\kappa = \omega$.

Theorem 7 *(a)* (Sacks [1976]*)* $\alpha_{KP_n}(R)$, $R \subseteq \omega$, *can be any countable* Σ_n*-admissible ordinal greater than* ω*.*
(b) (David [1982], Beller in Beller, Jensen and Welch [1982]) $\alpha_{ZF}(R)$, $R \subseteq \omega$, *can be any countable* α *such that* $L_\alpha \models ZF$*.*

Theorem 7(a) can be proved much like Theorem 1. For Theorem 7(b) note that it suffices to first find R_0 such that β an L-cardinal implies that $L_\beta[R_0] \not\models ZF$ and then apply Theorem 4 (relativized to R_0). The former is not hard to arrange using only the statement of Jensen's Coding Theorem. (Of course historically Theorem 7(b) was proved directly as Theorem 4 was not available.)

Next suppose that κ is regular and uncountable.

Theorem 8 (Friedman [1982]) $\alpha = \alpha_{KP_n}(x)$ *for some* $x \subseteq \kappa$ *if and only if* $\alpha < \kappa^+$, α *is* Σ_n*-admissible, cofinality* $(\alpha) = \kappa$ *and* L_α *is closed under the function* $\beta \mapsto \beta^{<\kappa}$*.*

The difficult part of Theorem 8 is the necessity of the stated condition, which draws heavily on Jensen's fine structure theory. The sufficiency is based on an almost disjoint forcing argument, not unlike Theorem 7(a).

Theorem 9 (David and Friedman [1985]) *$\alpha = \alpha_{ZF}(x)$ for some $x \subseteq \kappa$ if and only if $\kappa < \alpha < \kappa^+$, $L_\alpha \models ZF$ and there are $\beta < \alpha$, $\langle X_n \mid n \in \omega \rangle$ such that*

(i) $\forall \gamma < \kappa \forall f : \gamma \to \beta$ (f bounded $\to f \in L_\alpha$),

(ii) $X_n \in L_\alpha$, L_α-Card (X_n) is less than β for all n, and $L_\alpha = \cup \{X_n \mid n \in \omega\}$, and

(iii) β is a regular cardinal in L_α.

The proof of Theorem 9 makes use of almost disjoint forcing, the Covering Theorem (relativized to some $L_\alpha[x]$) and Jensen's fine structure theory.

When κ is singular of cofinality ω then methods from infinitary model theory come into play.

Theorem 10 (Friedman [1981a]) *$\alpha = \alpha_{KP}(x)$ for some $x \subseteq \kappa$ if and only if*

(i) $\kappa < \alpha < \kappa^+$,

(ii) if there is a largest L_α-cardinal γ then cofinality $(\gamma) = \omega$, and

(iii) there is a 1-1 function $f : L_\alpha \to \kappa$ such that $f^{-1}[\delta] \in L_\alpha$ for each $\delta < \kappa$.

Under the conditions stated in Theorem 10, a version of the Barwise Compactness Theorem is established, which can then be used to obtain the desired x. A related result appears in Magidor, Shelah and Stavi [1984].

For $n > 1$ a surprising thing occurs: for any $x \subseteq \aleph_\omega$, $x \in L_{\alpha_{KP_2}}$! And an even stronger fact holds for $x \subseteq \aleph_{\omega_1}$, namely $x \in L_{\alpha_{KP}}(x)$. Both of these facts follow from an effective version of Jensen's Covering Theorem. This puts severe restrictions on the possible values for $\alpha_{KP_n}(x)$, $x \subseteq \kappa$ for $n > 1$, κ singular of cofinality ω and for $n \geq 1$, κ singular of uncountable cofinality (as well as for $\alpha_{ZF}(x)$). The reader is referred to Friedman [1981], David and Friedman [1985] for complete characterizations.

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