Projective measure without projective Baire

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Abstract

We prove that it is consistent relative to a Mahlo cardinal that all sets of reals definable from countable sequences of ordinals are Lebesgue measurable, but at the same time, there is a Δ^1_3 set without the Baire property. To this end, we introduce a notion of stratified forcing and stratified extension and prove an iteration theorem for these classes of forcings. Moreover we introduce a variant of Shelah's amalgamation technique that preserves stratification. The complexity of the set which provides a counterexample to the Baire property is optimal.

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Chapter 1

Introduction

A central theme in descriptive set theory of the reals is that of the relation between definability and regularity. That is, one asks which sets in the projective hierarchy have regularity, in the sense of e.g. being Lebesgue measurable, having the Baire property or the perfect set property.

In recent years, regularity in this sense has been associated to the study of measurability with respect to an ideal on the real numbers and a forcing notion associated to this ideal, such as Sacks, Miller, Laver, Cohen, Random, Mathias and others. For the purpose of this article, we will study only the examples of the null and the meager ideals, with the associated Cohen and Random forcing and the regularity properties of being Lebesgue measurable and the Baire property. Nevertheless, the techniques developed here hold the promise of being applicable to the study of all these ideals.

The broad question we are concerned with in this article is: where in the projective hierarchy does the least irregular set appear? Clearly this is independent of ZFC; under $V = L$, irregular sets appear on the lowest possible level, $\mathbf{\Delta}^1_2$, and in Solovay's model, all sets are regular. It is also possible that the least irregular set appears at some higher level $n > 2$, in inner models of $\mathbf{\Pi}^1_{n-2}$-determinacy which have a $\mathbf{\Delta}^1_n$-good well-ordering of the reals (e.g. under sharps for $n = 3$ and in the minimal canonical inner model with $n-3$ Woodin cardinals for $n > 3$). Observe that in all the above examples, the effects on different ideals are always similar, i.e. the least non-measurable set appears at the same level as the least one lacking the Baire property.

So a slightly more subtle question arises: can we manipulate the behavior independently, for distinct notions of regularity? E.g. Can we have a model where all projective sets are Lebesgue measurable but there is a projective set without the Baire property? That some interaction is possible can be seen from [Bar84]: if all $\mathbf{\Sigma}^1_2$ sets of reals are Lebesgue measurable, then also they have the Baire property.

A seminal result was Shelah's [She84], where he finds two models of interest to our question: In the first, all sets have the Baire property, but there is a set which is not Lebesgue measurable, starting from just the consistency of ZFC. He proves that if all $\mathbf{\Sigma}^1_3$ sets are Lebesgue measurable, then ω_1 must be inaccessible to reals; thus, in this model, there is a non-measurable, projective set.

In the second, all sets are Lebesgue measurable but there is a set which does not have the Baire property. We improve this by making the counterexample projective:

Theorem 1.1. *Starting from "$V = L$ and there exists a Mahlo cardinal", we can force to obtain a model where all projective sets are Lebesgue measurable, but there is a (lightface) Δ^1_3 set without the Baire property.*

The main tool developed in [She84] to selectively attack a specific ideal is *amalgamation*, which allows to construct partial orders which admit certain automorphisms.

In our case we want a partial order where all partial isomorphisms of Random subalgebras extend to an automorphism of the whole forcing — in fact, we want to build an iteration where this is true for every tail segment. Thus, we obtain a model where all projective sets are measurable.[1]

In [Dav82], it is shown how iterations of Jensen-coding may be used to make a set of reals projective. This is done by adding and then coding branches through large constructible Suslin trees, where the coding is projective, i.e. localized. This localization is often referred to as "René David's trick" or "killing universes".

Our iteration will be of length κ, where κ is the least Mahlo in L. In this iteration we collapse every cardinal below κ to ω. Thus, we add many Cohen reals, which easily yield a set without the Baire property, which we call Γ^0. For every real s added by the iteration, we have to code a set of branches of κ^{++}-Suslin trees of L, projectively, into a real. From this code it will possible to determine in a projective way whether $s \in \Gamma^0$. This makes Γ^0 into a Δ^1_3 set.

Also, for every pair of Random reals added by a tail of the iteration, we have to amalgamate. Of course it is crucial to show that we can "catch our tail", i.e. that all reals in the final model appear in some initial segment of the iteration so that we have in fact treated all reals s and all pairs of random reals of the final model.

[1] In $L({}^{\omega}\mathrm{On})$ or in $L(\mathbb{R})$ of our extension, all sets will be measurable, as in Solovay's model, and there will be a projective set without the Baire property. Deliberately, we only consider the bigger model, where choice holds, and its projective sets.

To be able to code and amalgamate alternatingly in this iteration, we isolate the concept of iterations of stratified extensions of forcing, which is discussed in section 3 and 5 (the concept of stratified extension is needed since the components of our forcing need not be stratified—luckily, initial segment are stratified in a *coherent way*). This makes sure a tail of regular cardinals below κ is preserved in every initial stage of the iteration.

In section 6, we devise a version of amalgamation which is also a stratified extension, and therefore preserves stratification. In section 7, after having discussed stratified extension and amalgamation, we motivate and describe the iteration in detail. We also prove that in the final model, every projective set of reals is measurable. That κ is not collapsed and thus becomes ω_1 in the final model does not follow directly from the theory of stratified forcing. Instead, a kind of "κ-properness" is shown in a kind of Δ-systems type argument involving some ideas from stratification, an archetypical use $\diamondsuit$, a special kind of $\square$-sequence discussed in 4 and the fact that enough cardinals are preserved in earlier stages of the iteration; this is the argument of 7.4. Here, we also show that the κ stage of the iteration adds no new reals and that we indeed "catch our tail". To allow this proof to go through we need a version of (localized) Jensen coding which uses *Easton support*, and this is developed in section 4. Finally in the last section we show that Γ^0 is indeed Δ^1_3.

We advise the reader to skim through section 7, especially the more detailed sketch of the iteration given at the beginning. Also, the beginning of each chapter contains a small introduction.

Chapter 2

Notation and preliminaries

We first define *strong projection* (from [Abr10]), *strong sub-order* and *independence*. These are practical in understanding how amalgamation is a stratified extension.

Definition 2.1. We say Q *is a strong sub-order of* P if and only if Q is a complete sub-order of P and for every $p \in P$ and $q \in Q$ such that $q \leq \pi(p)$, we have $q \cdot p \in P$, where π denotes the canonical projection from r.o.(P) to r.o.(Q).

This is related to the notion of projection[1] and strong projection: We say $\pi\colon P \to Q$ is a *projection* if and only if

1. $p \leq p' \Rightarrow \pi(p) \leq \pi(p')$,
2. $\operatorname{ran}(\pi) = Q$,
3. if $q \in Q$ and $q \leq \pi(p)$, there is $\bar{p} \in P$ such that $\bar{p} \leq p$ and $\pi(\bar{p}) \leq q$.

Observe this implies that $\pi(1_P) = 1_Q$. In [Abr10], π is defined to be a *strong projection* if and only if it satisfies the first two requirements above and the following strengthening of the third requirement:

$3'$. If $q \leq \pi(p)$, there is $\bar{p} \leq p$ such that

   a. $\pi(\bar{p}) = q$,
   b. for any $r \in P$, if $r \leq p$ and $\pi(r) \leq q$ then $r \leq \bar{p}$.

[1][Abr10] defines (ordinary) projection with 3. replaced by the stronger: if $q \leq \pi(p)$, there is $\bar{p} \leq p$ such that $\pi(\bar{p}) = q$.

This uniquely determines $\bar{p}$, and we denote it by $q \cdot p$.

If $\pi\colon P \to Q$ is a projection, $\pi[G]$ generates a Q-generic Filter whenever G is a P-generic Filter, moreover in fact r.o.(Q) is a complete sub-algebra of r.o.(P). If $\pi\colon P \to Q$ is a strong projection, the map i sending $q \in Q$ to $i(q) = q \cdot 1_P$ is a complete embedding and we can assume that Q is a subset of P. It follows from 3′b. that $\forall p \in P \quad p \leq i(\pi(p))$.

When Q is a complete sub-order of P, we say $q \in Q$ is a *reduction (to Q) of* $p \in P$ if and only if for all $q' \in Q$, if $q' \leq q$ then q' and p are compatible.

Lemma 2.2. *Let Q be a complete sub-order of P and let π be the canonical projection $\pi\colon$ r.o.$(P) \to$ r.o.(Q). Say $p \in P$ and $q \in Q$ is a reduction of p such that $q \geq p$; then $q = \pi(p)$. If $\bar{\pi}\colon P \to Q$ is a strong projection, then $\bar{\pi}$ coincides with the canonical projection on P.*

Observe that if $\pi\colon$ r.o.$(P) \to$ r.o.(Q) is the canonical projection, then $\pi \restriction P$ is a strong projection if and only if for every $p \in P$ and $q \in Q$ such that $q \leq \pi(p)$ we have $p \cdot q \in P$. All of the above gives us:

Lemma 2.3. *The following are equivalent:*

- *Q is a strong sub-order of P.*
- *There is a strong projection $\pi\colon P \to Q$.*
- *The restriction of the canonical projection $\pi\colon$ r.o.$(P) \to$ r.o.(Q) to P is the unique strong projection from P to Q.*

Also observe that when Q is a strong sub-order of P which is a strong sub-order of R with $\pi_P\colon R \to P$ a strong projection, then 1_Q forces $\pi_P \restriction P : Q$ is a strong projection from $P : Q$ to $R : Q$.

Imagine an iteration $R = (Q_0 \times Q_1) * \dot{Q}_2$. Then in an extension by Q_0, the pre-order Q_1 is a complete sub-order of the tail $R : Q_0 = Q_1 * \dot{Q}_2$. This special situation is captured well by the following:

Definition 2.4. Let Q and C be sub-orders of P with strong projections $\pi_Q\colon P \to Q$ and $\pi_C\colon P \to C$. We say *C is independent over Q in P* if and only if for all $c \in C$ and $p \in P$ such that $c \leq \pi_C(p)$, we have $\pi_Q(p \cdot c) = \pi_Q(p)$.

For a P-name $\dot{C}$, we say *$\dot{C}$ is independent in P over Q* if and only if $\dot{C}$ is a name for a generic of an independent complete sub-order of P; i.e. there is a complete sub-order R_C of P (with a strong projection $\pi_C\colon P \to R_C$) such that R_C is a dense in $\langle \dot{C} \rangle^{\text{r.o.}(P)}$ and R_C is independent in P over Q.

By $\langle \dot{C} \rangle^{\text{r.o.}(P)}$ we mean the smallest Boolean subalgebra of r.o.(P) which contains all the boolean values occurring in the name $\dot{C}$. Thus $\langle \dot{C} \rangle^{\text{r.o.}(P)}$ is a ground model object.

We use the following in 7.4 (p. 123), via the notion of "remoteness" (see also lemma 5.33 and section 5.5).

Lemma 2.5. *If C is independent over Q in P, 1_Q forces that C is a complete sub-order of $P : Q$ and π_C is a strong projection. If $\dot{C}$ is a P-name which is independent over Q, then $\dot{C}$ is not in V^Q.*

Iterations. We say $\bar{Q}^\theta = (P_\iota, \dot{Q}_\iota)_{\iota<\theta}$ is an iteration if and only if for each $\iota < \theta$,

1. $\Vdash_{P_\iota} \dot{Q}_\iota$ is a pre-order
2. P_ι consists of sequences p such that $\mathrm{dom}(p) = \iota$ and for each $\nu < \iota$, $p(\nu)$ is a P_ν-name such that
$$1_{P_\nu} \Vdash p(\nu) \in \dot{Q}_\nu. \tag{2.1}$$
3. The ordering of P_ι is given by:
$$r \leq p \iff \forall \nu < \iota \quad r \restriction \nu \Vdash_{P_\nu} r(\nu) \leq_{\dot{Q}_\nu} p(\nu). \tag{2.2}$$

We state this as some would not agree with (2.1). Fix an iteration $\bar{Q}^{\theta+1}$.

Definition 2.6. 1. We call a sequence p with $\mathrm{dom}(p) = \theta$ a *thread through (or in)* $\bar{Q}^\theta$ if and only if it satisfies (2.1). The set of threads through $\bar{Q}^\theta$ we shall sometimes denote by $\prod \bar{Q}^\theta$. It is endowed with the ordering given by (2.2) (for $r, p \in \prod \bar{Q}^\theta$).

2. We also use the term *thread* in a second, related sense: if $\bar{p} \in \prod_{\eta<\iota<\theta} P_\iota$ for some $\eta < \theta$—i.e $\bar{p} = (p_\iota)_{\iota\in(\eta,\theta)}$ and for each $\iota \in \mathrm{dom}(\bar{p})$ we have $p_\iota \in P_\iota$—we say $\bar{p}$ forms or defines or simply *is a thread (through $\bar{Q}^\theta$)* if and only if
$$\forall \iota, \bar{\iota} \in \mathrm{dom}(\bar{p}) \quad \iota \leq \bar{\iota} \Rightarrow \pi_\iota(p_{\bar{\iota}}) = p_\iota. \tag{2.3}$$

The point is that a thread in the first sense yields one in the second sense and vice versa.

Definition 2.7. More generally we will also call sequence $(P_\iota)_{\iota\leq\theta}$ an iteration when in comes strong projections $\pi_\iota^{\bar{\iota}}\colon P_{\bar{\iota}} \to P_\iota$, for each $\iota < \bar{\iota} \leq \theta$, and for each (limit) ι, P_ι consists of threads in the second sense.

Clearly an iteration in the second sense can be written as an iteration in the first sense and vice versa.

Ideals. If c is a Borel-code, we write B_c for the Borel set coded by c. Of course given two models of set theory, both containing a Borel code c, it may be that c codes a different set in each model.

Definition 2.8. Say r.o.(Q) is a complete sub-algebra of r.o.(P). Let $\dot{G}$ be is the canonical P-name for the P-generic over V and $\pi\colon \text{r.o.}(P) \to \text{r.o.}(Q)$ the canonical projection. Let $\dot{I}$ be a P-name for an ideal on $\mathbb{R}$ in the extension via P. For a P-name $\dot{r}$ and $p \in P$, we say is *p forces $\dot{r}$ is $\dot{I}$-generic over V^Q*, just if $p \Vdash_P \dot{r} \in \mathbb{R}$ and for every Q-name for a Borel code $\dot{c}$,

$$p \Vdash_P B_{\dot{c}} \in \dot{I} \Rightarrow \dot{r} \notin B_{\dot{c}}.$$

We say p forces $\dot{r}$ is *fully* $\dot{I}$-generic over V^Q if and only if p forces $\dot{r}$ is $\dot{I}$-generic over V^Q and in addition, for every Q-name $\dot{c}$ such that $\pi(p) \Vdash_Q \dot{c}$ is a Borel code,

$$p \Vdash_P \dot{r} \notin B_{\dot{c}} \Rightarrow p \Vdash_P B_{\dot{c}} \in \dot{I}.$$

In other words, p does not force anything non-trivial about $\dot{r}$. We say *$\dot{r}$ is (fully) $\dot{I}$-generic* just if 1_P forces $\dot{r}$ is (fully) $\dot{I}$-generic. Instead of "$\dot{I}$-generic",

- If $\dot{I}$ is a name for the ideal of sets with measure zero, we say *Random over V^Q*.
- If $\dot{I}$ is a name for the ideal of meager sets, we say *Cohen over V^Q*.
- If $\dot{I}$ is a name for $\mathcal{P}_{<\omega}(\mathbb{R})^{V[\pi(\dot{G})]}$—or equivalently, for $\mathcal{P}(\mathbb{R}^{V[\pi(\dot{G})]})$—we say $\dot{r} \notin V^Q$ or *$\dot{r}$ is not in V^Q*.
- If $\dot{I}$ is a name for the ideal of sets which are bounded by a real in $V[\pi(\dot{G})]$ in the sense of eventual domination, we say *unbounded over V^Q*.

The terms *p forces $\dot{r}$ is fully Random over V^Q* and *fully Cohen over V^Q* are to be understood analogously.

Lemma 2.9. *Let P and Q be arbitrary partial orders and let $\dot{r}$ be a P-name for a real. If $\dot{r}$ is unbounded over V, viewing $\dot{r}$ as a $P \times Q$ name via the natural embedding, $\dot{r}$ is unbounded over V^Q.*

For a proof, see [JR93, lemma 3.3, p. 392].

Chapter 3

Stratified Forcing

In this section we assume $V = L[A]$ for some class A. We define *stratified partial orders*, show such orders preserve cofinalities, give some examples and show that stratification is preserved under composition. We also define diagonal support and state that iterations whose components are stratified are themselves stratified.[1] The proof is left out, since we prove a slightly more general theorem in section 5 where we deal with iterations with stratified initial segments but where the components aren't necessarily stratified.

We present the definition of stratification in two parts: the first we dub *quasi-closure*. We treat this first part separately from the remaining axioms of stratification for the following reasons: firstly, the proofs that each of these two groups of axioms is preserved in iterations are not only different but virtually independent of each other.

Secondly, we hope that the reader will agree that quasi-closure is interesting in its own right. This view is in stark contrast to the fact that quasi-closure alone is not a very useful property— in fact, every partial order is quasi-closed. One should think of it as an incomplete notion, to which some other property has to be added in order to render it non-trivial. Stratification is one example of this, closely connected to the notion of centered forcing. There may be other examples, as well.

Before we define quasi-closure, we introduce pre-closure systems; analogously we will define pre-stratification systems. We can reuse these notions when we define *quasi-closed and stratified extension*; see section 5 on page61.

[1]Most of these definitions are heavily inspired by [Fri94]; see also [Fri00].

3.1 Quasi-Closure

Throughout, let $\langle R, \leq\rangle$ be a pre-order and let I be an intervall of regular cardinals, that is $I = \mathbf{Reg} \cap [\lambda_0, \lambda_1)$ or $I = \mathbf{Reg} \cap [\lambda_0, \infty)$ for some regular λ_0, λ_1. We introduce the notion of R being *stratified on* I, which implies that for any $\lambda \in I$, any ordinal of cofinality greater than λ will remain so after forcing with R. We have to use another property of R to show cardinals greater $(\lambda_1)^+$ are preserved (in our application, R will have an appropriate chain condition). Moreover, we want to allow for R to collapse some cardinals up to and including λ_0.

From now on, we write $\Pi^T_1(z)$ for the set of formulas provably equivalent to a $\Pi^A_1(z)$ formula in some weak version of ZFC, T. For concreteness, say T states that the universe is closed under every function F which is definable by a Σ^A_1 formula, say Φ such that ZFC $\wedge\, V = L[A]$ proves "$\forall x \exists y \Phi(x, y)$".[2] We only mention this to point put that unfortunately, these classes of formulas are not closed under bounded quantification. In practice, the reader will see T is of little relevance and we shall sometimes omit the superscript.

We now make a few convenient definitions that facilitate the treatment of quasi-closed partial orders, which we define afterward.

Definition 3.1. We say $\mathbf{s} = (\mathbf{D}, c, \preccurlyeq^\lambda)_{\lambda \in I}$ is a pre-closure system for R on I if and only if $\mathbf{D} \subseteq \mathbf{Reg} \times V \times R^2$ is a $\Pi^T_1(c)$ class and for every $\lambda \in I$, $x \in V$, $p, q, r \in R$

(C 1) if $p \leq q \in \mathbf{D}(\lambda, x, r)$ then $p \in \mathbf{D}(\lambda, x, r)$.

(C 2) The relation $\preccurlyeq^\lambda$ is a preorder on R and $p \preccurlyeq^\lambda q \Rightarrow p \leq q$.

(C 3) If $p \leq q \leq r$ and $p \preccurlyeq^\lambda r$ then $p \preccurlyeq^\lambda q$.

(C 4) If $\bar\lambda \in I$, $\bar\lambda \geq \lambda$ then $q \preccurlyeq^{\bar\lambda} p \Rightarrow q \preccurlyeq^\lambda p$.

As a notational convenience, define $\preccurlyeq^0$ to mean $\leq_R$. Clause (C 3) can be dropped if one is not interested in iterations. Observe that by (C 3), $\preccurlyeq^\lambda$ is well-defined with respect to equivalence modulo $\approx$ (remember we say $p \approx q \iff p \leq q$ and $q \leq p$).

Think of each of the relations $\preccurlyeq^\lambda$ as a notion of *direct extension*, as it is often called in the case of e.g. Prikry-like forcings. Intuitively, $p \preccurlyeq^\lambda q$ expresses that p extends q but some part "below λ" is left unchanged. Think of $\mathbf{D}$ as providing a kind of strategy, as with strategically closed forcing.

[2]This includes all rudimentary functions and the function $x \mapsto \mathrm{tcl}(x)$ assigning to x its transitive closure; also, T implies Δ^A_0-separation. Alternatively, we could take T to be the theory of all Σ^A_1-elementary sub-models of $L[A]$.

Together, this additional structure on R allows us to express that certain sequences have lower bounds in R. The missing ingredient and distinct flavor of quasi-closure is the condition that these sequences be *definable* in a sense. The main point is that the definability and the use of $\mathbf{D}$ are intertwined in that they are coordinated by a common object $\bar{w}$ which we shall call the *strategic guide and canonical witness.*

For the next two definitions, fix a pre-closure system $\mathbf{s}$ for R on I. All the notions in the next two definitions have their meaning *with respect to* $\mathbf{s}$.

Definition 3.2. Let $\bar{p} = (p_\xi)_{\xi<\rho}$ be a sequence of conditions in R and $\rho \leq \lambda$. We say $\bar{w}$ is a (λ, x)-*strategic guide for* $\bar{p}$ if and only if $\bar{w} = (w_\xi)_{\xi<\rho}$ is a sequence of the same length as $\bar{p}$ and for a tail of $\xi < \rho$

1. for some $\lambda' \in I$, $p_{\xi+1} \in \mathbf{D}(\lambda', (\{x, c\}, \bar{w} \restriction \xi + 1), p_\xi)$ and $p_{\xi+1} \preccurlyeq^{\lambda'} p_\xi$.

2. $p_{\xi+1} \preccurlyeq^{\lambda} p_\xi$.

3. if ξ is a limit, p_ξ is a greatest lower bound of $(p_\nu)_{\nu<\xi}$.

Definition 3.3. 1. We say a sequence $\bar{w} = (w_\xi)_{\xi<\rho}$ is $\Pi^T_1(z)$ if for some $\Pi^T_1(z)$ formula Ψ we have $w = w_\xi \iff \Psi(w, \xi)$. We say G is a $\Sigma^T_1(z)$ function if $G(x) = y$ is a $\Sigma^T_1(z)$ formula and G is a partial function.

2. Let $\bar{p} = (p_\xi)_{\xi<\rho}$ be a sequence of conditions in R and $\rho \leq \lambda$. We say $\bar{w} = (w_\xi)_{\xi<\rho}$ is a (λ, x)-*canonical witness for* $\bar{p}$ if and only if $\bar{w}$ is $\Pi^T_1(\lambda \cup \{x, c\})$ and for some $\Sigma^T_1(\lambda \cup \{(x, c)\})$ (partial) function G, we have $p_\xi = G(\bar{w} \restriction \xi + 1)$ for every $\xi < \rho$.

3. We say $\bar{p}$ is (λ, x)-adequate if and only if $\rho \leq \lambda$ and there is $\bar{w}$ which is both a strategic guide and a canonical witness for $\bar{p}$.

4. If $\bar{p}$ is (λ, x)-adequate for some x we say $\bar{p}$ is λ-*adequate.*

Note that a sequence $\bar{p}$ has a canonical witness simply if it is $\mathbf{\Sigma}^T_2$. Also note there is some flexibility for with respect to the definability requirement we out in G; one could probably do with rudimentary G or with one specific function.

Definition 3.4. We say $\langle R, \mathbf{s} \rangle$ is *quasi-closed on* I if and only if for any x and $\lambda \in I$,

(C I) For any $p \in R$ there is $q \in \mathbf{D}(\lambda, x, p)$ such that $q \preccurlyeq^\lambda p$. In addition we can demand that $q \preccurlyeq^{\bar{\lambda}} 1_R$ for any $\bar{\lambda} \in I$ such that $p \preccurlyeq^{\bar{\lambda}} 1_R$.

(C II) Every λ-adequate sequence $\bar{p} = (p_\xi)_{\xi<\rho}$ in R has a greatest lower bound p in R and for all $\xi < \rho$, $p \preccurlyeq^\lambda p_\xi$. If $\bar{\lambda} \in I$ is such that for each $\xi < \rho$, $p_\xi \preccurlyeq^{\bar{\lambda}} 1_R$, then $p \preccurlyeq^{\bar{\lambda}} 1_R$.

We also use the expression R *is quasi-closed as witnessed by* $\mathbf{s}$. If we omit $\mathbf{s}$ and no pre-closure system can be deduced from the context, we mean that there exists a pre-closure system $\mathbf{s}$ such that $\langle R, \mathbf{s}\rangle$ is quasi-closed. When we say *quasi-closed on* $[\lambda_0, \lambda_1)$, we mean of course quasi-closed on $[\lambda_0, \lambda_1) \cap \mathbf{Reg}$ etc.

Clause (C I) and the last sentence of clause (C II) are useful regarding infinite iterations of quasi-closed forcings.

Remark 3.5. For arbitrary R, just define $p \preccurlyeq^\lambda q$ if and only if $p = q$ and $\mathbf{D}(\lambda, x, p) = \{p\}$ for all regular $\lambda \geq \lambda_0$ and all x. Then R is quasi-closed. Quasi-closure becomes non-trivial under the additional hypothesis that certain questions about the generic extension can be decided by strengthening a condition in the sense of $\preccurlyeq^\lambda$, for some λ. Stratified forcing satisfies such a hypothesis.

Remark 3.6. Say R is λ^+-closed; then R trivially satisfies all the conditions of 3.4 for this one λ. The same is true if R is λ^+-strategic: for if $\sigma\colon R \to R$ is a strategy for R, define $\mathbf{D}(\lambda, x, p) = \{q \in R \mid q \leq \sigma(p)\}$. $\mathbf{D}$ is clearly $\Pi_1^T(\{\sigma\})$. We can define $\preccurlyeq^\lambda$ to be the same as $\leq$. Of course every strategic and thus every adequate sequence has a greatest lower bound.

This is not vacuous, in that there are non-trivial adequate sequences. In fact, every sequence $\bar{p}$ of length less than λ^+ which adheres to σ is λ-adequate: For fix $\bar{p}$ of length less than λ^+. By re-indexing, assume the length of $\bar{p}$ is λ. Since $\bar{p}$ is $\Pi_1^T(\{\bar{p}\})$, and since $\mathbf{D}$ does not depend on x at all, $\bar{p}$ is $(\lambda, \{\bar{p}\})$-adequate.

These are our first examples of forcings which non-trivially satisfy the definition of quasi-closed (albeit for just one fixed λ), since any statement about the generic can be decided by extending in the sense of $\preccurlyeq^\lambda$.

3.2 A word about definability and set forcing

The concept of quasi-closure is more natural in a class forcing context. Since we only apply it for set forcing, we can make do with $\Pi_1^T(z)$ (as opposed taking into account $\Pi_n^T(z)$ for all $n > 1$). We still have to use a form of Π_1-uniformisation, implicit in the construction of canonical witnesses (see 3.14). In class forcing, this uniformisation can be achieved using the fact that conditions form a class, and the "height" of each condition in an adequate sequence effectively represents the canonical witness. Using this technique, I do not know a proof that κ is not collapsed. See also [Fri00], chapter 8.2, p. 175.

Think of x as a tuple of constants which can be used in in the definition of an adequate sequence $\bar{p}$. From now on we shall always assume that c is among the constants given by x. We can and will assume that some large enough $L_\mu[A]$ is among the constants given by x, where μ is a cardinal and $R \in L_\mu[A]$ (i.e. assume that $L_\mu[A]$ is in or is recursive in c). This allows us to bound quantifiers of certain statements and argue that they are $\Pi^T_1(\{x\} \cup \lambda)$. Intuitively, this use of c is analogous to the use of a large structure with predicates in the context of proper forcing. In our application, it will suffice to set $c = \{\kappa^{+++}\}$.

3.3 Stratification

Definition 3.7. We say $\mathbf{S} = (\mathbf{D}, \preccurlyeq^\lambda, \preccurlyeq^\lambda, \mathbf{C}^\lambda)_{\lambda \in I}$ is a pre-stratification system for R on I if and only if $(\mathbf{D}, c, \preccurlyeq^\lambda)_{\lambda \in I}$ is a pre-closure system for R on I and for every $\lambda \in I$ the following conditions are met:

(S 1) The binary relation $\preccurlyeq^\lambda$ on R satisfies $p \leq q \Rightarrow p \preccurlyeq^\lambda q$.

(S 2) If $p \leq q \preccurlyeq^\lambda r$ then $p \preccurlyeq^\lambda r$.[3]

(S 3) If $\lambda \leq \bar{\lambda}$ and $\bar{\lambda} \in I$ then $p \preccurlyeq^\lambda q \Rightarrow p \preccurlyeq^{\bar{\lambda}} q$.

(S 4) *Density:* $\mathbf{C}^\lambda \subseteq R \times \lambda$ is a binary relation such that $\mathrm{dom}(\mathbf{C}^\lambda)$ is dense in R. Moreover, for any $\lambda' \in I \cap \lambda$ and $p \in R$, there is $q \preccurlyeq^{\lambda'} p$ such that $q \in \mathrm{dom}(\mathbf{C}^\lambda)$.

(S 5) *Continuity*: If $\lambda' \in I \cap \lambda$ and p is a greatest lower bound of the λ'-adequate sequence $\bar{p} = (p_\xi)_{\xi<\rho}$ and for each $\xi < \rho$, $p_\xi \in \mathrm{dom}(\mathbf{C}^\lambda)$, *then* $p \in \mathrm{dom}(\mathbf{C}^\lambda)$.[4] If in addition $\bar{q}$ is another λ'-adequate sequence of length ρ with greatest lower bound q and for each $\xi < \rho$, $\mathbf{C}^\lambda(p_\xi) \cap \mathbf{C}^\lambda(q_\xi) \neq \emptyset$, then $\mathbf{C}^\lambda(p) \cap \mathbf{C}^\lambda(q) \neq \emptyset$.

[3] Note that we don't assume $\preccurlyeq^\lambda$ to be transitive, since this does not seem to be preserved by composition. If $\preccurlyeq^\lambda$ were transitive, condition (S 2) would follow from (S 1). We need (S 2) for lemma 3.15. We need that $\preccurlyeq^\lambda$ is reflexive (i.e. $p \preccurlyeq^\lambda p$ for all p) for 5.18($\lhd_s$4). In the context of (S 2), reflexivity is the same as the last part of (S 1).

[4] In any application I know, we could ask this for all $\lambda' \in I$, not just those smaller than λ.

The last part of condition (S 1), all of (S 3) and and the "moreover" part of (S 4) can be dropped if one is not interested in infinite iterations. Don't think that $\curlyeqprec^\lambda$ is a pre-order or well-defined on the separative quotient of R, although (S 2) guarantees some regularity with respect to $\approx$.

Definition 3.8. We say a pre-order $\langle R, \leq \rangle$ is *stratified on I* as witnessed by $\mathbf{S} = (\mathbf{D}, \preccurlyeq^\lambda, \curlyeqprec^\lambda, \mathbf{C}^\lambda)_{\lambda \in I}$ if and only if $\mathbf{S}$ is a pre-stratification system on I, $\langle R, \preccurlyeq^\lambda, \mathbf{D} \rangle_{\lambda \in I}$ is quasi-closed, and for each $\lambda \in I$ the following conditions hold:

(S I) *Expansion*: If $p \curlyeqprec^\lambda d$ and $d \preccurlyeq^\lambda 1_R$, then in fact $p \leq d$.

(S II) *Interpolation*: If $d \leq r$, there is $p \preccurlyeq^\lambda r$ such that $p \curlyeqprec^\lambda d$. In addition, whenever $\bar{\lambda} \in I$ and $d \preccurlyeq^{\bar{\lambda}} 1_R$, then also $p \preccurlyeq^{\bar{\lambda}} 1_R$.

(S III) *Centering*: If $p \curlyeqprec^\lambda d$ and and $\mathbf{C}^\lambda(p) \cap \mathbf{C}^\lambda(d) \neq \emptyset$ then p and d are compatible. In fact, there is w such that for any $\lambda' \in I \cap \lambda$, $w \preccurlyeq^{\lambda'} p$ and $w \preccurlyeq^{\lambda'} d$.

If we omit $\mathbf{S}$ and no pre-stratification system can be deduced from the context, we mean that there exists a pre-stratification system $\mathbf{S}$ witnessing that R is stratified. When we say *stratified on* $[\lambda_0, \lambda_1)$, we mean of course stratified on $[\lambda_0, \lambda_1) \cap \mathbf{Reg}$ etc.

Conditions (S 5) and (S I) are important to preserve stratification in (infinite) iterations. The second part of (S III) was introduced to allow for amalgamation (see section 6), but is also useful to control the diagonal support in iterations (see below).

We illustrate definition 3.8 with some examples.

Example 3.9. A simple observation is that for any pre-order R, R is stratified above $|R|$. A little more generally, if R is λ_0-centered, then R is stratified on $[\lambda_0, \infty)$: for if $\lambda \geq \lambda_0$, we can simply define $p \preccurlyeq^\lambda q$ just if $p = q$. Similarly, $\mathbf{D}(\lambda, x, p) = R$ for all $p \in R$. Thus, quasi-closure and continuity become vacuous. Moreover, let $g : R \to \lambda_0$ be a a function such that if $g(p) = g(q)$ then p and q are compatible. Set $\mathbf{C}^\lambda(p) = \{g(p)\}$ for any $p \in R$. Lastly, define $p \curlyeqprec^\lambda q$ to hold for *any* pair p, q. Then the only non-vacuous condition in the definition of stratification is *centering*, which holds for every $\lambda \geq \lambda_0$ since g witnessed that R was centered.

This example has a corollary:

Corollary 3.10. *If a pre-ordered set R is stratified, we can always assume that for $\lambda \geq |R|$, $\mathbf{D}$, $\preccurlyeq^\lambda$, $\curlyeqprec^\lambda$ and $\mathbf{C}^\lambda$ take the simple form discussed above in example 3.9.*

Observe that (C 4) and (S 3) remain valid if we modify a given pre-stratification system in such a way as to ensure that the above assumption holds. A more interesting example:

Example 3.11. Say $R = P * \dot{Q}$ where P is $(\lambda_0)^+$-centered and $(\lambda_0)^+$-closed and $\Vdash_P \dot{Q}$ is λ_0-centered and λ_0-closed. Then R is stratified on **Reg** — ignoring (S 5). If the centering functions for P and $\dot{Q}_\xi$ in the extension are continuous in the sense of (S 5)—and it seems that for many centered forcings, this is the case—R is actually stratified.

Define $\mathbf{D}$ as in the previous example. For $\lambda < \lambda_0$, define $\preccurlyeq^\lambda$ to be identical to $\leq_R$; define $p \precsim^\lambda q$ if and only if $p = q$ and let $\mathbf{C}^\lambda(p) = \lambda$ for every $p \in R$. Then *Interpolation* and *centering* hold at λ for trivial reasons, and quasi-closure at λ expresses the fact that R is closed under sequences of length at most λ. For $\lambda = \lambda_0$, fix a name for a centering function $\dot{g}$; set $(p, \dot{q}) \preccurlyeq^\lambda (p', \dot{q}')$ if and only if $(p, \dot{q}) \leq (p', \dot{q}')$ and $\dot{q} = \dot{q}'$; set $(p, \dot{q}) \precsim^\lambda (p', \dot{q}')$ if and only if $p \leq_P p'$. Let $\chi \in \mathbf{C}^\lambda(p, \dot{q})$ if and only if $p \Vdash \dot{g}(\dot{q}) = \check{\chi}$. Lastly, R has a subset R' which is $(\lambda_0)^+$ centered and $\preccurlyeq^{\lambda_0}$-dense. This allows us to define a stratification above $(\lambda_0)^+$, in a similar way to the previous example.

Finally, we can discuss preservation of cofinalities and the GCH.

Definition 3.12. In the following, we fix a regular cardinal $\lambda \in I$ and drop the superscripts on $\mathbf{C}, \preccurlyeq$ and $\precsim$.

1. Let $D, D^* \subseteq R$, and $r \in R$. We say r *λ-reduces D to D^** (often, we don't mention the prefix λ) exactly if
   (a) $D^* \subseteq \mathrm{dom}(\mathbf{C})$ and $|D^*| \leq \lambda$;
   (b) for each $d \in D^*$, $r \precsim d$;
   (c) for any $q \in \mathrm{dom}(\mathbf{C}) \cap D$, if $q \leq r$, there is $d \in D^*$ such that $\mathbf{C}(q) \cap \mathbf{C}(d) \neq 0$.
2. Let $\dot{\alpha}$ be a name for an element in the ground model V, and let $r \in R$. We say $\dot{\alpha}$ is λ-chromatic below r just if there is a function H with $\mathrm{dom}\, H \subseteq \lambda$ such that if $q \leq r$ decides $\dot{\alpha}$ and $q \in \mathrm{dom}(\mathbf{C})$, then $\mathbf{C}(q) \cap \mathrm{dom}(H) \neq 0$ and for all $\chi \in \mathbf{C}(q) \cap \mathrm{dom}(H)$, $q \Vdash \dot{\alpha} = H(\chi)$ (to be pedantically precise, we mean the "standard name" for $H(\chi)$). We call such H a *λ-spectrum (of $\dot{\alpha}$)*.
3. If $\dot{s}$ is a name and $p \Vdash \dot{s} \colon \lambda \to V$, then we say *$\dot{s}$ is λ-chromatic (with λ-spectrum $(H_\xi)_{\xi<\lambda}$) below* p if and only if for each $\xi < \lambda$, $\dot{s}(\check{\xi})$ is chromatic with spectrum H_ξ below p.

   For notational convenience, we say $\dot{x}$ is 0-chromatic below p if for some x, $p \Vdash_R \dot{x} = \check{x}$.

Observe that if for some ground model set x, $p \Vdash \dot{\alpha} = \check{x}$ (i.e. α is 0-chromatic), then $\dot{\alpha}$ is in fact λ-chromatic for every regular λ, and the function with domain λ and constant value x is a λ-spectrum.

To illustrate λ-reduction, observe that if r λ-reduces D to D^*, then D^* is a subset of D^* of size λ which is predense below r: for if $q \leq r$, we can assume that $q \in \text{dom}(\mathbf{C}^\lambda)$, so that there is $d \in D^*$ with $\mathbf{C}^\lambda(d) \cap \mathbf{C}^\lambda(q) \neq \emptyset$. By (S 2), $q \preccurlyeq^\lambda d$ and so by *Centering* (S III), q and d are compatible.

Now say Say R is stratified in I and $\lambda \in I$.

Theorem 3.13. *Let D be a dense subset of R and $p \in D$. Then there is $r \preccurlyeq^\lambda p$ such that r λ-reduces D to some D^*.*

Proof. We build an adequate sequence $\bar{p} = (p_0)_{\xi \leq \lambda}$, starting with $p_0 = p$. We shall first sketch a construction such that $r = p_\lambda$ is the desired condition, without specifying a canonical witness; then we argue how this construction can be carried out so as to obtain a canonical witness at the same time. This will serve as a blueprint for later constructions, where we shall not explicitly carry out the construction of $\bar{w}$, as it is entirely analogous to the case at hand.

Let $x = (p_0, \preccurlyeq^\lambda, \preccurlyeq^\lambda, \mathbf{C}^\lambda, R, \mathbf{D}, c)$, where c contains any parameters in the definition of $\mathbf{D}$. Say we have constructed $\bar{w} \restriction \nu$ and $\bar{p} \restriction \nu$. If $\nu \leq \lambda$ is a limit ordinal, assume by induction that $\bar{p} \restriction \nu$ is adequate and let p_ν be a greatest lower bound. Now say $\nu = \xi + 1$. Choose, in a manner yet to be specified $w_{\xi+1}$, $p_{\xi+1}$, p^*_ξ and d_ξ which satisfy the following:

1. $p^*_\xi \in \mathbf{D}(\lambda, (x, \bar{w} \restriction \xi + 1), p_\xi)$, and $p_{\xi+1} \preccurlyeq^\lambda p^*_\xi$.
2. (a) If there is no $d \leq p^*_\xi$ such that $d \in D$ and $\xi \in \mathbf{C}^\lambda(d)$ we demand $p_{\xi+1} = p^*_\xi$;

   (b) else $p_{\xi+1}$ and d_ξ satisfy: $d_\xi \leq p^*_\xi$, $d_\xi \in D$ and $\xi \in \mathbf{C}^\lambda(d_\xi)$; moreover $p_{\xi+1} \preccurlyeq p^*_\xi$ and $p_{\xi+1} \preccurlyeq^\lambda d_\xi$.

Observe this does not even mention $w_{\xi+1}$; its role will be explained by lemma 3.14, below. If the first alternative of item 2 obtains, the d_ξ we choose is completely irrelevant for the rest of the construction. It is clear that $\bar{p}$ will have $\bar{w}$ as a strategic guide: since $p_{\xi+1} \leq p^*_\xi$, by item 1 and (C 1), $p_{\xi+1} \in \mathbf{D}(\lambda, x, p_\xi)$ and $p_{\xi+1} \preccurlyeq^\lambda p_\xi$.

Let $D^* = \{d_\xi \mid \xi < \rho \text{ and 2b obtained at stage } \xi + 1\}$. Clearly, p_λ reduces D to D^*. For if $q \leq p_\lambda$, $q \in D$ and $\xi \in \mathbf{C}^\lambda(q)$, then q witnesses that at step $\xi + 1$ of the construction of $\bar{p}$, 2b obtained. So we have $d_\xi \in D$ with $\xi \in \mathbf{C}^\lambda(d_\xi)$. Moreover, $q \leq p_\lambda \preccurlyeq^\lambda d_\xi$. In order to conclude that this construction works, we need to show that for every $\nu \leq \lambda$, $\bar{p} \restriction \nu$ is (λ, x)-adequate. For this it is enough to explain how precisely we made our choices in the above construction:

Lemma 3.14. *In the previous,* $w_{\xi+1}$*,* $p_{\xi+1}$*,* p^*_ξ *and* d_ξ *can be chosen so that* $\bar{w}$ *is a* (λ, x)*-canonical witness for* $\bar{p}$*.*

Proof. Observe that at successor stages, $p_{\xi+1}$, p^*_ξ and d_ξ are chosen so as to satisfy a property which is Π^T_1 in parameters x and $\bar{w} \restriction \xi + 1$. That is, we may choose $\Pi^T_1(x)$ formula Φ_0 such that $\Phi_0(p_{\xi+1}, p^*_\xi, d_\xi, \bar{w} \restriction \xi + 1)$ holds just if 1 and 2 hold.

In what follows, the choice of $\bar{w}$ will be such that for each $\xi < \lambda$, w_ξ is a quintuple, and $\bar{p}$ is obtained from $\bar{w}$ by projecting to the first coordinate. In fact, we will have $w_{\xi+1} = (p_\xi, M, p^*_\xi, d_\xi, \bar{w} \restriction \xi+1)$ where M is a model allowing us to uniformize Φ_0 and the rest of the coordinates are as in the successor step of the above construction. At limits ν we shall have $w_\nu = (p_\nu, M, p^*, d, \bar{w} \restriction \nu)$, where p^* and d are just dummies. Using the initial segment $\bar{w} \restriction \xi$ as a last coordinate in w_ξ makes $\bar{w}$ $\Pi^T_1(x)$ in the end (see below).

Let $\Phi_1(p, p^*, d, \tilde{w})$ be the formula expressing that if $\nu = \mathrm{dom}(\tilde{w})$ is a limit then p is the greatest lower bound in R of the sequence obtained from $\tilde{w}$ by projecting to the first coordinate (and for clarity, $p^* = d = \emptyset$ if you want); and if $\nu = \xi + 1$ then $\Phi_0(p, p^*, d, \tilde{w})$ holds.

Now let $\Phi(w, \tilde{w})$ be the $\Pi^T_1(x)$ formula expressing: $w = (p, M, p^*, d, \tilde{w})$ is such that

1. M is transitive, $M \prec_{\Sigma_1} L[A]$ and $p, p^*, d, \tilde{w} \in M$,
2. for any initial segment of M containing $p, p^*, d, \bar{w}$ we have $N \not\prec_{\Sigma_1} M$,
3. $M \models (p, p^*, d)$ is $\leq_{L[A}$-least such that $\Phi_1(p, p^*, d, \tilde{w})$.

We may choose $\bar{w}$ recursively such that for each $\xi < \lambda$, $\Phi(w_\xi, \bar{w} \restriction \xi)$ holds: Observe that if such w_ξ exists, it is unique. For limit ν, we may assume by induction that $\bar{w} \restriction \nu$ is a canonical witness for $\bar{p} \restriction \nu$, allowing us to infer $\bar{p} \restriction \nu$ is adequate and that p_ν and hence w_ν exists. At successor stages $\xi + 1$, $w_{\xi+1}$ always exists, and so $\bar{w}$ is well-defined.

Lastly, we show $\bar{w}$ is $\Pi^T_1(x)$: this is because $w = w_\xi$ if and only if w is a quintuple with last coordinate $\bar{w}$ such that $\Phi(w, \bar{w})$ holds, and $\bar{w}$ is a sequence such that for each $\xi \in \mathrm{dom}(\bar{w})$, $\Phi(\bar{w}(\xi), \bar{w} \restriction \xi)$. This finishes the proof of the lemma. □

Having shown that $\bar{p}$ may be chosen as a (λ, x)-adequate sequence, we are finished with the proof of the theorem. □

Lemma 3.15. *For each $\xi < \lambda$, let D_ξ be an open dense subset of R. Let*

$$X = \{q \in R \mid \exists D^* \quad \forall \xi < \lambda \quad q \ \lambda\text{-reduces } D_\xi \text{ to } D^*\}.$$

Then X is dense in $\langle R, \preccurlyeq^\lambda \rangle$ and open in $\langle R, \leq \rangle$.

If $\dot{s}$ is a name such that $p \Vdash \dot{s} \colon \lambda \to V$, the set of q such that $\dot{s}$ is λ-chromatic below q is dense in $\langle R(\leq p), \preccurlyeq^\lambda \rangle$ and open.

Proof. Let $\bar{D} = (D_\xi)_{\xi<\lambda}$ be a sequence of dense open subsets of R. Build a sequence as before: let

$$x = \langle p_0, \preccurlyeq^\lambda, \preccurlyeq^\lambda, \mathbf{C}^\lambda, \mathcal{P}(R), \bar{D}, y \rangle.$$

At successor steps ξ, choose p_ξ and w_ξ such that $p_\xi \preccurlyeq^\lambda p_{\xi-1}$ and $p_\xi \in \mathbf{D}(\lambda, (x, \bar{w} \restriction \xi+1), p_{\xi-1})$ and such that we can pick D^*_ξ such that p_ξ reduces D_ξ to D^*_ξ. As in lemma 3.14, argue this can be done in such a way that the resulting sequence $\bar{p}$ is (λ, x)-adequate. So a greatest lower bound p_λ exists and for each $\xi < \lambda$, p_λ reduces D_ξ to $\bigcup_{\xi<\lambda} D^*_\xi$.

Now let $p \Vdash \dot{s} \colon \lambda \to V$. Let D_ξ be the set of conditions $p \in R$ which decide $\dot{f}(\xi)$. As above, find q reducing all D_ξ to D^*. We now find a spectrum for $\dot{s}$: For $\chi < \lambda$, if $w \leq q$ decides $\dot{s}(\xi)$ and $\chi \in \mathbf{C}^\lambda(w)$, there is also $d \in D^*$ which decides $\dot{s}(\xi)$ and such that $\chi \in \mathbf{C}^\lambda(d)$. Fix z such that $d \Vdash f(\xi) = \check{z}$. Then we may set $H_\xi(\chi) = z$. It is easy to check that for each $\xi < \lambda$, H_ξ is a spectrum for $\dot{s}(\xi)$ (and thus $(H_\xi)_{\xi<\lambda}$ is a spectrum for $\dot{s}$): Say $w \leq q$ decides $\dot{s}(\xi)$ and fix some $\chi \in \mathbf{C}^\lambda(w)$. Then there is $d \in D^*$ with $\chi \in \mathbf{C}^\lambda(d)$ such that $d \Vdash s(\xi) = H_\xi(\chi)$. As $d \in D^*$, $q \preccurlyeq^\lambda d$. So as $\chi \in \mathbf{C}^\lambda(w) \cap \mathbf{C}^\lambda(d)$, w and d are compatible and thus $w \Vdash s(\xi) = H_\xi(\chi)$. □

Corollary 3.16. *Cofinalities greater than λ remain greater than λ after forcing with R and $(2^\lambda)^V = (2^\lambda)^{V[G]}$ for any R-generic V.*

3.4 Composition of stratified forcing

In the main theorem of this section, theorem 3.17 below, we show stratification is preserved by composition. In the proof, we use "guessing systems", which we shall motivate now, before we state and prove the theorem.

Say P is stratified and $\dot{Q}$ is forced by P to be stratified on I, and let $\lambda \in I$ be fixed. We know P has a centering relation $\mathbf{C}$ and $\dot{Q}$ is forced to have a centering relation $\dot{\mathbf{C}}$ in the extension. Similar to the proof that composition of centered forcing stays centered, we want to gain some control over $\dot{\mathbf{C}}$ in the ground model. If we ignore the requirements 3.7(S 4) *density* and 3.8(S 5) *continuity*, we could define $\bar{\mathbf{C}}$ on $P * \dot{Q}$ in the following way:

$$(\chi, \xi) \in \bar{\mathbf{C}}(d, \dot{d}) \iff \chi \in \mathbf{C}(d) \text{ and } d \Vdash \check{\xi} \in \dot{\mathbf{C}}(\dot{d})$$

Then $\operatorname{dom}(\bar{\mathbf{C}})$ is dense and 3.8(S III) *centering* holds.
The following definition also satisfies 3.7(S 4) *density*: let

$$(\chi, X) \in \bar{\mathbf{C}}(d, \dot{d}) \iff \Big(\chi \in \mathbf{C}(d) \text{ and for some } \dot{\xi} \text{ and } \lambda' \in \mathbf{Reg} \cap [\lambda_0, \lambda),$$
$$X \text{ is a } \lambda'\text{-spectrum for } \dot{\xi} \text{ below } d \text{ and } d \Vdash \dot{\xi} \in \dot{\mathbf{C}}(\dot{d})\Big). \quad (3.1)$$

Let's check 3.7(S 4) *density* holds: Given a condition $\bar{p} = (p, \dot{p})$ and $\lambda' \in \mathbf{Reg} \cap [\lambda_0, \lambda)$, we can find $\bar{d} = (d, \dot{d})$ such that $\bar{d} \mathrel{\bar{\preccurlyeq}}^{\lambda'} \bar{p}$, $d \in \operatorname{dom}(\mathbf{C})$ and $d \Vdash \dot{d} \in \operatorname{dom}(\dot{\mathbf{C}})$. Moreover, we can assume that for some name $\dot{\chi}$, $d \Vdash \dot{\chi} \in \dot{\mathbf{C}}(\dot{d})$ and $\dot{\chi}$ is λ'-chromatic. We have $\bar{d} \in \operatorname{dom}(\bar{\mathbf{C}})$. Let's also check that 3.8(S III) *centering* holds: say $\bar{d} \mathrel{\bar{\precsim}}^{\lambda} \bar{p}$ and $(\chi, X) \in \bar{\mathbf{C}}(\bar{p}) \cap \bar{\mathbf{C}}(\bar{d})$. First, observe that p and d are compatible. Fix $\dot{\chi}_0$, $\dot{\chi}_1$ such that both $p \Vdash \dot{\chi}_0 \in \dot{\mathbf{C}}(\dot{p})$ and $d \Vdash \dot{\chi}_1 \in \dot{\mathbf{C}}(\dot{d})$ and X is a spectrum for $\dot{\chi}_0$ below p and for $\dot{\chi}_1$ below d. As $p \cdot d \Vdash \dot{\chi}_0 = \dot{\chi}_1 \in \dot{\mathbf{C}}(\dot{d}) \cap \dot{\mathbf{C}}(\dot{p})$, by *centering* for $\dot{Q}$ in the extension, $p \cdot d \Vdash \dot{d}$ and $\dot{p}$ are compatible, whence $\bar{d}$ and $\bar{p}$ are compatible.

To show stratification is preserved at limits, we will have to use *Continuity* of the centering function; Unfortunately, the approach described above does not yield a *continuous* centering function in the sense of (S 5). For say $\bar{d}_\xi = (d_\xi, \dot{d}_\xi)$ form a λ'-adequate sequence of length ρ, and for each $\xi < \rho$, $d_\xi \Vdash \dot{\chi}_\xi \in \dot{\mathbf{C}}(\dot{d}_\xi)$ and X_ξ is a λ_ξ-spectrum for $\dot{\chi}_\xi$ below d_ξ. By *Continuity* for the components of the forcing, if $(d, \dot{d})$ is a greatest lower bound, we know $d \Vdash \dot{d} \in \operatorname{dom}(\dot{\mathbf{C}})$; but there is no reason to assume that there exists a P-name $\dot{\gamma}$, such that $d \Vdash \dot{\gamma} \in \dot{\mathbf{C}}(\dot{d})$ and $\dot{\gamma}$ is λ''-chromatic for some $\lambda'' < \lambda$.

The solution to this problem is to allow a more general set of values for $\bar{C}(\bar{d})$: in the situation described above, e.g. the sequence $(X_\xi)_{\xi<\rho}$ be used in much the same way as the single spectrum X. This leads to the notion of a guessing system, which will be precisely defined in 3.18.

Theorem 3.17. *Say P is stratified on I and $\dot{Q}$ is forced by P to be stratified on I. Then $\bar{P} = P * \dot{Q}$ is stratified (on I).*

Proof. Say stratification of P is witnessed by $\mathbf{D}, \mathbf{C}^\lambda, \preccurlyeq^\lambda, \precsim^\lambda$ for each regular $\lambda \in I$, and we have class $\dot{\mathbf{D}}$, definable with parameter $\dot{c}$ and names $\dot{\mathbf{C}}^\lambda, \dot{\preccurlyeq}^\lambda$, $\dot{\precsim}^\lambda$ for λ regular which are forced by P to witness the stratification of $\dot{Q}$. We now define $\bar{\mathbf{D}}, \bar{\mathbf{C}}^\lambda, \bar{\preccurlyeq}^\lambda$ and $\bar{\precsim}^\lambda$ for regular $\lambda \geq \lambda_0$ to witness stratification of $P * \dot{Q}$.

The auxiliary orderings

Let $\lambda \in I$ be regular. We say $(p, \dot{q}) \mathrel{\bar{\preccurlyeq}}^\lambda (u, \dot{v})$ if and only if $p \preccurlyeq^\lambda u$ and $p \Vdash_P \dot{q} \mathrel{\dot{\preccurlyeq}}^\lambda \dot{v}$. This defines a pre-order stronger than the natural ordering on

$P * \dot{Q}$ (i.e. 3.1(C 2) holds). Define $\bar{p} \mathrel{\bar{\precsim}}^\lambda \bar{q}$ if and only if $p \precsim^\lambda q$ and if $p \cdot q \neq 0$, $p \cdot q \Vdash_P \dot{p} \precsim^\lambda \dot{q}$.

The ordering axioms

Let $\bar{p} = (p, \dot{p})$, $\bar{q} = (q, \dot{q})$ and $\bar{r} = (r, \dot{r})$ be conditions in $\bar{P}$.

We check that 3.1(C 3) holds: Say $\bar{p} \leq_{\bar{P}} \bar{q} \leq_{\bar{P}} \bar{r}$ and $\bar{p} \mathrel{\bar{\preccurlyeq}}^\lambda \bar{r}$. Then $p \preccurlyeq^\lambda r$ by 3.1(C 3) for P. Moreover, p forces 3.1(C 3) for $\dot{Q}$ as well as $\dot{p} \leq \dot{q} \leq \dot{r}$ and $\dot{p} \mathrel{\dot{\preccurlyeq}}^\lambda \dot{r}$. So $p \Vdash_P \dot{p} \mathrel{\dot{\preccurlyeq}}^\lambda \dot{q}$, and we conclude $\bar{p} \mathrel{\bar{\preccurlyeq}}^\lambda \bar{q}$.

Check that 3.7(S 2) holds: Say $\bar{p} \leq_{\bar{P}} \bar{q} \mathrel{\bar{\precsim}}^\lambda \bar{r}$. By (S 2) for P, $p \precsim^\lambda r$. If $p \cdot r \neq 0$,

$$p \cdot r \Vdash_P \dot{p} \leq_{\dot{Q}} \dot{q} \precsim^\lambda \dot{r},$$

and so $p \cdot r \Vdash_P \dot{p} \precsim^\lambda \dot{r}$. Thus $\bar{p} \mathrel{\bar{\precsim}}^\lambda \bar{r}$.

Next, check 3.7(S I). Say $\bar{p} \mathrel{\bar{\precsim}}^\lambda \bar{q}$ and $\bar{q} \mathrel{\bar{\preccurlyeq}}^\lambda 1_{\bar{P}}$. By (S I) for P, $p \leq q$. So $p \Vdash \dot{p} \mathrel{\dot{\precsim}}^\lambda q \mathrel{\dot{\preccurlyeq}}^\lambda 1_{\dot{Q}}$, so by (S I) applied in the extension, $p \Vdash \dot{p} \leq_{\dot{Q}} \dot{q}$, whence $\bar{p} \leq_{\bar{P}} \bar{q}$. We leave it to the reader to check 3.1(C 4) and 3.8(S 3).

Quasi-Closure

Define $(p, \dot{p}) \in \bar{\mathbf{D}}(\lambda, x, (q, \dot{q}))$ if and only if $p \in \mathbf{D}(\lambda, x, q)$ and $p \Vdash \dot{p} \in \dot{\mathbf{D}}(\lambda, x, \dot{q})$. It is straightforward to see that this definition is $\Pi^T_1((c, \dot{c}))$. Clearly, this defines a "dense and open" set, i.e. (C 1) and (C I) are satisfied.

Now say $(p_\xi, \dot{q}_\xi)_{\xi<\rho}$ is (λ, x)-adequate. We show this sequence has a greatest lower bound. Let $\bar{w}$ be a strategic guide and a canonical witness for $(p_\xi, \dot{q}_\xi)_{\xi<\rho}$. We can immediately infer by the definition of $\bar{\mathbf{D}}$ that $\bar{w}$ is a strategic guide for $(p_\xi)_{\xi<\rho}$. It is also clear that $\bar{w}$ is a canonical witness, since p_ξ can be obtained from $(p_\xi, \dot{q}_\xi)$ by projecting to the first coordinate, and this map is Δ_0. Thus there is a greatest lower bound p_ρ of $(p_\xi)_{\xi<\rho}$.

It is easy to see now that $p_\rho \Vdash_P$"$(\dot{q}_\xi)_{\xi<\rho}$ is (λ, x)-adequate": Fix a $\Pi^T_1(\lambda \cup \{x\})$ formula $\Phi(x, y)$ such that for $\xi < \rho$ we have

$$\Phi(x, \xi) \iff x = w_\xi.$$

Then the relativization $\Phi(x, y)^{L[A]}$ witnesses that 1_P forces that $\bar{w}$ is $\Pi_1(\lambda \cup \{x\})$ in the extension by P, as well. For a $\Sigma^T_1(\lambda \cup \{x\})$-function G, $(p_\xi, \dot{q}_\xi) = G(\bar{w} \restriction \xi + 1)$ for each $\xi < \rho$, and so $\dot{q}_\xi = \pi_1(G(\bar{w} \restriction \xi + 1))$, where π_1 is the projection to the second coordinate. As $G(x) = y$ is Σ^T_1, clearly the relativized formula $(\pi_1(G(x)) = y)^{L[A]}$ is also $\Sigma^T_1(\lambda \cup \{x\})$ in the extension by P. So $\bar{w}$ is forced to be canonical witness. Moreover, it is clear that p_ρ forces that $\bar{w}$ is a strategic guide for $(\dot{q}_\xi)_{\xi<\rho}$, by the definition of $\bar{\mathbf{D}}$.

So we can find $\dot{q}_\rho$ such that $p_\rho \Vdash_P$"$\dot{q}_\rho$ is a greatest lower bound of $(\dot{q}_\xi)_{\xi<\rho}$", whence $(p_\rho, \dot{q}_\rho)$ is a greatest lower bound of the original sequence. Leaving the last sentence of (C II) to the reader, we conclude that $\langle P*\dot{Q}, \bar{\preccurlyeq}^\lambda, (c, \dot{c}), \bar{\mathbf{D}}\rangle$ is λ-quasi-closed above on I.

To define $\bar{\mathbf{C}}^\lambda$, we first define the notion of a *guessing system*. Roughly speaking, a guessing system consists of conditions which are organized in levels; the conditions on the bottom have a $\bar{\mathbf{C}}^\lambda$-value in the sense of (3.1). Conditions on higher levels are greatest lower bounds of conditions on the levels below, and we have some control over their $\dot{\mathbf{C}}^\lambda$-value by *continuity* for $\dot{Q}$.

Definition 3.18. Say $(p, \dot{q}) \in P * \dot{Q}$ and λ is regular and uncountable. A *λ-guessing system for $\dot{q}$ below p* is a quadruple $(T_g, H_g, \lambda_g, q_g)$ such that

1. T_g is a tree, $T_g \subseteq {}^{<\omega}\gamma$, where $\gamma = \mathrm{width}(T) < \lambda$ and $\lhd$ (initial segment) is reversely well founded on T_g. The root of T_g is $\emptyset$ (i.e. the empty sequence).
2. For $s \in T_g$, $\rho_g(s) = \{\xi \mid s ^\frown \xi \in T_g\}$ is an ordinal. Write T_g^0 for the set of $\lhd$-maximal $s \in T_g$, i.e. $T_g^0 = \{s \in T_g \mid \rho_g(s) = 0\}$.
3. q_g is a function from T_g into the set of P-names for conditions in $\dot{Q}$ and $\lambda_g \colon T_g \to \lambda \cap \mathbf{Reg}$.
4. For $s \in T_g \setminus T_g^0$, $\{q_g(s ^\frown \xi)\}_{\xi<\rho_g(s)}$ is a $\lambda_g(s)$-adequate sequence and p forces that $q_g(s)$ is a greatest lower bound of $\{q_g(s ^\frown \xi)\}_{\xi<\rho_g(s)}$.
5. $\mathrm{dom}(H_g) = T_g^0$.
6. For $s \in T_g^0$, there is a P-name $\dot{\chi}$ such that $p \Vdash_P \dot{\chi} \in \mathbf{C}^\lambda(q_g(s))$ and $H_g(s)$ is a $\lambda_g(s)$-spectrum of $\dot{\chi}$ below p.
7. $q_g(\emptyset) = \dot{q}$.

Now we are ready to define $\bar{\mathbf{C}}^\lambda$: let $s \in \bar{\mathbf{C}}^\lambda(p, \dot{q})$ if and only if *either*

(a) $s \in \mathbf{C}^\lambda(p)$ and $p \Vdash \dot{q} \dot{\preccurlyeq}^\lambda 1_{\dot{Q}}$ holds *or else*

(b) if $\lambda > \min I$, $s = (\chi, T_g, H_g, \lambda_g)$ where $\chi \in \mathbf{C}^\lambda(p)$ and for some q_g, $(T_g, H_g, \lambda_g, q_g)$ is a λ-guessing system for $\dot{q}$ below p.

(c) if $\lambda = \min I$, $s = (\chi, \xi)$, where $\chi \in \mathbf{C}^\lambda(p)$ and $p \Vdash_P \check{\xi} \in \dot{\mathbf{C}}^\lambda(\dot{q})$.

It is straightforward to check that $\mathrm{ran}(\bar{\mathbf{C}}^\lambda)$ has size at most λ. Thus we may assume $\bar{\mathbf{C}}^\lambda \subseteq (P * \dot{Q}) \times \lambda$, although this is not literally the case.

We have finally defined the stratification of $\bar{P} = P * \dot{Q}$. Let's check the remaining axioms.

Continuity

Say $\lambda' \in I$, $\lambda' < \lambda$), both $\bar{p} = (p_\xi, \dot{p}_\xi)_\xi$ and $\bar{q} = (q_\xi, \dot{q}_\xi)_\xi$ are λ'-adequate sequences of length ρ and for each $\xi < \rho$, $\bar{\mathbf{C}}^\lambda(p_\xi, \dot{p}_\xi) \cap \bar{\mathbf{C}}^\lambda(q_\xi, \dot{q}_\xi) \neq \emptyset$. Moreover, let $(p, \dot{p})$ and $(q, \dot{q})$ denote greatest lower bounds of $\bar{p}$ and $\bar{q}$, respectively.

First, by *Continuity* for P, we can find $\chi \in \mathbf{C}^\lambda(p) \cap \mathbf{C}^\lambda(q)$. For each $\xi < \rho$, fix $(T^\xi_g, H^\xi_g, \lambda^\xi_g)$ such that for some χ',

$$(\chi', T^\xi_g, H^\xi_g, \lambda^\xi_g) \in \bar{\mathbf{C}}^\lambda(p_\xi, \dot{p}_\xi) \cap \mathbf{C}^\lambda(q_\xi, \dot{q}_\xi).$$

and find p^ξ_g such that $(T^\xi_g, H^\xi_g, \lambda^\xi_g, p^\xi_g)$ is a guessing system for $\dot{p}_\xi$ below p_ξ.

Now construct a guessing system $(T_g, H_g, \lambda_g, p_g)$ for $\dot{p}$ below p, showing $(p, \dot{p}) \in \mathrm{dom}(\bar{\mathbf{C}}^\lambda)$. It will be clear from the construction that T_g, H_g and λ_g do not depend on the sequence of p^ξ_g, $\xi < \rho$. Let $s \in T_g$ if and only if $s = \emptyset$ or $\xi \frown s \in T^\xi_g$. Let $\lambda_g(\emptyset) = \lambda'$, and of course $p_g(\emptyset) = \dot{p}$. Now let $s \in T_g \setminus \{\emptyset\}$ be given and define $\lambda_g(s)$, $p_g(s)$ and, in the case that $s \in T^0_g$, also define $H_g(s)$. Find s' such that $s = \xi \frown s'$. Let $\lambda_g(s) = \lambda^\xi_g(s')$ and let $H_g(s) = H^\xi_g(s')$ if $s \in T^0_g$ (or equivalently, if $s' \in (T^\xi_g)^0$). Let $p_g(s) = p^\xi_g(s')$.

To check that $(T_g, H_g, \lambda_g, p_g)$ is a guessing system, first observe that $\lhd$ is reversely well-founded on T_g. Moreover, $\rho_g(\emptyset) = \rho$ is an ordinal and $\lambda_g(\emptyset) < \lambda$. Also, clause 4. holds for $s = \emptyset$, by construction. The rest of the conditions are straightforward to check; they hold by construction and because for each $\xi < \lambda'$, $(T^\xi_g, H^\xi_g, \lambda^\xi_g, p^\xi_g)$ is a guessing system.

If we carry out the same construction for $(q, \dot{q})$, we obtain q_g such that $(T_g, H_g, \lambda_g, q_g)$ is a guessing system for $\dot{q}$ below q. Thus,

$$(\chi, T_g, H_g, \lambda_g) \in \bar{\mathbf{C}}^\lambda(p, \dot{p}) \cap \bar{\mathbf{C}}^\lambda(q, \dot{q}).$$

Interpolation

Say $(d, \dot{d}) \leq_{\bar{P}} (r, \dot{r})$. First find $p \in P$ such that $p \precsim^\lambda d$ and $p \preccurlyeq^\lambda r$. If $p \cdot d \neq 0$, then $p \cdot d \Vdash_P \dot{d} \leq_{\dot{Q}} \dot{r}$, so we can find $\dot{p}$ such that $p \cdot d \Vdash_P \dot{p} \dot{\precsim}^\lambda \dot{d}$ and $p \Vdash_P \dot{p} \dot{\preccurlyeq}^\lambda \dot{r}$.

Centering

Say $\bar{p} \bar{\precsim}^\lambda \bar{d}$, where $\bar{p} = (p, \dot{p})$ and $\bar{d} = (d, \dot{d})$, and assume $\bar{\mathbf{C}}^\lambda(\bar{p}) \cap \bar{\mathbf{C}}^\lambda(\bar{d}) \neq \emptyset$. First assume we can find $(\chi, T_g, \lambda_g, H_g) \in \bar{\mathbf{C}}^\lambda(\bar{p}) \cap \bar{\mathbf{C}}^\lambda(\bar{d})$ (i.e. (b) holds in the definition of $\bar{\mathbf{C}}^\lambda$). As $\chi \in \mathbf{C}^\lambda(p) \cap \mathbf{C}^\lambda(d)$, by *centering* for P there exists w such that for all regular $\lambda' \in 0 \cup I \cap \lambda$, both $w \preccurlyeq^{\lambda'} p$ and $p \precsim^{\lambda'} d$.

Now fix p_g and d_g such that $(T_g, \lambda_g, H_g, p_g)$ is a guessing system for $\dot{p}$ below p and $(T_g, \lambda_g, H_g, d_g)$ is a guessing system for $\dot{d}$ below d. We show by induction on the rank of s (in the sense of the reversed $\lhd$-order) that for each $s \in T_g$,

$$p \cdot d \Vdash_P \dot{\mathbf{C}}^\lambda(p_g(s)) \cap \dot{\mathbf{C}}^\lambda(d_g(s)) \neq 0. \tag{3.2}$$

First, let $s \in T_g^0$. By definition 3.18, 6. we can find P-names $\dot\alpha$ and $\dot\beta$ such that both have spectrum $H_g(s)$ below p and d, respectively, and moreover:

$$p \Vdash_P \dot\alpha \in \dot{\mathbf{C}}^\lambda(p_g(s))$$

and

$$d \Vdash_P \dot\beta \in \dot{\mathbf{C}}^\lambda(d_g(s)).$$

Thus, as $\dot\alpha$ and $\dot\beta$ have a common spectrum below $p \cdot d$, (3.2) holds.

For s of greater rank, we may assume by induction that for each $\xi < \rho_g(s)$,

$$p \cdot d \Vdash_P \dot{\mathbf{C}}^\lambda(p_g(s ^\frown \xi)) \cap \dot{\mathbf{C}}^\lambda(d_g(s ^\frown \xi)) \neq 0. \tag{3.3}$$

As p forces that

$$\begin{array}{l}\{p_g(s ^\frown \xi)\}_{\xi<\rho_g(s)} \text{ is a } \lambda_g(s)\text{-adequate sequence and } p_g(s) \text{ is a} \\ \text{greatest lower bound of } \{p_g(s ^\frown \xi)\}_{\xi<\rho_g(s)},\end{array} \tag{3.4}$$

and as d forces the corresponding statement for $d_g(s)$ and $\{d_g(s ^\frown \xi)\}_{\xi<\rho_g(s)}$, *Continuity* for $\dot Q$ in the extension allows us to infer (3.2) for this s. This finishes the inductive proof on the rank of s.

Finally, (3.2) holds for $s = \emptyset$, so as $p_g(\emptyset) = \dot p$ and $d_g(\emptyset) = \dot d$, by *centering* for $\dot Q$ in the extension, $w \Vdash$ there exists $\dot w$ such that for all regular $\lambda' \in 0 \cup I \cap \lambda$, both $\dot w \dot\preccurlyeq^{\lambda'} \dot p$ and $\dot w \dot\preccurlyeq^{\lambda'} \dot d$. Then $\bar w = (w, \dot w)$ is as desired.

Now secondly assume we have $\chi \in \bar{\mathbf{C}}^\lambda(\bar p) \cap \bar{\mathbf{C}}^\lambda(\bar d)$ and (a) holds in the definition of $\bar{\mathbf{C}}^\lambda$. In this case $\chi \in \mathbf{C}^\lambda(p) \cap \mathbf{C}^\lambda(d)$. Let $w \in P$ such that $w \preccurlyeq^{<\lambda} p$ and $w \preccurlyeq^{<\lambda} d$. By assumption, $w \Vdash \dot d \dot\preccurlyeq^\lambda 1_{\dot Q}$ and $\dot p \dot\preccurlyeq^\lambda \dot d$. By *expansion* (S I) for $\dot Q$, we conclude $w \Vdash \dot p \leq \dot d$. We claim $\bar w = (w, \dot p)$ is the desired lower bound: $\bar w \mathrel{\bar\preccurlyeq}^{<\lambda} \bar p$ holds because $\dot\preccurlyeq^\lambda$ is a pre-order. We show $\bar w \mathrel{\bar\preccurlyeq}^{<\lambda} \bar d$: we have $w \Vdash \dot p \leq \dot d \leq 1_{\dot Q}$ and by assumption $w \Vdash \dot p \dot\preccurlyeq^\lambda 1_{\dot Q}$. So by (C 3), we conclude $w \Vdash \dot p \dot\preccurlyeq^{<\lambda} \dot d$ and are done.

Density

Let $(p_0, \dot q_0) \in \bar R$. First, assume $minI < \lambda$ and fix a regular $\lambda' \in I \cap \lambda$. By *Density* for $\dot Q$ in the extension, we can find P-names $\dot\chi$ and $\dot q_1$ such that $\Vdash_P$"$\dot q_1 \dot\preccurlyeq^{\lambda'} \dot q_1$ and $\dot\chi \in \mathbf{C}^\lambda(\dot q_1)$". By lemma 3.15, we can find $p_1 \preccurlyeq^{\lambda'} p_0$ such that $\dot\chi$ is λ'-chromatic below p_1, and by *Density* for P we can find $p_2 \preccurlyeq^{\lambda'} p_1$ and ζ such that $\zeta \in \mathbf{C}^\lambda(p_2)$.

Let $T_g = \{\emptyset\}$, $q_g(\emptyset) = \dot{q}_1$, $\lambda_g(\emptyset) = \lambda'$ and let $H_g(\emptyset)$ be a λ'-spectrum of $\dot{\chi}$ below p_2. Thus $(T_g, \lambda_g, H_g, q_g)$ is a guessing system for $\dot{q}_1$ below p_2—the only non-trivial clause is (6.), which holds as $H_g(\emptyset)$ is a λ'-spectrum of $\dot{\chi}$ below p_1 and $p_2 \leq_P p_1$. So $(\zeta, T_g, \lambda_g, H_g) \in \bar{\mathbf{C}}^\lambda(p_2, \dot{q}_1)$, and $(p_2, \dot{q}_1) \bar{\precsim}^{\lambda'} (p_0, \dot{q}_0)$.

It remains to show $\mathrm{dom}(\mathbf{C}^\lambda)$ is dense in the case that $minI = \lambda$. Find $(p_1, \dot{q}_1) \leq_{\bar{R}} (p_0, \dot{q}_0)$ such that for some ordinals $\zeta, \chi < \lambda$, $\zeta \in \mathbf{C}^\lambda(p)$ and $p_1 \Vdash_P \check{\chi} \in \dot{\mathbf{C}}^\lambda(\dot{q}_1)$. Then $(\zeta, \chi) \in \bar{\mathbf{C}}^\lambda(p_1, \dot{q}_1)$. □

3.5 Stratified iteration and diagonal support

We now proceed to show the notion of stratified forcing is iterable, if the right support is used. To this end, let's define stratified iteration with diagonal support.

To motivate this, imagine we want to take a product of forcings $P_\xi * \dot{Q}_\xi$, of the type of example 3.11. The present approach to showing these forcings preserve cofinalities makes use of the fact that for large enough λ_0, P_ξ is closed under sequences of length λ_0 while $\dot{Q}_\xi$ has a (strong form of) $(\lambda_0)^+$-chain condition. If we want to preserve the latter, we should use support of size less than λ_0; if we want to preserve the first, our choice would be to use support of size λ_0. This calls for a kind of mixed support: i.e. define

$$\Pi^d_{\xi<\lambda_0}(P_\xi * \dot{Q}_\xi)$$

to be the set of all sequences $(p(\xi), \dot{q}(\xi))_{\xi<\lambda_0} \in \Pi_{\xi<\lambda_0}(P_\xi * \dot{Q}_\xi)$ such that for all but less than λ_0 many ξ,

$$p(\xi) \Vdash_P \dot{q}(\xi) = \dot{1}_\xi. \tag{3.5}$$

Using the stratification of $P * \dot{Q}$, (3.5) may be written as

$$(p(\xi), \dot{q}(\xi)) \precsim^{\lambda_0} 1_{P_\xi * \dot{Q}_\xi}.$$

The use of the term "diagonal" is motivated by the intuition that we allow large support on P_ξ, which we regard as the "upper" part, and small support on $\dot{Q}_\xi$, which we regard as the "lower" part of the forcing $P_\xi * \dot{Q}_\xi$.

Definition 3.19. 1. We say the iteration $\bar{Q}^\theta = \langle P_\nu; \dot{Q}_\nu, \leq_\nu, \dot{1}_\nu \rangle_{\nu<\theta}$ *has stratified components* if and only if for every $\nu < \theta$, $\dot{Q}_\nu$ is a P_ν-name and P_ν forces $\dot{Q}_\nu$ is a stratified partial order as witnessed by the system

$$\bar{\mathbf{S}} = (\dot{\precsim}^\lambda{}_\nu, \dot{c}_\nu, \dot{\bar{\precsim}}^\lambda{}_\nu, \dot{\mathbf{D}}_\nu, \dot{\mathbf{C}}_\nu)_{\lambda \in \mathbf{Reg}, \nu<\theta}.$$

(which is called its *stratification*). Formally, the reader may wish to replace $\dot{\mathbf{D}}_\nu$ in the above by a name for the Gödel number of a formula defining the class $\dot{\mathbf{D}}_\nu$ with parameter $\dot{c}_\nu$. Moreover, we demand that for all regular λ there is $\bar{\lambda} < \lambda^+$ such that $\mathrm{supp}_\lambda(p) \subseteq \bar{\lambda}$, where $\mathrm{supp}_\lambda(p)$ is defined as

$$\mathrm{supp}^\lambda(p) = \{\xi \mid p \restriction \xi \nVdash_\xi p(\xi) \dot{\preccurlyeq}_\xi \dot{1}_\xi\}.$$

2. P_θ is the *diagonal support limit* of the iteration with stratified components $\bar{Q}^\theta$ with stratification $\bar{\mathbf{S}}$ if and only if P_θ is the set of all threads though $\bar{Q}^\theta$ such that for each regular λ, $\mathrm{supp}_\lambda(p)$ has size less than λ.

3. We say $\bar{Q}_\theta$ is an iteration with diagonal support if for all limit $\nu < \theta$, P_ν is the diagonal support limit of $\bar{Q}\nu$.

We omit the proof of the following theorem, since it will follow from theorem 5.23 and lemma 5.20 as corollary 5.26. In the proof of the main theorem, we will need to use these stronger lemmas, theorem 3.20 does not suffice.

Theorem 3.20. *Say $\bar{Q} = \langle P_\nu, \dot{Q}_\nu \rangle_{\nu<\theta}$ is an iteration with stratified components and diagonal support. Then P_θ is stratified.*

Chapter 4

Easton Supported Jensen Coding

In this section we shall discuss Easton supported Jensen coding. To avoid repetition, see the beginning of section 7 (definition 7.1 and the preceding discussion) for a comprehensive motivation.

4.1 A variant of Square

The variant of square we discuss in this section is a technical prerequisite which we use to obtain a smooth transition from inaccessible to singular coding at certain points in our construction, an approach we shall refer to as *fake inaccessible coding.* We will say more about this when we use it.

Lemma 4.1. *There is a class* $(E_\alpha)_{\alpha\in\mathbf{Card}}$ *such that for all* $\alpha \in \mathbf{Card}$ *which are not Mahlo,* E_α *is club in* α*,* $E_\alpha \subseteq \mathbf{Sing}$ *and whenever* β *is a limit point of* E_α *we have* $E_\beta = E_\alpha \cap \beta$ *and* $E_\alpha \in J^{A\cap\alpha}_{\delta+2}$ *whenever* $J^{A\cap\alpha}_{\delta} \models$*"*$\alpha$ *is not Mahlo."*

Proof. Let $(C_\alpha)_{\alpha\in\mathbf{Sing}}$ be the standard global square on singulars, constructed as in [Sch14, 11.63, p. 228]. For $\alpha \in \mathbf{Sing}$, let $\eta^*(\alpha)$ be the maximal limit ordinal such that α is regular in $J^{A\cap\alpha}_{\eta^*(\alpha)}$ and let $M^*_\alpha = J^{A\cap\alpha}_{\eta^*(\alpha)}$. Observe that if β is a limit point of C_α, there is $\sigma\colon M^*_\beta \to M^*_\alpha$ which is (at least) Σ_0-elementary such that $\mathrm{crit}(\sigma) = \beta$ and $\sigma(\beta) = \alpha$.

Say α is not Mahlo. Let η be least such that $J^{A\cap\alpha}_\eta \models$"$\alpha$ not Mahlo", let $m_\alpha = J^{A\cap\alpha}_\eta$, $\eta(\alpha) = \eta + 1$ and let $M_\alpha = J^{A\cap\alpha}_{\eta(\alpha)}$.[1]

Case 1 If $\alpha \in \mathbf{Sing}^{M_\alpha}$ we let $E_\alpha = C_\alpha$, where the latter comes from the standard square.

[1]Observe we could write $\eta(\alpha) = \eta$ above. Then m_α would be minimal with a definable club of definably singular cardinals. This is still enough to make the rest of the argument go through, as our goal was to witness the non-Mahlo-ness in a way that is preserved with Σ_0-embeddings with large enough range.

Case 2 Otherwise, if α is $\mathbf{\Sigma}_1$-singular in M_α, to ensure coherency we define E_α to be the tail of C_α obtained by requiring that the maps witnessing that $\xi \in C_\alpha$ have m_α in their range.

In detail: Note that $\rho_1(M^*_\alpha) = \alpha$. Let $p(\alpha)$ be the 1st standard parameter and let

$$W(\alpha) = \{W^{\nu, p_1(M^*_\alpha)}_{M^*_\alpha} \mid \nu \in p_1(M^*_\alpha)\}$$

be the appropriate "solidity witness" (following the notation in [SZ10]). Observe that by construction, we can find a minimal $\theta(\alpha) < \alpha$ such that $h^{M^*_\alpha}_{\Sigma_1}(\theta(\alpha) \cup \{p(\alpha)\})$ is unbounded in α.[2] Let E^*_α consist of those $\beta \in \alpha \cap \mathbf{Sing}$ such that there is a Σ_0-elementary map $\sigma\colon J^{A\cap\beta}_{\bar{\eta}} \to M^*_\alpha$ such that $\{\alpha, p(\alpha), m_\alpha\} \cup W(\alpha) \subseteq \mathrm{ran}(\sigma)$, $\mathrm{crit}(\sigma) = \beta$ and $\sigma(\beta) = \alpha$. As in the proof of $\square$, we show that if E^*_α is bounded in α, $\mathrm{cf}\,(\alpha) = \omega$. In this case, we can set $E_\alpha = C_\alpha$.

Fact 4.2. If E^*_α is bounded in α, $\mathrm{cf}\,(\alpha) = \omega$.

Sketch of the proof. We need to find an embedding σ witnessing $\beta \in E^*_\alpha$ for some large enough β. Let $\xi < \alpha$ be given and let $M \prec M_\alpha$ be a countable elementary submodel such that $\{\xi, m_\alpha\} \subseteq M$ and let $\pi\colon \bar{M} \to M$ be the inverse of the collapsing map. Let E be the $(\mathrm{crit}(\pi), \beta)$-extender, where $\beta = \sup(\mathrm{ran}(\pi) \cap \alpha)$, derived from π. Let $\sigma\colon \mathrm{Ult}(\bar{M}, E) \to M_\alpha$ be the factor map. Check that σ is as required, in particular it has critical point β and $m_\alpha \in \mathrm{ran}(\sigma)$. $\square$

If E^*_α is unbounded in α, we define two sequences as follows. Let $\gamma_0 = \min E^*_\alpha$. Given γ_ξ, let δ_ξ be the least δ such that $h^{M^*_\alpha}_{\Sigma_{n(\alpha)}}(\delta \cup \{p(\alpha)\}) \setminus (\gamma_\xi + 1) \neq \emptyset$. Let

$$\gamma_{\xi+1} = \min[E^*_\alpha \setminus h^{M^*_\alpha}_{\Sigma_{n(\alpha)}}(\delta_\xi \cup \{p(\alpha)\})].$$

Here, assume that $h^{M_\alpha}_{\Sigma_{n(\alpha)}}$ to be defined such that its range grows by one element for every ordinal, slightly abusing notation.

At limit points λ, let $\gamma_\lambda = \bigcup_{\xi<\lambda} \gamma_\xi$, if this yields a point below α. Otherwise set $\lambda = \bar{\theta}(\alpha)$ and stop the construction. Observe that $\delta_\xi < \theta(\alpha)$ and the δ_ξ are increasing, so the ordertype $\bar{\theta}$ of the sequence constructed is at most $\theta(\alpha) < \alpha$. Set

$$E_\alpha = \{\gamma_\xi \mid \xi < \bar{\theta}(\alpha)\} \cap \mathbf{Card}.$$

[2]Pick a minimally definable function F witnessing that α is singular; the parameter of F, if any, can be absorbed into $\theta(\alpha)$ since M_α projects to α; now the Skolem hull is a fortiori unbounded.

Note the only difference to $\square$ is the requirement $m_\alpha \in \mathrm{ran}(\sigma)$ when $\eta^*(\alpha) = \eta(\alpha)$, i.e. $M_\alpha = M^*_\alpha$.

Case 3 α is $\mathbf{\Sigma}_1$-regular over M_α. So

$$E_\alpha = \{\beta < \alpha \mid h^{M_\alpha}_{\Sigma_1}(\beta \cup \{m_\alpha\}) \cap \alpha = \beta\}$$

defines a club. Observe that for some $C \in m_\alpha$, we have $m_\alpha \models C \subseteq$ **Sing** and C is club in α; moreover, $\{\alpha, C\} \subseteq h^{M_\alpha}_{\Sigma_n}(\emptyset \cup \{m_\alpha\})$ (α is the cardinity of m_α). So for every n and $\beta \in E^n_\alpha$, we have $\beta \in C$ by elementarity, and thus $\beta \in$ **Sing**.[3]

It remains to see that $(E_\alpha)_{\alpha\in\mathbf{Sing}}$ is coherent. So let $\alpha \in$ **Sing** and β be a limit of E_α. We must check that β falls into the same case as α.

Assume α falls into case 1. We know that there is a Σ_0-elementary $\sigma\colon M^*_\beta \to M^*_\alpha$. Since $\eta^*(\alpha) < \eta(\alpha)$, it must be the case that $\eta^*(\beta) < \eta(\beta)$: otherwise, m_β is a J-structure which is a model of "β is not Mahlo" and by elementarity $\sigma(m_\beta)$ is a J-structure and $\sigma(m_\beta) \models$"α is not Mahlo," contradicting $\eta^*(\alpha) < \eta(\alpha)$. Thus β also falls into case 1 and coherency follows by the coherency of $\square$.

Assume α falls into case 2. It follows that $\eta^*(\alpha) = \eta(\alpha)$. We know that there is $\sigma\colon J^{A\cap\beta}_{\bar\eta} \to M^*_\alpha$ as in the definition. Standard fine structural arguments (e.g. see [Sch14]) show that $\eta^*(\beta) = \bar\eta$, i.e. $J^{A\cap\beta}_{\bar\eta} = M^*_\beta$ and β is Σ_1-singular in M^*_β. We have $m_\alpha \in \mathrm{ran}(\sigma)$, so by Σ_0-elementarity $\sigma^{-1}(m_\alpha)$ must be a J-structure and $\sigma^{-1}(m_\alpha) = J^{A\cap\beta}_{\eta^*(\beta)-1}$, for otherwise the rudimentary closure of $m_\alpha \cup \{m_\alpha\}$ would have to exist in M_α. Thus $\eta^*(\beta) = \eta(\beta)$, for by a similar argument, $J^{A\cap\beta}_{\eta^*(\beta)-2} \models$"$\beta$ is Mahlo." Thus β falls into case 1. Now the same arguments as in the proof of $\square$ show that $E_\beta = E_\alpha \cap \beta$ (maps factor because $W(\alpha) \subseteq \mathrm{ran}(\sigma)$).

Now assume α falls into case 3. Let $\sigma\colon J^{A\cap\beta}_{\bar\eta} \to M_\alpha$ be the inverse of the collapsing map of $h^{M_\alpha}_{\Sigma_1}(\beta \cup \{m_\alpha\})$, and let $\bar m = \sigma^{-1}(m_\alpha)$. Clearly $\sigma(\beta) = \alpha$, $\mathrm{crit}(\sigma) = \beta$. By elementarity, $\bar m$ is the first J-structure witnessing that β is not Mahlo. Thus, $\bar\eta = \eta(\beta)$. Also, by elementarity and since β is a limit point of E_α, β is $\mathbf{\Sigma}_1$-regular in $M_\beta = J^{A\cap\beta}_{\bar\eta}$. Thus β falls into case 2. It is easy to check that $E_\beta = E_\alpha \cap \beta$.

$\square$

[3] The same argument would work in case 1.

4.2 Notation

Definition 4.3 (Notation)**.** In the following we shall use a convenient partition of the ordinals into 4 components: for $0 \leq i \leq 4$, write $[\mathrm{On}]_i$ for the set of ξ which is equal to i modulo 4. Write $(\xi)_i$ for the ξ-th element of $[\mathrm{On}]_i$. Given a set (or class) B, write $[B]_i = \{(\xi)_i \mid \xi \in B\}$; this is consistent with the notation $[\mathrm{On}]_i$ for the i-th component. When we say A *codes* B *on its* i*-th component*, we mean $A \cap [\mathrm{On}]_i = [B]_i$; that is $\xi \in B \iff (\xi)_i \in A$. We also write $[A]_i^{-1}$ for $\{\xi \mid (\xi)_i \in A\}$, so that $[A]_i^{-1}$ is the set coded by the i-th component of A. We shall sometimes write $B \oplus C$ for $[B]_0 \cup [C]_1$.

If s and t are strings of 0's and 1's, we shall write $s^\frown t$ for the concatenation of s and t.[4] If $X \subset \mathrm{On}$, we write $s \frown X$ for $s \frown \chi_X$, the concatenation of s with the characteristic function of X. Write $s \frown i$ for $s \frown t$ where t is the string with length 1 and value i.

Let $\langle .\,, . \rangle$ denote the Gödel pairing function.

We work in the following settting, which captures the essence of the situation we shall find ourselves in at (some of the) successor stages in our iteration. Again, see the beginning of section 7 (definition 7.1 and the preceding discussion) for a comprehensive motivation.

Say κ is the only Mahlo and $V = L[G^o][\bar{B}^-]$, where, $G^o \subseteq \mathbf{H}(\kappa^+)$ and $\bar{B}^- \subseteq \mathbf{H}(\kappa^{++})$. We also assume $\mathbf{H}(\kappa^{++}) = L_{\kappa^{++}}[G^o]$ (intuitively, $\bar{B}^-$ contributes no subsets of κ^+ — in fact, it comes from a κ^{++}-distributive forcing). Lastly, for some reals x_0, $\mathbf{Card}^V = \mathbf{Card}^{L[x_0]}$.

Let $\bar{T}(\sigma, n, i)$ be a constructible, canonically definable sequence of κ^{++}-Suslin trees, indexed by $(\sigma, n, i) \in \kappa \times \omega \times 2$. Further, assume that in $L[G^o][\bar{B}^- A]$, there is a set I of size κ and a real r such that $\bar{B}^-(\sigma, n, i)$ is a branch through $T(\sigma, n, i)$ if and only if $r(n) = i$ and $\sigma \in I$; otherwise, we assume $\bar{B}^-(\sigma, n, i) = \emptyset$, and $T(\sigma, n, i)$ remains κ^{++}-Suslin in $L[G^o][\bar{B}^-]$.[5]

Find $A_0 \subseteq \kappa^{++}$ such that for each cardinal $\alpha \leq \kappa^{++}$, we have $\mathbf{H}(\alpha) = L_\alpha[A_0]$. We can also assume that $A_0 \cap [\mathrm{On}]_0 = [I]_0$, $A_0 \cap [\mathrm{On}]_1 = [x_0 \oplus \{2n + i \mid r(n) = i\}]_1$ and $A_0 \cap [\kappa^+, \kappa^{++}) \cap [\mathrm{On}]_1$ is the set $\{(\#(\sigma, n, i, t))_1 \mid r(n) = i, \sigma \in I, t \in \bar{B}(\sigma, n, i) \setminus \mathbf{H}(\kappa^+)\}$. Thus we have $V = L[G^o][\bar{B}^-] = L[A_0]$ and for each cardinal α, we have $\mathbf{H}(\alpha) = L_\alpha[A_0]$. Moreover, $\mathbf{Card}^V = \mathbf{Card}^{L[A_0 \cap \omega]}$.

[4] That is, supposing $\mathrm{dom}(s) = \alpha$ and $\mathrm{dom}(t) = \beta$, we have $\mathrm{dom}(s \frown t) = \alpha + \beta$, $(s \frown t) \restriction \alpha = s$ and $s \frown t(\alpha + \xi) = t(\xi)$ for $\xi \in [\alpha, \beta)$.

[5] The current proof could be formulated without this last requirement, but this would be cumbersome. Also, to show both quasi-closure of our iteration P_κ and that r is projective in the generic extension via P_κ, we must work in a model where the requisite trees remain Suslin when we define Easton supported Jensen coding.

We shall write s_{κ^+} for the characteristic function of $A_0 \cap [\kappa^+, \kappa^{++})$, as we will sometimes treat this set similar to conditions in our forcing (the top "string"). Our goal is to find a forcing P which codes $L[A_0]$ into a subset of ω_1, using David's Trick (or Killing Universes), with Easton support. This means that if G is P-generic over $L[A]$, $L[G] = L[G \cap \omega_1]$ and A_0 is "locally" definable in $L[G]$. It is then easy to code $L[G \cap \omega_1]$ into a real.

4.3 Coding structures and coding apparatus

We now define the building blocks of our forcing. These consist of three types of almost-disjoint coding (for successor cardinals, for inaccessible cardinals, and for singular cardinals) using a specific *coding apparatus*.

We will make use of our partition in the following way: the coding of A_0 by G uses $[\mathrm{On}]_0$, the "successor coding" of $G \cap [\alpha^+, \alpha^{++})$ uses $[\mathrm{On}]_1$, the "inaccessible limit coding" (from $G \cap [\alpha, \alpha^+)$ to $G \cap \alpha$ for inaccessible α) uses $[\mathrm{On}]_2$, except at the Mahlo, where $[\mathrm{On}]_3$ is used. Finally, the singular limit coding (from $G \cap [\alpha, \alpha^+)$ to $G \cap \alpha$ for singular α) uses $[\mathrm{On}]_3$.

The definition of the building blocks will be relative to a set A, which stands in lieu of A_0 above. We will later not use A_0 itself, but a slightly adapted $A = A_p$, depending on $p \in P$ (and in notation from below, on α). This is due to the nature of the Mahlo coding from κ^+ into κ and can be circumvented by doing the coding as a three step iteration.

Definition 4.4 (The locally regular case)**.** Let α be an uncountable cardinal $\leq \kappa^{++}$ and let $A \subseteq \mathrm{On}$. Let $\langle \diamondsuit_\alpha ; \alpha \in \mathrm{On} \rangle$ be the global $\diamondsuit$-sequence of L concentrating on the inaccessibles below κ (i.e. the following holds in L: if $X \subseteq \kappa$, the set $\{\alpha \mid \diamondsuit_\alpha = X \cap \alpha\} \cap \mathbf{Inacc}$ is stationary). For a more uniform notation, write $E_\kappa = \emptyset$.

Basic Strings Let $S_\alpha = S(A)_\alpha$ denote the set of $s \colon [\alpha, |s|) \to 2$, where $\alpha \leq |s| < \alpha^+$. We abuse notation by writing $\emptyset_\alpha$ for the empty string at α and note $|\emptyset_\alpha| = \alpha$.

Steering Ordinals For $\xi \in [\alpha, \alpha^+)$, define $\mu_0^\xi = \mu(A)_0^\xi$ and $\mu_0^{<\xi}$ simultaneously by induction: Let $\mu_0^{<\alpha}$ be least μ such that $E_\alpha, \diamondsuit_\alpha \in L_\mu[A \cap \alpha]$[6]

[6]We want E_α in the coding structures of locally inaccessibles, whence E_α is hit and we can easily distinguish between conditions as they turn out in the distributivity proof and those as built to show extendibility. The presence of $\diamondsuit_\alpha$ is important for a kind of κ-chain condition in 7.6, when we show that κ is not collapsed in our iteration and stays Mahlo until the last stage.

and for $\xi > \alpha$ let
$$\mu_0^{<\xi} = \sup_{\nu<\xi} \mu^\nu.$$

Define $\sigma(\xi)$ to be least above $\mu_0^{<\xi}$ such that $L_{\sigma(s)} \models$"α is the greatest cardinal." Let $\mu_0^\xi = \sigma(\xi) + \alpha$.

Note that by induction $\xi \leq \mu_0^{<\xi}$. If $\alpha \in \mathbf{Reg}$, we write μ^ξ and $\mu^{<\xi}$ for μ_0^ξ and $\mu_0^{<\xi}$. We write μ_0^s and $\mu_0^{<s}$ for $\mu_0^{|s|}$ and $\mu_0^{<|s|}$; similarly for μ^s and $\mu^{<s}$.

Coding structures We let $\mathcal{A}_0^s = \mathcal{A}(A)_0^s$ denote $L_{\mu_0^s}[A \cap \alpha, s]$. If $\alpha \in \mathbf{Reg}$, we write $\mathcal{A}^s$ for $\mathcal{A}_0^s$ and $\mathcal{A}^\xi$ for $\mathcal{A}_0^\xi$.

Coding apparatus For cardinals $\alpha < \kappa$, $s \in S_\alpha$ such that α is regular in $\mathcal{A}_0^s$, let
$$H_i^s = H(A)_i^s = h_{\Sigma_1}^{\mathcal{A}_0^s}(i \cup \{A \cap \alpha, D_\alpha, E_\alpha, s\})$$
and let $f^s(i)$ be the order type of $H_i^s \cap \mathrm{On}$. If $\alpha = \beta^+$ for $\beta \in \mathbf{Card}$, let B^s be $\{i \in [\beta, \alpha) \mid H_i^s \cap \alpha = i\}$ and let $b^s = \{\langle i, f^s(i)\rangle \mid i \in B^s\}$ (where $\langle .\,, .\rangle$ denotes the Gödel pairing function). If α is inaccessible in $\mathcal{A}_0^s$, let $B^s = \{\beta^+ \mid H_\beta^s \cap \alpha = \beta, \beta \in \mathbf{Card} \cap \alpha\}$ and let $b^s = \mathrm{ran}(f^s \restriction B^s)$.[7]

The intuition is that we want to code s via almost-disjoint coding, using the almost disjoint family $\{b^{s\restriction\nu} \mid \nu \in [\alpha, |s|)\}$ (but in fact, s will grow at the same time). The particular form of the b^s is very convenient when $\alpha \in$ **Sing** (see the proof of extendibility 4.14). For successor α, using the pair $\langle i, \mathrm{otp}(H_i \cap \mathrm{On})\rangle$ is vital to ensure that the b^s are almost disjoint. Observe that by definition, for $s \in S_\alpha$ when $\alpha < \kappa$ is locally inaccessible (that is, inaccessible in $\mathcal{A}_0^s$) we have $\|\xi\|^- \in E_\alpha$ for any all $\xi \in b^s$. For $s \in S_\alpha$ when α is inaccessible and $\diamondsuit_\alpha$ is club in α, also $\|\xi\|^- \in \diamondsuit_\alpha$ for any all $\xi \in b^s$.[8]

As we will see later, fake inaccessible coding makes it necessary that we be able to put all of b^s into the support of a condition in P; otherwise the coding structures will change as conditions are extended (see below). Thus we want b^s to be Easton. This, together with coherency issues arising in the proof of *extendibility* at singular α (see 4.14 below), is why we use E_α. In said proof, we shall have no control over what happens at limits of E_α, which is why we use successors for B^s.

[7]For $\alpha < \kappa$ inaccessible in $\mathcal{A}_0^s$, we could also just let $B^s = \{\beta^+ \mid \beta \in \diamondsuit_\alpha \cap E_\alpha\}$ if this is unbounded in α and $B^s = \{\beta^+ \mid \beta \in E_\alpha\}$ otherwise. We could let $B^s = \mathbf{Succ} \cap \kappa$ for $s \in S_\kappa$.

[8]As mentioned before, this is needed in the proof of 7.6.

Now we define coding structures for singular α. This is complicated by the fact that a discontinuity arises when α is not seen to be singular by a structure arising in the proof of distributivity. In this proof, we will build conditions p which inevitably use inaccessible coding at α which are really singular. This is the phenomenon we call "fake inaccessible coding." The idea behind my solution is that we make singular coding structures (say, at α) large enough to "see" the fake inaccessible coding. This requires that we can distinguish wether or not fake inaccessible coding has occurred at all. Also the coding structures now depend on the condition *below* α and we have to be careful they be stable with respect to extending p.

Definition 4.5 (The singular case)**.** Let α be a limit cardinal now. Except for the first definition, we will always assume α is singular. The following definitions will be relative to some (partial) function $p\colon \alpha \to 2$. We extend the definition of support to such p by $\mathrm{supp}(p) = \mathbf{Card} \cap \mathrm{dom}(p)$.

Given p as above and a function f with $\mathrm{dom}(f) \subseteq \mathbf{Card}$ such that $f \in \prod_{\beta \in \mathrm{dom}(f)} [\beta, \beta^+)$, we let f_p be the partial function where $f_p(\beta)$ is the least $\eta \in [\mathrm{On}]_3$ such that $f(\beta) < \eta$ and $p(\eta) = 1$.

Regular decoding First, define $s(A,p) = s(p) \in S_\alpha$. To this end, define a sequence $(s_\gamma)_\gamma < \gamma(p)$ of basic strings in S_α. Let $k = 1$ if $\alpha \in \mathbf{Succ} \cap \kappa^+ + 1$ and $k = 2$ if $\alpha \in \mathbf{Inacc} \cap \kappa + 1$. Let $s_0 = \emptyset$. Given s_γ, first assume $b^{s_\gamma} \subseteq \mathrm{dom}(p)$ and α is regular in $\mathcal{A}_0^{s_\gamma}$. In this case let $s_{\gamma+1}$ be $s_\gamma \frown j$, where j is such that $p((i)_k) = j$ for cofinally many $i \in b^{s_\gamma}$ (and $s \frown t$ denotes concatenation of strings of 0's and 1's). This is well-defined as long as α is regular in $\mathcal{A}_0^{s_\gamma}$ (b^s is not defined otherwise).

If $b^{s_\gamma} \not\subseteq \mathrm{dom}(p)$, or if $\mathcal{A}_0^{s_\gamma} \models \alpha$ is singular, let $s(p) = s_\gamma$ and $\gamma(p) = \gamma$. Write $\mu(p)$ for $\mu^{s_{\gamma(p)}}$.

Steering Ordinals at singulars Inductively define $\mu(A,p)^\xi$, $\tilde{\mu}(A,p)^\xi$ and $\mu(A,p)^{<\xi}$ or more simply, $\mu(p)^\xi$, $\tilde{\mu}(p)^\xi$ and $\mu(p)^{<\xi}$ by induction on $\xi \in [\alpha, \alpha^+)$:

If $p((\beta^+)_2) = 1$ for a cofinal set of $\beta \in E_\alpha$, we let $\mu(p)^{<\alpha} = \mu(p)$ and we say p *uses fake inaccessible coding*[9].

Otherwise, let $\mu(p)^{<\alpha}$ be the least μ such that $E_\alpha, \diamondsuit_\alpha \in L_\mu[A \cap \alpha]$, i.e. $\mu(p)^{<\alpha} = \mu^{<\alpha}$. We say in this case that p *immediately uses singular coding.*[10]

[9] Conditions with this property have to be dealt with when we prove quasi-closure.

[10] such conditions are built in a straightforward manner in the proof of extendibility 4.14.

For $\xi > \alpha$, let

$$\mu(p)^{<\xi} = \sup_{\nu<\xi} \mu^{\nu}.$$

and for $\xi \geq \alpha$ let, first let $\sigma(p)^{\xi} > \mu^{<\xi}$ be least such that $L_{\sigma(\xi)} \models$"α is the greatest cardinal and $\alpha \in$ **Sing**". Now let $\tilde{\mu}(p)^{\xi} = \sigma(p)^{\xi} + \alpha$ and $\mu(p)^{\xi} = \sigma(p)^{s} + \omega \cdot \alpha$. Notice that by induction $\xi \leq \mu(p)^{<\xi}$ (equality may hold if ξ is a limit) and $\mu(p)^{\xi} \geq \mu_0^{\xi}$.

Again, write $\mu(p)^{s}, \tilde{\mu}(p)^{s}, \mu(p)^{<s}$ for $\mu(p)^{|s|}, \tilde{\mu}(p)^{|s|}, \mu(p)^{<|s|}$.

We say p *recognizes the singularity of* α if and only if α is singular in $\mathcal{A}_0^{s(p)}$. Observe this is never relevant when p immediately uses singular coding.

Coding structures at singulars We let

$$\mathcal{B}(A,p)^{s} = \mathcal{B}(p)^{s} = L_{\mu(p)^{s}}[A \cap \alpha, s(p), s]$$
$$\tilde{\mathcal{B}}(p)^{s} = L_{\tilde{\mu}(p)^{s}}[A \cap \alpha, s(p), s],$$

where we set $s(p) = \emptyset$ if p immediately uses singular coding. Note that $\mathcal{B}(p)^{s} \models \alpha \in$ **Sing** whenever $|s| > 0$ or p immediately uses singular coding. Again note that $E_{\alpha} \in \mathcal{B}(p)^{\emptyset_{\alpha}}$ and $\mathcal{A}_0^{s} \subseteq \mathcal{B}(p)^{s}$.

Coding apparatus at singulars For $s \in S_{\alpha}$, let

$$H(A,p)_i = H(p)_i = h_{\Sigma_1}^{B(p)^{s}}(i \cup \{A \cap \alpha, s(p), s\})$$

and let $f(p)^{s}(i)$ be the order type of $H(p)_i \cap \text{On}$. Note again that $f(p)^{\emptyset_{\alpha}} = f^{\emptyset_{\alpha}}$.

Singular coding with delays We define $t(A,p) = t(p) \in S_{\alpha}$. To this end, we define a sequence $(t(p)_{\xi})_{\xi<\delta(p)}$. Let $t(p)_0 = \emptyset$. For limit δ, let $t(p)_{\delta} = \bigcup_{\delta'<\delta} t(p)_{\delta'}$. Now suppose $t = t(p)_{\delta}$ is defined. For $\gamma \in$ **Card** let $f_p^{t}(\gamma)$ be the least η such that $f(p)^{t}(\gamma) \leq \eta < \gamma^{+}$ and $p((\eta)_3) = 1$. If $f_p^{t}(\beta^{+})$ is undefined for cofinally many $\beta \in E_{\alpha}$, let $\delta(p) = \delta$ and $t(p) = t(p)_{\delta}$. Otherwise, if possible define $X = X(p,\delta) \subseteq \alpha$ by $\eta \in X$ if and only if $p((f_p^{s_{\gamma}}(\beta) + 1 + \eta)_3) = j$ for a tail of successor cardinals $\beta < \alpha$. If $X(p,\delta)$ is undefined, again stop the construction at γ and let $\delta(p) = \delta$ and $t(p) = t_{\delta}$[11]

[11]This will never occur. Note that it follows from later definitions that the construction of $t(p)$ finishes only for one reason, namely that $f_p^{t}(\beta^{+})$ is undefined for cofinally many $\beta \in E_{\alpha}$. For since we require $p \restriction \alpha$ exactly codes p_{α}, the construction cannot stop before we have $t(p) = p_{\alpha}$, where we halt for the given reason, by the requirement that $p \restriction \alpha \in \mathcal{B}(p_{<\alpha})^{p_{\alpha}}$.

If $[X(\xi)]_0^{-1}$ precodes a string $t \in S_\alpha$ which end-extends $t(p)_\delta$ and such that $f_p^{s_\gamma} \in \mathcal{B}(p)^t$, let $t(p)_{\delta+1} = t$.

Otherwise let $t = t(p)_\delta{}^\frown[X]_3$. If $f_p^{s_\gamma} \in \mathcal{B}(p)^t$, let $t(p)_{\delta+1} = t$. If not, again stop the construction at γ and let $\delta(p) = \delta$ and $t(p) = t_\delta$ (we will later see this case never occurs). We say *p exactly codes t* if and only if $t = t(p)$.

Definition 4.6 (Decoding)**.** We now describe the process of *decoding*. This will be run in the generic extension $L[A_0][G]$, the P-generic extension and there, $s_0 = \bigcup_{p\in G} p_{\omega_1}$ will yield A_0 via this process. We shall also run this process "locally" in the transitive collapse $\bar{M}$ of certain (small) elementary submodels M. In this case, the reader should anticipate that a condition of the forcing may be sufficiently generic over M to ensure correct coding e.g. in the sense that if $\kappa^{++} \in M$, the decoding process relative to M yields $A_0 \cap M$ — whereas of course conditions will never code anything non-trivial over $L[A_0]$ as they are bounded. If the following is run over $\bar{M}$, all definitions should be relativized to $\bar{M}$, so that κ is replaced by the least Mahlo in $\bar{M}$, coding structures are as defined in $\bar{M}$ etc.

The following definition is by induction on cardinals $\beta \in [\beta_0, \kappa^+)$. Assume $\beta_0 \in \mathbf{Card}$, $A' \subseteq \beta_0$ and $p_{\beta_0} \colon [\beta_0, {\beta_0}^+) \to 2$, or $p_{\beta_0} \colon \beta_0 \to 2$ and $\beta_0 = \kappa$ (or, respectively, the least Mahlo in $\bar{M}$). We inductively construct $A^* \subseteq \kappa$, setting $A^* \cap \beta = A'$.

If $\beta \in [\beta_0, \kappa)$ and we have constructec $A^* \cap \beta$ and $p_\beta \colon [\beta, \beta^+) \to 2$, define $p_{\beta^+} \colon [\beta^+, \beta^{++}) \to 2$ via the decoding process at regulars described in definition 4.5 from p_β, i.e. let $p_{\beta^+} = s(A^* \cap \beta, p_\beta)$. Also, let $\xi \in A^* \cap [\beta, \beta^{+3}) \iff p_\beta((\langle \xi, \nu \rangle)_0) = 1$ for β^+-many $\nu \in [\beta, \beta^+)$.

At limit $\beta < \kappa$, let $p_{<\beta} = \bigcup_{\beta' < \beta} s_{\beta'}$ and let $s_\beta = s(A^* \cap \beta, p_{<\beta})$ when β is inaccessible and $s_\beta = t(A^* \cap \beta, p_{<\beta})$ when β is singular.

When β is the least Mahlo κ, first extract $A_0^* \cap \kappa$ and the generic for the Mahlo-coding from A^*: let $A_0^* \cap \kappa = [A^* \cap \kappa]_0^{-1}$, let $p_{<\kappa} = [A^* \cap \kappa]_1^{-1}$, and let $p_\kappa = s(A_0^* \cap \kappa, p_{<\kappa})$ (when $\beta = \beta_0 = \kappa$ and we are at the beginning of the induction, we just set $A_0^* \cap \kappa = A'$). Let $\xi \in A_0^* \cap [\kappa, \kappa^+) \iff p_\kappa((\langle \xi, \nu \rangle)_0) = 1$ for κ^+-many $\nu \in [\kappa, \kappa^+)$.

For $\beta = \kappa^+$, we continue exactly as we did below κ, but with A_0^* instead of A^*: let $p_{\beta^+} = p_{\kappa^{++}} = s(A_0^* \cap \kappa^+, p_{\kappa^+})$ and let $A_0^* \cap [\kappa^+, \kappa^{++})$ be the set whose characteristic function is p_{κ^+}. The set $A_0^* \subseteq \kappa^{++}$ is the outcome of the decoding procedure run up to κ^{++} and we write $A^*(p_{\beta_0}) = A^*(A', p_{\beta_0})$ for this set.

Definition 4.7 (Strings)**.** We now define $S_\alpha^* = S(A)_\alpha^*$. Let $\Phi(r)$ be the statement "$r(n) = i \Rightarrow$ for cofinally many $\sigma < \bar{\kappa}$, $\bar{T}(\sigma, n, i)$ has a branch, where $\bar{T}$ is the canonical $\bar{\kappa}$-sequence of $\bar{\kappa}^{++L}$-Suslin trees and $\bar{\kappa}$ is the least Mahlo in L." We also say r *is coded by branches* for $\Phi(r)$.

We say $s \in S^*_\alpha$ if and only if $s \in S_\alpha$ and for all $\zeta \leq |s|$ and all $\eta, \bar{\kappa}$ such that $N = L_\eta[A \cap \alpha, s \restriction \zeta] \vDash$"$\zeta = \alpha^+$, $\bar{\kappa}$ is the least Mahlo in L, $\bar{\kappa}^{++}$ exists, $\mathbf{Card} = \mathbf{Card}^{L[A \cap \omega]}$ and ZF$^-$ holds", we have that $L[A^*] \vDash \Phi(r)$, where $A^* = A^*(A \cap \alpha, s \restriction \zeta)$ is the predicate Jensen-coded by $s \restriction \zeta$ over N. When $s \in S_\alpha$, we call N as in the hypothesis a *test model for* s. Thus, $s \in S^*_\alpha$ if and only if for every test model N for s, $L^N[A^*(s \restriction \alpha^{+N})] \models r$ is coded by branches. [12]

Let $S_{<\kappa}$ denote the set of $s\colon [0, |s|) \to 2$, where $|s| < \kappa$. We say $s \in S^*_{<\kappa}$ if and only if $s \in S_{<\kappa}$ and for all $\bar{\kappa} \in \mathbf{Card} \cap |s| + 1$ and all η such that $N = L_\eta[A_0 \cap \bar{\kappa}, s \restriction \bar{\kappa}] \vDash$ "$\bar{\kappa}$ is the least Mahlo in L, $\bar{\kappa}^{++}$ exists, $\mathbf{Card} = \mathbf{Card}^{L[A_0 \cap \omega]}$ and ZF$^-$ holds", we have that $N \vDash \Phi(r)$. Similarly to the above, when $s \in S_{<\kappa}$ we call N as in the hypothesis a $<\kappa$*-test model for* s.

Definition 4.8 (Building blocks)**.** We now define three partial orders, each of which codes a set via almost disjoint coding using the almost disjoint family $(b^t)_t$ and simultaneously localizes it. These partial orders serve as building blocks for the final forcing.

Successor coding Let either $\alpha = \kappa^+$, $\beta = \kappa$ and $s = s_{\kappa^+}$ or let $\beta \in \mathbf{Card} \cap \kappa$, $\alpha = \beta^+$ and $s \in S^*_\alpha$ and let $A \subset \alpha$. We define the partial order $R^s = R(A)^s$ to consist of conditions $p = (p_\beta, p^*)$ such that $p \in S^*_\beta$ and $p^* \subseteq \{b^{s \restriction \xi} \mid \xi \in [\alpha, |s|)\} \cup |p|$ of size at most β. It is ordered by: $(q, q^*) \leq (p, p^*)$ if and only if q end-extends p, $p^* \subseteq q^*$ and

1. If $b^{s \restriction \xi} \in p^*$ and $s(\xi) = 0$ then for any $\gamma \in (|p|, |q|) \cap b^{s \restriction \xi}$ we have $q((\gamma)_1) = 0$.
2. If $\xi \in p^* \cap |s|$ and $\xi \in A$ then if γ is such that $\langle \xi, \gamma \rangle \in (|p|, |q|)$, we have $q((\langle \xi, \gamma \rangle)_0) = 0$.

Inaccessible coding Let $\alpha \leq \kappa$ be inaccessible, $s \in S^*_\alpha$. We define the partial order $R^s = R(A)^s$ to consist of conditions (p, p^*) such that p is a partial function $p\colon \alpha \to 2$ and $p^* \subseteq \{b^{s \restriction \xi} \setminus \eta \mid \xi \in [\alpha, |s|), \eta < \alpha\} \cup \alpha$ of size less than α and $\alpha \cap p^*$ is an ordinal. We write $\rho(p^*)$ for this ordinal. For $\alpha = \kappa$, we additionally demand that $p_{<\kappa} \in S^*_{<\kappa}$. R^s is ordered by: $(q, q^*) \leq (p, p^*)$ if and only if q end extends p, $p^* \subseteq q^*$ and

[12] Observe that requiring $N \models \text{ZF}^- \wedge \exists \bar{\kappa}^{++}$ ensures that L^N thinks that the canonical $\bar{\kappa}$-sequence of $\bar{\kappa}^{++}$-Suslin trees exists, but this is not the only reason to make this requirement. Rather, we will this requirement and $s \restriction \zeta$ to ensure ζ collapses to α "quickly". In effect, this allows us to use s to thin out the set of test models. We could do the same without ZF$^-$ but that would take an argument.

1. If $b = b^{s\restriction\xi} \setminus \eta$, $b \in q^* \setminus p^*$ then

$$b \cap (\rho(p^*) \cup \sup \mathrm{dom}(p)) \subseteq \bigcup\{b' \mid b' \in p^*\}.$$

2. If $b^{s\restriction\xi} \setminus \eta \in p^*$ and $s(\xi) = 0$ then for any $\gamma \in (|p|, |q|) \cap (b^{s\restriction\xi} \setminus \eta)$ we have $q((\gamma)_2) = 0$.

The additional construct $\rho(p^*)$ and its use in the definition of $\leq$ for inaccessible coding is necessary to preserve requirement (**??**) of the definition of $P(A_0)$ at the limit (see also definition 4.21, item (D 1)), Also, together with the similar use of $\sup(\mathrm{dom}(p_{<\alpha}))$ this becomes important in the proof that κ is not collapsed in the limit of our iteration (see 7.6). With the intuition that our iteration can *almost* be decomposed into an upper and a lower part, the idea is that this device ensures some independence of upper and lower parts in that argument, since restraints are the only possible interaction. This also means that the part of the restraint below $\rho(p^*) \cup \sup \mathrm{dom}(p_{<\alpha})$ as well as $\rho(p^*)$ itself must part of the value of the "centering function" $\mathbf{C}^\lambda(p)$ (see 4.9). The use of $\sup(\mathrm{dom}(p_{<\alpha}))$ can be eliminated (but renders the argument slightly more elegant). The use of $\sup(\mathrm{dom}(p_{<\alpha}))$ cannot obviate that of $\rho(p^*)$ because of the last clause in (C I).

4.4 Definition of the forcing $P(A_0)$

Definition 4.9 (The conditions)**.** Let $\mathbf{Card}'$ for now denote $(\mathbf{Card} \setminus \omega) \cup \emptyset\emptyset$. A condition in $P = P(A_0)$ is a sequence

$$p = (p_{<\alpha}, p_\alpha, p^*_\alpha)_{\alpha \in \mathbf{Card}' \cap \kappa^+ + 1}.$$

For limit cardinals $\alpha < \kappa$, we demand $p_{<\alpha} = \bigcup_{\delta<\alpha} p_\delta$ (so $p_{<\alpha}$ is redundant unless $\alpha = \kappa$).

For p as above to be a condition, we demand:

1. $p_{\kappa^+} = s_{\kappa^+}$ (where s_{κ^+} is the characteristic function of $A \cap [\kappa^+, \kappa^{++}]$) and $p_\alpha \in S^*_\alpha$ for all $\alpha \leq \kappa$, while $p_{<\kappa} \in S^*_{<\kappa}$ (this is redundant by the definition of the Mahlo coding). We write $\mathrm{supp}(p) = \{\alpha \mid p_\alpha \neq \emptyset_\alpha\}$ and $\alpha(p) = \sup(\mathrm{supp}(p) \cap \kappa)$.

2. $\mathrm{supp}(p) \cap \kappa$ is an Easton subset of $|p_{<\kappa}|$. For $\alpha \in \{\kappa, \kappa^+\}$ we let $\mathcal{A}^{p_\alpha}$ be defined relative to A_0. For $0 < \alpha \leq |p_{<\kappa}|$, we let $\mathcal{A}^{p_\alpha}$ and $\mathcal{B}(p_{<\alpha})^{p_\alpha}$ be defined relative to

$$A_p = A_0 \oplus \{\xi < \alpha \mid p_{<\kappa}(\xi) = 1\}.$$

Observe that we will order P in such a way that coding structures never change when p is extended, that is, the dependence on p is "static". In the following, R^s is defined relative to these coding structures.

3. For all $\alpha \in \operatorname{supp}(p)$, $(p_\alpha, p^*_{\alpha^+}) \in R^{p_{\alpha^+}}$, where we let R^{p_ω} denote the standard almost disjoint coding of p_ω by a real relative to some convenient almost disjoint family in $L[A_0 \cap \omega]$.

4. For all inaccessible $\alpha \leq \kappa$, $(p_{<\alpha}, p^*_\alpha) \in R^{p_\alpha}$, remembering $p_{<\alpha} = \bigcup_{\alpha'<\alpha} p_\alpha$ for $\alpha < \kappa$.

5. For all singular $\alpha < \kappa$, $p \restriction \alpha \in \mathcal{B}(p_{<\alpha})^{p_\alpha}$, $t(p_{<\alpha}) = p_\alpha$ and if $p_{<\alpha}$ is unbounded below α and uses fake inaccessible coding, then $p_{<\alpha}$ recognizes singularity of α.[13]

6. For $\alpha \in \mathbf{Reg}$, there is a $\gamma < \alpha$ such that for all $\delta \in \mathbf{Inacc} \setminus \alpha + 1$ and all $(\xi, \eta) \in p^*_\delta$ we have $b^{p_\delta \restriction \xi} \setminus \eta \cap \alpha \subseteq \gamma$.

Note again we have $p_{\kappa^+} = s_\kappa$ for every condition $p \in P$. We say $q \leq p$ if and only if for all $\alpha \in \operatorname{supp}(p)$, $(q_\alpha, q_{\alpha^+}) \leq (p_\alpha, p_{\alpha^+})$ in $R^{p_{\alpha^+}}$ and for all inaccessible $\alpha \in \operatorname{supp}(p)$, $(q_{<\alpha}, q_\alpha) \leq (p_{<\alpha}, p_\alpha)$ in R^{p_α}.

Observe that the limit coding from κ^+ into κ takes a slight detour, via $p_{<\kappa}$; The reason for this is that the coding into κ is easier if we use strings (mainly because of the proof that κ isn't collapsed, 7.6). In contrast, $p_{<\alpha}$ can be seen as a shorthand for $\bigcup_{\delta<\alpha} p_\delta$; we use these "broken strings" to code into inaccessible $\alpha < \kappa$, which is the natural choice.

We will need that being a condition, and in fact all of the definitions in 4.4 and 4.5 are absolute for Σ^A_1-correct models. In fact, all these notions are Boolean combinations of Σ^A_1 statements.[14]

For $p \in P$ and $\alpha \in \mathbf{Card}$, let $B^\alpha_p \subseteq \alpha$ be defined by

$$B^\alpha_p = \bigcup \{ b \cap \alpha \mid b = b^s \setminus \eta \in p^*_\gamma, \gamma \in \mathbf{Inacc} \setminus \alpha + 1 \},$$

and let $b_p \in \prod_{\alpha \in \mathbf{Card} \cap \kappa^{++}} \alpha$ be defined by

$$b_p(\alpha) = \sup B^\alpha_p$$

[13] We let the fake inaccessible coding stop when the singularity of α is realized; without this natural stopping point, the coding structure for singulars will change when a condition is extended below α, i.e. $\mathcal{B}(q_{<\alpha})^{\emptyset_\alpha} \neq \mathcal{B}(p_{<\alpha})^{\emptyset_\alpha}$ for $q \leq p$, because q might carry new information in $t(q_{<\alpha})$. Intuitively, p hasn't exhausted the room for fake inaccessible coding.

[14] more generally, it would suffice that they be provably Δ^A_2 in a weak enough theory.

That is, B_p^α is the set of coding ordinals used by inaccessibles above α, and b_p locally bounds the height of this set.

Remark 4.10. Note that 6 of 4.9 (the definition of P), is equivalent to asking $b_p(\alpha) < \alpha$ for every $\alpha \in \mathbf{Reg}$.

4.5 Doing the same in three steps.

To aid the readers intuition, we describe the same forcing as a three-step iteration. This illustrates further the special role of $p_{<\kappa}$ and the detour in the Mahlo coding, but will not be needed for the rest of the proof.

1. The first step is to force with the successor coding R^{A_0}. We will later see this forcing is fully stratified. Let G_1 be the generic and let

$$A_1 = \{\xi \in [\kappa, \kappa^+) \mid \exists (p, p^*) \in P \; p(\xi) = 1\} \cup A_0 \cap \kappa.$$

   Observe that A_1 has the following property: For any $N = L_\eta[A_1 \restriction \zeta]$ such that $\zeta \geq \kappa$ and

2. The second step is to force with the Mahlo coding R^{A_1} in $L[A_1] = L[A_0][G_1]$. We will later see this too, preserves all cardinals and κ is still Mahlo. Let G_2 be the generic and let

$$A_2 = \{\xi \in \kappa \mid \exists (p, p^*) \in P \; p(\xi) = 1\}.$$

Observe that A_2 satisfies the following:

$$\text{for all } \bar\kappa \in \mathbf{Card},\ \eta \in \mathrm{On} \text{ such that } N = L_\eta[A_2 \restriction \bar\kappa] \models \text{“}\bar\kappa \text{ is the least Mahlo, } \bar\kappa^{++} \text{ exists and } \mathrm{ZF}^- \text{ holds”, we have that } N \models \Phi(r). \tag{4.1}$$

Lastly, we define $P(A_2)$ which codes A_2 by a subset of ω_1. For this sake, let A denote A_2, and let all coding structures be defined relative to this A.

Definition 4.11 (The conditions for coding below κ)**.** A condition in $P(A)$ is a function $p\colon \mathbf{Card} \cap \kappa \to V$, $p(\alpha) = (p_\alpha, p^*_\alpha)$ such that

1. $(p_\kappa, p^*_{\kappa^+}) \in R^{s_{\kappa^+}}$.

2. $\mathrm{supp}(p) = \{\alpha < \kappa \mid p_\alpha \neq \emptyset\}$ is an Easton set.

3. For all $\alpha \in \mathrm{supp}(p)$, $(p_\alpha, p^*_{\alpha^+}) \in R^{p_{\alpha^+}}$.

4. For all inaccessible $\alpha \in \operatorname{supp}(p)$, $(p_{<\alpha}, p_\alpha) \in R^{p_\alpha}$, where we define $p_{<\alpha} = \bigcup_{\alpha' < \alpha} p_\alpha$ for any α.

5. For all singular $\alpha \in \operatorname{supp}(p)$, $p \restriction \alpha \in \mathcal{B}(p_{<\alpha})^{p_\alpha}$, $t(p_{<\alpha}) = p_\alpha$ and if $p_{<\alpha}$ is unbounded below α and uses fake inaccessible coding, then $p_{<\alpha}$ recognizes singularity of α.[15]

6. For $\alpha \in \mathbf{Reg}$, there is a $\gamma < \alpha$ such that for all $\delta \in \mathbf{Inacc} \setminus \alpha + 1$ and all $(\xi, \eta) \in p^*_\delta$ we have $b^{p_\delta \restriction \xi} \setminus \eta \cap \alpha \subseteq \gamma$.

We say $q \leq p$ if and only if for all $\alpha \in \operatorname{supp}(p)$, $(q_\alpha, q_{\alpha^+}) \leq (p_\alpha, p_{\alpha^+})$ in $R^{p_{\alpha^+}}$ and for all inaccessible $\alpha \in \operatorname{supp}(p)$, $(q_{<\alpha}, q_\alpha) \leq (p_{<\alpha}, p_\alpha)$ in R^{p_α}.

This ends our description of P as a three-step iteration. We find it more convenient to talk about P instead of this iteration, and this is the forcing we work with in all of the following.

4.6 Extendibility

Lemma 4.12 (Extendibility for the Mahlo coding). *Let $t \in S_\kappa$ or*

$$t\colon [\kappa, \kappa^+) \to 2.$$

For any $\alpha < \kappa$ and any $p \in R^t$ there is $q \in R^t$ such that $q \leq p$ and $|q_{<\kappa}| \geq \alpha$. In addition, we can demand that $q_{<\kappa}(\nu) = 0$ for $\nu \in [|p_{<\kappa}|, |q_{<\kappa}|) \cap \mathbf{Card}$.

Proof. Let α be the least counterexample. If $\alpha \notin \mathbf{Card}$ it suffices to extend $p_{<\kappa}$ to $p^1_{<\kappa}$ with $|p^1_{<\kappa}| \geq \|\alpha\|$. We can then further extend $p^1_{<\kappa}$ by appending 0s to obtain q. If $\alpha \in \mathbf{Succ}$, the proof is similar (as no test model will think that α is Mahlo). So assume α is a limit cardinal.

Let $C \subset \alpha$ be club in α, $\operatorname{otp} C = \operatorname{cf} \alpha$, $C \subseteq \mathbf{Sing}$ and let $(\beta_\xi)_{\xi \leq \rho}$ be the increasing enumeration of $C \cup \{\alpha\}$. Moreover we demand that $\beta_0 > \rho$ if α is singular. Let $p^0 \leq p$ such that $|p^0_{<\kappa}| = \beta_0$ and build a descending chain of conditions. Assume you have p^ξ such that $|p^\xi_{<\kappa}| = \beta_\xi$. Extend to get p' with $|p'| = \beta_{\xi+1}$. Let $p^{\xi+1}$ be obtained from p' by shifting values of $p'_{<\kappa}$ above β_ξ away from the cardinals in a gentle manner, putting a 1 on β_ξ and padding with 0s, as follows: let $p^{\xi+1}_{<\kappa} \restriction \beta_\xi = p'_{<\kappa} \restriction \beta_\xi = p^\xi_{<\kappa} \restriction \beta_\xi$ and for $\nu \in [\beta_\xi, \beta_{\xi+1})$, let

$$p^{\xi+1}_{<\kappa}(\nu) = \begin{cases} p'_{<\kappa}(\delta + k) & \text{if } \nu = \delta + k + 1 \text{ for some } \delta \in \mathbf{Card},\ k \in \omega, \\ 1 & \text{if } \nu = \beta_\xi, \\ 0 & \text{if } \nu \in \mathbf{Card} \cap (\beta_\xi, \beta_{\xi+1}). \\ p'_{<\kappa}(\nu) & \text{otherwise.} \end{cases}$$

[15] See footnote 13

Observe that $p^{\xi+1} \leq p^\xi$ since no restraints $b \in (p^\xi)^*$ are violated, as we have $\delta + k \notin [b]_3$ for $\delta \in \mathbf{Card}$, $k \in \omega$. We still have $p^{\xi+1}_{<\kappa} \in S^*_{<\kappa}$, as $L_\eta[A \cap \bar\kappa, p^{\xi+1}_{<\kappa} \restriction \bar\kappa] = L_\eta[A \cap \bar\kappa, p'_{<\kappa} \restriction \bar\kappa]$ for all relevant $\eta, \bar\kappa$. At limit $\xi \leq \rho$, if $N = L_\eta[p^\xi_{<\kappa} \restriction \bar\kappa]$, note that $C \cap \beta_\xi \in N$ and so $N \models \bar\kappa$ is not Mahlo. Thus $q = p^\rho$ is as desired. □

Lemma 4.13 (Extendibility for the successor coding.)**.** *For each $p \in P$, $\xi \in \kappa^+$ there is $q \leq p$ such that $\xi \in \mathrm{dom}(q_{<\kappa^+})$. Equivalently, let $\alpha \in$ $\mathbf{Card} \cap |p_{<\kappa}| \cup \{\kappa\}$, $A \subseteq \mathrm{On}$ when $\alpha = \kappa$ and $A = A_q$ otherwise and let $s \in S(A)^*_{\alpha^+}$ or $s = s_{\kappa^+}$. Any $p = (p_\alpha, p^*) \in R(A)^s$ can be extended to $q \leq p$ such that $|q| \geq \xi$, for any $\xi \in [\alpha, \alpha^+) \cap \mathbf{Card}$.*

Proof. The two statements are equivalent by lemma 4.12. To avoid repetition, we leave the proof of this lemma as a by-product of lemma 4.21: any $q \in \mathbf{D}^{L[A]}(0, p, \{\xi\})$ does the trick. □

The next lemma allows us to extend conditions at a singular cardinal α without violating that the extension be coded exactly below α. By the last item below, we may at the same time capture a set X locally (in a sense); this included for completeness and is not needed in the rest of the proof.

Lemma 4.14 (Extendibility at singulars)**.** *Let $p \in P$, $\alpha \in \mathbf{Sing} \cap |p_{<\kappa}|$ and $s \in S(A_p)_\alpha$ such that $p_\alpha \lhd s$, further say $X \subseteq \mathrm{On}$ and $X \in \mathcal{B}(p_{<\alpha})^s$. We can find $q \in P$, such that*

- *$q \restriction (\alpha, \infty] = p \restriction (\alpha, \infty]$,*
- *$q_\alpha = s$,*
- *for each each β which is a limit point of E_α (in fact, for all $\beta = \delta^+$ for some $\delta \in E_\alpha$), $X \cap \beta \in \mathcal{B}(q_{<\alpha})^{q_\beta}$.*

We write $A = A_p$ for the discussion of this lemma. Before we proof this lemma, we make two technical observations:

Lemma 4.15. *Let $s, t \in S_\alpha$, p be a condition such that $p \restriction \alpha \in \mathcal{B}(p_{<\alpha})^s$ and s a proper initial segment of t. Then $f(p_{<\alpha})^s \in \mathcal{B}(p_{<\alpha})^t$. In fact, $f(p_{<\alpha})^s \in$ $L_{\mu(p_{<\alpha})^s+\alpha+1}[A \cap \alpha, s(p), s]$.*

Proof of lemma 4.15. This is because the Skolem hulls are definable over $\mathcal{B}(p_{<\alpha})^s$, and the transitive collapse of the Skolem hull of $\gamma < \alpha$ is constructible at most $f(p_{<\alpha})^s(\gamma) + 1$ steps above $\mathcal{B}(p_{<\alpha})^s$, by the recursive definition of the transitive collapse. Thus, $f(p_{<\alpha})^s$ is a definable subset of $L_{\nu(p_{<\alpha})^s+\alpha}[A \cap \alpha, s]$, and $f(p_{<\alpha})^s \in L_{\nu(p_{<\alpha})^s+\alpha+1}[A \cap \alpha, s]$. Since possibly, $\nu(p_{<\alpha})^s + \alpha = \tilde\nu(p_{<\alpha})^t$, it needn't be the case that $f(p_{<\alpha})^s \in \tilde{\mathcal{B}}(p_{<\alpha})^t$ but clearly, $f(p_{<\alpha})^s \in \mathcal{B}(p_{<\alpha})^t$. □

Note that if the length of $\mathrm{lh}(t) > \mathrm{lh}(s) + 1$, then we even have $f(p_{<\alpha})^s \in \mathcal{B}(p_{<\alpha})^{<t}$.

Lemma 4.16. *If $\beta \in \mathbf{Card} \cap \alpha+1$, $t \in S_\beta$, $\mathcal{B}(p_{<\beta})^t = \mathcal{B}(q_{<\beta})^t$ and $f(q \restriction \beta)^t = f(p \restriction \beta)^t$.*

Proof. Fix t and β as in the first statement. First, assume $p \restriction \beta$ does not use fake inaccessible coding. Then neither does q. In this case, $\mu(q_{<\beta})^{\emptyset_\beta} = \mu^{\emptyset_\beta}$ and $p \restriction \beta$ and $q \restriction \beta$ play no role at all in the definition of $f(q_{<\beta})^t$ and $f(p_{<\beta})^t$.

In the other case, when $p \restriction \beta$ uses inaccessible coding, by definition we have $b^{\bar{s}} \subseteq \mathrm{dom}(p)$ for any $\bar{s} \lhd s(p \restriction \beta)$. But then as $q \leq p$, $s(p \restriction \beta) = s(q \restriction \beta)$ and again $\mu(q_{<\beta})^{\emptyset_\beta} = \mu(p_{<\beta})^{\emptyset_\beta}$ by definition. Thus in this case as well, we have $f(q_{<\beta})^t = f(p_{<\beta})^t$. □

Observe that for limit cardinals $\beta < \alpha$, $[Y]_0^{-1} \cap \beta$ correspond to the Gödel-numbers of true Σ_1 sentences of $\tilde{\mathcal{B}}(p_{<\alpha})^s$ with parameters from $\beta \cup \{A \cap \alpha, s\}$. Similarly for $[Y]_1^{-1} \cap \beta$. Also observe that for large enough $\beta < \alpha$, we never have $H_\beta \cap \alpha = \beta$. This means also that $Y \cap \beta$ does not precode a $t \in S_\beta$, for $[Y]_0^{-1} \cap \beta$ codes a model where β^+ exists.

Proof of lemma 4.14. The proof is by induction on α. Pick $M = L_\mu[A \cap \alpha, s]$ so that μ is a limit ordinal, $\tilde{\mu}(p_{<\alpha})^s < \mu < \mu(p_{<\alpha})^s$ and $p, X, b_p^\alpha \in M$. Let

$$H_\beta = h_{\Sigma_1}^M(\beta \cup \{A \cap \alpha, s\}),$$

let $\pi_\beta \colon H_\beta \to \bar{M}_\beta$ be the transitive collapse. Let g be defined by $g(\beta) = (\beta^+)^{\bar{M}_\beta}$ for successor cardinals $\beta < \alpha$, noting that this is well defined since $\beta \in H_\beta$.

Pick Y such that $[Y]_0^{-1} = \mathbf{Th}_{\Sigma_1}^{\tilde{\mathcal{B}}(p_{<\alpha})^s}(\alpha \cup \{A \cap \alpha, s\})$, while $[Y]_1^{-1} = \mathbf{Th}_{\Sigma_1}^M(\alpha \cup \{A \cap \alpha, s\})$. Observe that Y pre-codes s. Moreover, demand that $[Y]_2^{-1} = X$.

The same construction works, wether or not $p_\alpha \neq \emptyset_\alpha$. For $\beta \in E_\alpha$, let $q_{\beta^+} = p_{\beta^+} {}^\frown 0^{g(\beta)} {}^\frown 1 {}^\frown ([Y]_3 \cap \beta)$ (where 0^ν denotes a string of 0's of length ν). For β which is a limit point of E_α, let $q_\beta = p_\beta {}^\frown ([Y]_3 \cap \beta)$. Note that clearly, if p didn't use fake inaccessible coding, neither does q.

We now show that this definition works for β large enough.

First, note that

$$H_\alpha = h_{\Sigma_1}^M(\alpha \cup \{A \cap \alpha, s\}) = M,$$

since M can be characterized as the minimal model such that the Σ_1 statement asserting the existence of $\tilde{\mathcal{B}}(p_{<\alpha})^s$ and certain statements describing the height of the M via ordinal addition hold inside M. Thus, $M = \bigcup_{\beta<\alpha} H_\beta$.

Lemma 4.17. *Assume β is large enough so that $p \in H_\beta$, and also large enough so that letting $\tilde{\beta} = \min H_\beta \cap [\beta, \alpha]$, we have $\tilde{\beta} \neq \alpha$. Then $\pi_\beta(p_{\tilde{\beta}}) = p_\beta$.*

Proof. If $\tilde{\beta} = \beta$, there is nothing more to proove.

Otherwise, first note that $E_{\tilde{\beta}} \in H_\beta$ by elementarity and $\pi_\beta(E_{\tilde{\beta}}) = E_{\tilde{\beta}} \cap \beta = E_\beta$. In fact, that $p \restriction \tilde{\beta}$ exactly codes $p_{\tilde{\beta}}$ is expressible as a Σ_1 statement inside $L_\alpha[A]$. By elementarity, this statement also holds of $\pi_\beta(p \restriction \tilde{\beta})$ and $\pi_\beta(p_{\tilde{\beta}})$ in $\bar{M}_\beta$ and is upwards absolute, so $\pi_\beta(p \restriction \tilde{\beta}) = p \restriction \beta$ exactly codes $\pi_\beta(p_{\tilde{\beta}})$. But since by definition of P, $p \restriction \beta$ exactly codes p_β, we have $p_\beta = \pi_\beta(p_{\tilde{\beta}})$. $\square$

Lemma 4.18. *The following hold:*

1. *$g \restriction \beta$ eventually dominates $f(q \restriction \beta)^{p_\beta}$ for large enough $\beta \in$ **Card** $\cap\, \alpha$.*
2. *If p_α is a proper initial segment of s, g eventually dominates $f(q \restriction \alpha)^{p_\alpha}$.*
3. *On the other hand $f(q_{<\alpha})^s$ eventually dominates g, in fact $g \in \mathcal{B}(q_{<\alpha})^s$.*

Proof. Let $\beta \leq \alpha$ be a limit cardinal large enough so that $p \in H_\beta$. In case $\beta = \alpha$, note that $p_\alpha \in M$. We may assume that either $\beta = \alpha$ or $H_\beta \cap \alpha \neq \beta$ (as α is singular in M).

Let $\tilde{\beta} = \min H_\beta \cap [\beta, \alpha]$. For the case $\alpha = \beta$, we may assume p_α is a proper initial segment of s. Then $f(p \restriction \alpha)^{p_{\tilde{\beta}}} \in M$ (by lemma 4.15 and the previous remark if $\beta = \alpha$), and in fact it is the solution to a Σ_1-formula in M with parameters $\tilde{\beta}$ and p - or p_α, in case $\beta = \alpha$. Thus $f(p \restriction \alpha)^{p_{\tilde{\beta}}} \in H_\beta$. Say $\gamma < \beta$ is a successor large enough so that $f(p \restriction \alpha)^{p_{\tilde{\beta}}} \in H_\gamma$. Observe $\gamma \in H_\gamma$. Thus by elementarity, $\bar{M}_\gamma \models$"$f(p \restriction \alpha)^{p_\beta}(\gamma)$ has size γ".

Finally, as $M \in \mathcal{B}(p_{<\alpha})^s$, clearly $g(\gamma) < \operatorname{otp}(\mathrm{On} \cap H_\gamma) < \operatorname{otp}(\mathrm{On} \cap H^s_\gamma)$ and so $g(\gamma) < f(p \restriction \alpha)^s(\gamma)$ for all $\gamma < \beta$. Thus g is slower than $f(p \restriction \alpha)^s$. $\square$

We now show that for any limit $\beta \leq \alpha$, $q \restriction \beta \in \mathcal{B}(p_{<\alpha})^{q_\beta}$ and exactly codes q_β.

First, let $\beta = \alpha$. Since $\tilde{\mathcal{B}}(p_{<\alpha})^s \in \bar{M} \in \mathcal{B}(p_{<\alpha})^s = \mathcal{B}(q_{<\alpha})^{q_\alpha}$, we have $Y \in \mathcal{B}(q_{<\alpha})^{q_\alpha}$and $g \in \mathcal{B}(q_{<\alpha})^{q_\alpha}$ (because the height of the latter model is a limit mutiple of β, we can argue as in lemma 4.15). So also $q \restriction \alpha \in \mathcal{B}(q_{<\alpha})^{q_\alpha}$.

Say $p_\alpha = s = q_\alpha$. Then we have seen that $f(p \restriction \alpha)^s$ dominates g, so $q \restriction \alpha$ exactly codes p_α just as $p \restriction \alpha$ does.

Now say p_α is a proper initial segment of $s = q_\alpha$. Moreover, we have seen that g eventually dominates $f(p \restriction \alpha)^{p_\beta}$. So $f(p \restriction \alpha)^{p_\alpha}_{q \restriction \alpha}$ is defined, the limit coding for $q \restriction \alpha$ goes on for one more step than for $p \restriction \alpha$ and we obtain Y pre-coding $s = q_\alpha$. Also, the argument of lemma 4.15 shows that $f(p \restriction \alpha)^{p_\beta}_{q \restriction \alpha} \in \mathcal{B}(p_{<\alpha})^s$. Observe that also $\mathcal{B}(p_{<\alpha})^s = \mathcal{B}(q_{<\alpha})^{q_\alpha}$ by lemma 4.16. Since also

$Y \in \mathcal{B}(q_{<\alpha})^{q_\alpha}$, $q \restriction \alpha$ codes q_α. As $q \restriction \alpha \in \mathcal{B}(q_{<\alpha})^{q_\alpha}$, the coding stops there and $q \restriction \alpha$ exactly codes q_α.

Now say $\beta < \alpha$. Remember $H_\beta \cap \alpha \neq \beta$. Clearly, $Y \cap \beta \in \mathcal{B}(q_{<\beta})^{q_\beta}$ as $q_\beta = p_\beta \frown [Y \cap \beta]_3$. The Σ_1-theory of $\bar{M}_\beta$ allows us to reconstruct g inside $\mathcal{B}(q_{<\beta})^{q_\beta}$ using an argument like that of lemma 4.15. Thus, $q \restriction \beta \in \mathcal{B}(q_{<\beta})^{q_\beta}$. Moreover, in this case we have also shown that $g \restriction \beta$ eventually dominates $f(p \restriction \alpha)^{p_\beta}$, and as before we obtain $Y \cap \beta$ in the next step of the limit coding (after the exact coding of p_β by $p_{<\beta}$, which could of course be trivial if $p_\beta = \emptyset_\beta$). We have seen $Y \cap \beta$ does not pre-code an element of S_β, so $q \restriction \beta$ exactly codes $q_\beta = p_\beta \frown [Y \cap \beta]_3$, provided that $f(p_{<\beta})^{p_\beta}_{q \restriction \beta} \in \mathcal{B}(q_{<\beta})^{q_\beta}$. This holds again by an argument similar to that of lemma 4.15. Again, as $q \restriction \beta \in \mathcal{B}(q_{<\beta})^{q_\beta}$, the coding stops there and $q \restriction \beta$ exactly codes q_β.

Finally, we obtain a q such that the desired properties hold except for the capturing of X, which only holds for a tail of β; we can use induction to get q that works for all β. Obviously, $q \leq p$ (we have never put 1s on any partition affected by restraints). For the beginning of the induction, assume α is the least limit cardinal. Argue as in the general limit case described above to get a condition q that works on a tail below α. Now make finitely many extensions to obtain a q that works everywhere. □

Note the subtle role of the proof of 4.14 in my choice to work with the E_β's: it seems I could have easily done with the standard C_β from □ instead. Fake inaccessible coding requires an Easton set, but this is also no reason to go beyond □. It is the mechanism of distinguishing between fake inaccessible coding and immediate singular coding which requires that E_β appear in the smallest relevant type of structure—where β is seen to be non-Mahlo, but not necessarily singular. Without this distinction, we could choose to always do fake inaccessible coding—but then the proof of 4.14 seems to fail.

4.7 The main theorem for quasi-closure of $P(A_0)$

We shall now define $\mathbf{D}$ and $\preccurlyeq^\lambda$ witnessing that P is quasi-closed.

Definition 4.19. We define $p \preccurlyeq^\lambda q$ just if $p \leq q$, for all $\delta \in \mathbf{Reg} \cap \lambda$ both $p_\delta = q_\delta$ and $p^*_{\delta+} = q^*_{\delta+}$. In accordance with the section 3.1, define $p \preccurlyeq^{<\lambda} q$ just if $p \leq q$ and both $p_\delta = q_\delta$ and $p^*_\delta = q^*_\delta$ for all $\delta \in \mathbf{Reg} \cap \lambda$ when λ is a limit cardinal.

The class $\mathbf{D}$ is defined in a more elementary way than might be expected from the literature (e.g. [BJW82] or [Fri00]), ensuring a more basic kind of genericity over certain Skolem-hulls, owing to the fact that we've assumed

$\mathbf{Card} = \mathbf{Card}^{L[A_0 \cap \omega]}$. The proof that this class is dense, i.e. (C I), takes the form of an intricate inductive argument using that quasi-closure already holds over smaller (in the sense of the induction) coding structures.

The next theorem treats two situations simultaneously: the first is a localized version of quasi-closure which will carry us through the singular limit step of the inductive argument we just mentioned (to show $\mathbf{D}$ is dense). The second is the global version of quasi-closure. The proof readily suggests such an aggregation. We shall work under the following assumption.

Assumption 4.20. Let $\beta \in \mathbf{Card} \cap \kappa$ or $\beta = \infty$. In the first case, let $q \in P$, $M = \mathcal{A}^{q_{\beta^+}}$; also demand that $|q_{<\kappa}| \geq \beta^+$ (we need to ask this so $A = A_q$ is defined, since $\mathcal{A}^{q_{\beta^+}} = \mathcal{A}(A)^{q_{\beta^+}}$). Define

$$P(q)^{\beta^+} = \{p \restriction \beta^+ \mid p \in P \wedge p \leq q \wedge p^{\beta^+} = q^{\beta^+}\},$$

and let $R = P(q)^{\beta^+}$. In the second case let $M = L[A]$ and $R = P$.

The definition of $P(q)^{\beta^+}$ was chosen so that for $p \in P(q)^{\beta^+}$, $p \cdot q \in P$ and $P(q)^{\beta^+}$ generically codes both $A \cap \beta$ and q^{β^+} over $\mathcal{A}^{q^{\beta^+}}$. In fact, $(p \cdot q) \restriction \beta^{++} = p$. Equivalently, $p \in R$ if and only if $p \leq q \restriction \beta^{++}$, $p^{\beta^+} = q^{\beta^+}$ and p obeys all restraints from q for inaccessibles $\delta > \beta$.

Definition 4.21. Given $p \in P$, $\lambda \in \mathbf{Reg}$, $\lambda < \beta \in \mathbf{Card}$, $x \in M$ arbitrary, we now define $\mathbf{D}^M_{[\lambda,\beta)}(p, \vec{x}) \subseteq P$. For $\delta \in \mathrm{On}$, let

$$H_\delta = H^M_\delta(p, \vec{x}) = h^M_{\Sigma_1}(\delta \cup \{\vec{x}\}),$$
$$H_{<\delta} = H^M_{<\delta}(p, \vec{x}) = h^M_{\Sigma_1}(\sup(\mathrm{supp}(p) \cap \delta) \cup \{\vec{x}\}).$$

We define $\mathbf{D}^M_{[\lambda,\beta)}(p, \vec{x})$ as the set of $q \in P$ such that if $\tau = \min(\mathrm{supp}(p) \cap [\lambda, \beta))$ exists then $q \preccurlyeq^\tau p$ and

(D 1) if $\tau > \lambda$ and $\tau \in \mathbf{Inacc}$ then $\rho(q^*_\tau) \geq \lambda$ (of course, this is vital to preserve 6 in the definition of $P(A_0)$; see also the last clause of (C I));

(D 2) for all $\delta \in [\tau, \beta)$ such that $\delta \in H_{<\delta} \cup \mathrm{supp}(p)$

(a) $|q_\delta| > H_\delta \cap \delta^+$; for $\delta = \kappa$ in addition, $|q_{<\kappa}| > \sup(H_{|p_{<\kappa}|} \cap \kappa)$.

(b) if $\nu \in H_{<\delta} \cap [\delta, \delta^+)$ then $b^{p_\delta \restriction \nu} \setminus \eta \in q^*_\delta$ for some $\eta \in [\lambda, \delta)$;

(c) if $\xi \in H_\delta \cap [\delta, \delta^+)$ there is $\nu > |p_\delta|$ such that $p_\delta((\langle \xi, \nu \rangle)_0) = 1$ if $\xi \in A$ and $p_\delta((\langle \xi, \nu \rangle)_0) = 0$ if $\xi \notin A_p$;

(d) if $b^{p_{\delta^+} \restriction \nu} \in p^*_{\delta^+}$ there is $\zeta > |p_\delta|$ such that $q_\delta((\zeta)_1) = p_{\delta^+}(\nu)$;

(e) if $\delta \in \mathbf{Inacc} \cap \kappa$ there is $\beta \in E_\delta \setminus \sup(\mathrm{supp}(p) \cap \delta))$ such that $q_{\beta^+}((\beta^+)_2) = 1$;[16]

(f) if $\delta \in \mathbf{Inacc} \cap \kappa^+$ and $b^{p_\delta \restriction \nu} \setminus \eta \in p^*_\delta$ then there is $\xi \in b^{p_\delta \restriction \nu} \setminus \eta$ such that $\xi > \sup(\mathrm{supp}(p) \cap \delta))$ and $q_{<\delta}((\xi)_2) = p_\delta(\nu)$ (note that this clause elegantly covers both the Mahlo coding and the inaccessible, non-Mahlo coding);

(g) if $\delta \in \mathbf{Inacc}$ and $b \in p^*_\delta \setminus \mathrm{On}$ then $b \cap \sup(\mathrm{supp}(p) \cap \delta) \subseteq \mathrm{dom}(q_{<\delta})$;

We also write

$$U(p) = U^M(p, \vec{x}) = \{\delta \in \mathbf{Card} \cap \delta \mid \delta \in H_{<\delta} \cup \mathrm{supp}(p)\}.$$

Finally, for the proof of quasi-closure we set $\mathbf{D}(\lambda, \vec{x}, p) = \mathbf{D}^{L[A]}_{[\lambda,\infty)}(p, \vec{x})$.

Remark 4.22. To show that P is stratified, we also need to put something like the first clause into the definition of $\mathrm{dom}(\mathbf{C}^\lambda)$. The easiest is to require $\lambda \in \mathrm{supp}(p)$.

Note that we define $\mathbf{D}^M_{[\lambda,\beta)}(p, \vec{x})$ as a subset of P, and for $p \in P$ rather than as a subset of R and for $p \in R$. This is a notational convenience we will make use of when we show that these sets are non-emty. To build sequences with greatest lower bounds, it is only the restriction to R of $\mathbf{D}^M_{[\lambda,\beta)}(p, \vec{x})$ which is useful.

We also introduce the following terminology, which provides good intuition and will be useful when we show the least Mahlo is not collapsed in our iteration (see 7.6).

Definition 4.23. Let H be any set and let $p, q \in P$, $q \leq p$. We say that q is basic generic for (H, p) at δ^+ if and only if

1. $|q_\delta| > H \cap \delta^+$;

2. if $\nu \in H \cap [\delta^+, \delta^{++})$ then $b^{p_{\delta^+} \restriction \nu} \in q^*_{\delta^+}$;

3. if $\nu \in H \cap [\delta^+, \delta^{++})$ then there is $\zeta > |p_\delta|$ such that $q_\delta((\zeta)_1) = p_{\delta^+}(\nu)$;

4. if $\xi \in H \cap [\delta, \delta^+)$ there is $\nu > |p_\delta|$ such that $q_\delta((\langle \xi, \nu \rangle)_0) = 1$ if $\xi \in A$ and $q_\delta((\langle \xi, \nu \rangle)_0) = 0$ if $\xi \notin A_p$;

[16] This somewhat technical requirement makes it easy to distinguish the fake inaccessible coding from the singular coding. Conditions which immediately use singular coding, i.e. those constructed in the extendibility lemma will have $q_{\beta^+}((\beta^+)_2) = 1$ on a tail of $\beta \in E_\delta$.

When $\delta \in \mathbf{Inacc}$, we say that q is basic generic for (H, p) at δ if and only if

1. $|q_{<\delta}| \geq \sup(H \cap \delta)$.
2. if $\nu \in H \cap [\delta, \delta^+)$ then $b^{p_\delta \restriction \nu} \setminus \eta \in q^*_\delta$ for some η;
3. if $\nu \in H \cap [\delta, \delta^+)$ then there is $\xi \in b^{p_\delta \restriction \nu} \setminus \eta$ such that $\xi \geq |p_{<\delta}|$ and $q_{<\delta}((\xi)_2) = p_\delta(\nu)$;
4. if $\xi \in H \cap \delta$ there is $\nu > |p_\delta|$ such that $q_{<\delta}((\langle \xi, \nu \rangle)_0) = 1$ if $\xi \in A$ and $q_{<\delta}((\langle \xi, \nu \rangle)_0) = 0$ if $\xi \notin A_p$;

The second definition we shall only use for $\delta = \kappa$ (in 7.6).

Theorem 4.24. *Let $\rho \leq \lambda \leq \beta$, $\lambda \in \mathbf{Reg}$. If $\beta < \infty$, let $(\beta_\xi)_{\xi<\rho} \in M$ be an increasing sequence of cardinals such that $\sup_{\xi<\rho} \beta_\xi = \beta$ and set $\beta_\xi = \infty$ for $\xi < \rho$ if $\beta = \infty$. Say $(p^\xi)_{\xi<\rho} \in M$ is a sequence of conditions in R which has $\bar{w} = (w^\xi)_{\xi<\rho} \in M$ s.t. $M \models \bar{w}$ is a (λ, x)-canonical witness. Moreover say $(p^\xi)_{\xi<\rho}$ is strategic in the following sense:*

(A) for any $\xi < \bar{\xi} < \rho$, $p^{\bar{\xi}} \preccurlyeq^\lambda p^\xi$,

(B) for any $\xi < \rho$, there is λ_ξ such that $p^\xi \in \mathbf{D}_{[\lambda_\xi, \beta_\xi)}(p^\xi, \{\bar{w} \restriction \xi + 1, x\})$ and $p^{\xi+1} \preccurlyeq^{\lambda_{\xi+1}} p^\xi$,

(C) in case $\beta < \infty$, letting $H = h^M_{\Sigma_1}(\beta_\xi \cup \{\bar{w} \restriction \xi + 1, x\})$, $p^{\xi+1}$ is (H, p^ξ)-generic at β^+, which means in the present case:

(1) $|p^{\xi+1}_\beta| > H \cap \beta^+$;

(2) if $\nu \in H \cap [\beta^+, |s|)$ then $b^{s \restriction \nu} \in (p^{\xi+1})^*_\beta$;

(3) if $\xi \in H \cap [\beta, \beta^+)$ there is $\nu > |p^\xi_\beta|$ such that $p^{\xi+1}_\beta((\langle \xi, \nu \rangle)_0) = 1$ if $\xi \in A_p$ and $p^{\xi+1}_\beta((\langle \xi, \nu \rangle)_0) = 0$ if $\xi \notin A_p$;

(4) if $b^{s \restriction \nu} \in (p^\xi)^*_{\beta^+}$ there is $\zeta > |p^\xi_\beta|$ such that $p^{\xi+1}_\beta((\zeta)_1) = s(\nu)$;

Then $\bar{p}$ has a greatest lower bound $p^\rho \in R \cap M$.

Corollary 4.25. *Say $\bar{p}$ is a (λ, x)-adequate sequence. Then $\bar{p}$ has a greatest lower bound.*

Proof of corollary. In case $\beta = \infty$, strategic in the sense of the theorem means exactly that $\bar{w}$ is a strategic witness, except for one point: we must check is that the proof of the theorem goes through even if the hypothesis that $\bar{p}$ is adequate is only satisfied on a tail, so we shall pay some attention to this in the proof of the theorem. □

Proof of theorem. The proof is split over several lemmas. Fix a sequence $(p^\xi)_{\xi<\rho}$ and $\bar{w}$ as in the hypothesis. For each $\xi < \rho$, pick $\lambda_\xi \in \mathbf{Reg}$ as in (B). Note we may assume $\lambda_\xi \geq \lambda$, for we may replace λ_ξ by λ whenever this assumption fails: firstly, (A) holds, and secondly, $\mathbf{D}^M_{[\lambda_\xi,\beta_\xi)}(p^\xi, \{x, \bar{w}\restriction\xi+1\}) \subseteq \mathbf{D}^M_{[\lambda,\beta_\xi)}(p^\xi, \{x, \bar{w}\restriction\xi+1\})$ if $\lambda_\xi \leq \lambda$.

Let $p = p^\rho$ be the obvious candidate for a greatest lower bound, i.e. for $\delta \in \bigcup_{\xi<\rho} \operatorname{supp}(p^\xi)$,

$$\begin{aligned}(p_\delta, p^*_\delta) &= (\bigcup_{\xi<\rho} p^\xi_\delta, \bigcup_{\xi<\rho} (p^\xi)^*_\delta),\\ p_{<\kappa} &= \bigcup_{\xi<\rho} p^\xi_{<\kappa}, \text{ if } \kappa \leq \beta.\end{aligned} \tag{4.2}$$

Most of this proof will now be devoted to check that for each $\delta \in \operatorname{supp}(p)$ both

$$p_\delta \in S^*_\delta \tag{4.3}$$

$$p \restriction \delta \in \mathcal{B}(p_{<\delta})^{p_\delta} \text{ if } \delta \in \mathbf{Sing}. \tag{4.4}$$

After that we conclude by checking that $p^*_{<\kappa} \in S^*_{<\kappa}$.

Let $\bar{\lambda}$ be minimal such that for an unbounded set of $\xi < \rho$, we have $\lambda_\xi \leq \bar{\lambda}$. Obviously, if $\delta \in \mathbf{Card} \cap \bar{\lambda}$, $p^\rho_\delta = p^\xi_\delta$ for some ξ, and so (4.3) and (4.4) hold. So let $\delta \geq \bar{\lambda}$. Let

$$X^\xi = \begin{cases} h^M_{\Sigma_1}(\delta \cup \{x, \bar{w}\restriction\xi+1\}) & \text{for } \delta < \beta,\\ h^M_{\Sigma_1}(\beta_\xi \cup \{x, \bar{w}\restriction\xi+1\}) & \text{for } \delta = \beta.\end{cases}$$

Let $X = \bigcup_{\xi<\rho} X^\xi$, let $\pi\colon \bar{X} \to X$ be the transitive collapse. Let $j = \pi^{-1}$. Write $\bar{A} = \bigcup_{\nu\in X} \pi(A \cap \nu)$ and (if $\beta < \infty$) $s = q_{\beta^+}$, $\bar{s} = \pi(s)$, so that

$$\bar{X} = \begin{cases} L_{\bar{\mu}}[\bar{A}] & \text{in case } \beta = \infty, M = L[A]\\ L_{\bar{\mu}}[\bar{A}, \bar{s}] & \text{when } \beta < \infty, M = \mathcal{A}^s.\end{cases}$$

We now embark on a series of lemmas which will be used several times when prooving (4.3) and (4.4) in various cases.

Lemma 4.26 (Definability)**.** *The sequence $(\pi(p^\xi))_{\xi<\rho}$ (or at least a tail) is a definable class in $\bar{X}$.*

Proof of lemma 4.26. The notion of canonical witness was chosen precisely to ensure this definability, and in a sense, the lemma is trivial. We proove the lemma under the weaker assumption that $\bar{w}$ is a (λ, x)-canonical witness for $(p_\xi)_{\xi<\rho}$ holds only for $\xi_0 < \xi < \rho$, as in the definition of quasi-closure 3.4.

We show that there is a formula Θ such that $M \models \Theta(\xi, p^*) \iff p^* = p_\xi$ for $\xi \in [\xi_0, \rho)$.

Let Ψ and G be as in the definition of canonical witness, i.e. Ψ is the formula defining $\bar{w}$ and G is the $\Sigma_1^T(\lambda \cup \{x\})$ function s.t. $p^\xi = G(\bar{w} \restriction \xi+1, \vec{x})$ for $\xi < \rho$. Let Γ be a $\Sigma_1^T(\lambda \cup \{x\})$ formula representing G. The formula $\Theta(\xi, p^*)$ is

$$\begin{aligned} \exists \bar{w}^* = (w^*_\nu)_{\nu<\xi+1} \text{ s.t. } \bar{w}^* \restriction \xi_0 = \bar{w} \restriction \xi_0 & \\ \wedge\ [\forall \nu \in [\xi_0, \xi+1] \quad \Psi(w^*_\nu, \bar{w}^* \restriction \nu, x)] & \\ \wedge\ \Gamma(p^*, (w^*_\nu)_{\nu<\xi+1}, x) & \end{aligned}$$

As $\rho \subseteq X \prec_{\Sigma_1} M$ and $\{p_\xi, \bar{w} \restriction \xi+1\} \subseteq X$ for each $\xi < \rho$, the same formula defines (a tail of) $\bar{p}$ in X. Now apply π. □

Observe that only δ such that

$$\delta \in \operatorname{supp}(p^\rho) \text{ or } \delta = \bigcup_{\xi<\rho} \sup(\operatorname{supp}(p^\xi) \cap \delta) \tag{4.5}$$

are relevant to the proof of (4.3) and (4.4). Thus we restrict our attention to such δ in the following.

We shall now show that because of (B), p^ρ enjoys a basic type of genericity over X. First, note that $\mathbf{Card} \cap X \cap (\kappa^+ + 1) \subseteq \operatorname{supp}(p^\rho)$, because of (D 2a) and the following:

Lemma 4.27. *If $\tilde{\gamma} \in \mathbf{Card} \cap X$, then $\tilde{\gamma} \in U^M(p^\xi, \{\bar{w} \restriction \xi+1, \vec{x}\})$ for large enough $\xi < \rho$.*

Proof. Fix ξ large enough such that $\tilde{\gamma} \in X^\xi$. If first alternative of (4.5) obtains, we may assume ξ is large enough such that $\delta \in \operatorname{supp}(p^\xi)$. We can assume $\tilde{\gamma} > \delta$ as trivially, $\operatorname{supp}(p^\xi) \subseteq U^M(p^\xi, \{\bar{w} \restriction \xi+1, \vec{x}\})$. Then $\sup(\operatorname{supp}(p^\xi) \cap \tilde{\gamma}) \geq \delta$, and so $\tilde{\gamma} \in H^M_{\tilde{\gamma}}(p^\xi, \{\bar{w} \restriction \xi+1, \vec{x}\})$. Now say the second alternative of (4.5) obtains. Then we may assume ξ is large enough so that $\tilde{\gamma} \in h^M_{\Sigma_1}((\sup(\operatorname{supp}(p^\xi) \cap \delta) \cup \{\bar{w} \restriction \xi+1, \vec{x}\})$. Since $\delta \leq \tilde{\gamma}$, we immediately conclude $\tilde{\gamma} \in H^M_{\tilde{\gamma}}(p^\xi, \{\bar{w} \restriction \xi+1, \vec{x}\})$. □

For $\gamma \in \mathbf{Card}^{\bar{X}}$, let

$$\bar{p}_\gamma = \bigcup_{\xi<\rho} \pi(p^\xi)_\gamma.$$

In fact, $j(\gamma)^+ \cap X = \bigcup_{\xi<\rho} |p^\xi_\sigma(\gamma)|$, i.e.

$$|\bar{p}_\gamma| = \gamma^{+L_{\bar{\mu}}}, \tag{4.6}$$

by the following:

Lemma 4.28. *If $\nu \in [\gamma, \gamma^{+\bar{X}})$ and either $\nu > \delta$ or the second alternative of* (4.5) *obtains, then $j(\nu) \in H^M_{<j(\gamma)}(p^\xi, \{\bar{w} \restriction \xi + 1, \vec{x}\})$, for large enough $\xi < \rho$.*

Proof. Exactly as the previous lemma. □

So clearly by (D 2a), we have $\nu < |\bar{p}^{\xi+1}_\gamma|$ for ξ, ν as above, prooving (4.6). Similarly, (D 2a) makes sure $|\bar{p}^\xi_\gamma|$ strictly increases with $\xi < \rho$.

Remark 4.29. Observe for (4.6) we can assume $\nu > \delta$ in the hypothesis of the lemma, but the full generality of the lemma will be needed in 4.30.

Thus, letting

$$\bar{p} = \bigcup_{\gamma \in \mathbf{Card}^{\bar{X}}} \bar{p}_\gamma$$

we see $\bar{p}\colon \bar{\mu} \to 2$ is a total function. We continue exploiting the basic genericity of $\bar{p}$ over $\bar{X}$, in the sense that $\bar{p}_\delta$ codes all of $\bar{A}$ and $\bar{p}$:

Lemma 4.30 (Local coding)**.** *Provided δ satisfies* (4.5) *and letting*

$$Y = L_{\bar{\mu}}[A \cap \delta, \bar{p}_\delta]$$

we have $\bar{X} \subseteq Y$ and $\bar{X}$ is a definable class in Y. In fact, for each $\nu < \bar{\mu}$, $\bar{A} \cap \nu$ and $\bar{p} \restriction \nu$ are definable in $(\mathcal{A}^{<\bar{p}_{\|\nu\|} \restriction \nu})^Y$ if $Y \models \|\nu\| \in \mathbf{Reg}$ and definable in $(\mathcal{B}(\bar{p}_{<\|\nu\|})^{<\bar{p}_{\|\nu\|} \restriction \nu})^Y$ if $Y \models \|\nu\| \in \mathbf{Sing}$.

Proof of lemma 4.30. We show the definability of $\bar{p}$ and $\bar{A}$ by checking they can be reconstructed from $\bar{A} \cap \delta$ and $\bar{p}_\delta$ by the decoding procedure run over $\bar{X}$ (see the last item of definition 4.5). We now describe this procedure, which is carried out in $L_{\bar{\mu}}[\bar{A} \cap \delta, \bar{p}_\delta]$, making three claims which we proove thereafter.[17] Observe that $\mathbf{Card}^{\bar{X}} = \mathbf{Card}^{L_{\bar{\mu}}}$.

(i) Say for $\gamma \in \mathbf{Sing}^{L_{\bar{\mu}}}$ we have already constructed $\bar{A} \cap \gamma$ and $\bar{p} \restriction \gamma$ inside $L_{\bar{\mu}}[\bar{A} \cap \delta, \bar{p}_\delta]$. We claim that $\bar{p} \restriction \gamma$ codes $\bar{p}_\gamma$ via singular limit coding using as coding apparatus the functions $f^{\bar{p}_\gamma \restriction \nu}$ as defined in $L_{\bar{\mu}}[\bar{A} \cap \gamma, \bar{p}_\gamma \restriction \nu]$, for $\nu < \gamma^{+L_{\bar{\mu}}}$. Observe this is well defined as $L_{\bar{\mu}}[\bar{A} \cap \gamma, \bar{p}_\gamma \restriction \nu] \subseteq \bar{X}$ and so γ is a singular cardinal in that model (this needn't be the case for $L_{\bar{\mu}}[\bar{A} \cap \gamma, \bar{p} \restriction \gamma, \bar{p}_\gamma \restriction \nu]$).

[17] A lot of the technical complexity of the following stems from the fact that we do not know the cardinals of $L_{\bar{\mu}}[A \cap \delta, \bar{p}^\rho_\delta]$. Thus we have to work in a mixture of models below. If the sets $\mathbf{D}$ are defined to provide stronger genericity over X, the construction can be carried out entirely over the natural model, namely $L_{\bar{\mu}}[A \cap \delta, \bar{p}^\rho_\delta]$.

(ii) Say for $\gamma \in \mathbf{Reg}^{L_{\bar\mu}}$ below the Mahlo of $L_{\bar\mu}$ (if there is one) and $\nu \in [\gamma, \gamma^{+L_{\bar\mu}})$ we have already constructed $\bar A \cap \gamma$ and $\bar p \restriction \nu$ inside $L_{\bar\mu}[\bar A \cap \delta, \bar p_\delta]$. Letting $k = 1$ if $\gamma \in \mathbf{Succ}^{L_{\bar\mu}}$ and $k = 2$ if γ is inaccessible in $L_{\bar\mu}$, for each $\nu \in [\gamma, \gamma^{+L_{\bar\mu}})$ we claim

$$\bar p_\gamma(\nu) = 1 \iff$$
$$\{\xi < \gamma \mid \xi \in b^{\bar p_\gamma \restriction \nu} \wedge \bar p((\xi)_k) = 1\} \text{ is unbounded below } \gamma,$$

where by $b^{\bar p_\gamma \restriction \nu}$ we mean $(b^{\bar p_\gamma \restriction \nu})^{L_{\bar\mu}[\bar A \cap \gamma, \bar p_\gamma \restriction \nu]}$.

(iii) Say for $\gamma = \bar\kappa$, the Mahlo of $L_{\bar\mu}$, and $\nu \in [\bar\kappa, \bar\kappa^{+L_{\bar\mu}})$, say we have already constructed $\bar A \cap \bar\kappa$ inside $L_{\bar\mu}[\bar A \cap \delta, \bar p_\delta]$. Let $\bar p_{<\bar\kappa} = [\bar A]_1$ and $\bar A_0 = [\bar A]_0$. For each $\nu \in [\bar\kappa, \bar\kappa^{+L_{\bar\mu}})$ we claim

$$\bar p_{\bar\kappa}(\nu) = 1 \iff$$
$$\{\xi < \bar\kappa \mid \xi \in b^{\bar p_{\bar\kappa} \restriction \nu} \wedge \bar p_{<\bar\kappa}((\xi)_3) = 1\} \text{ is unbounded below } \gamma,$$

where by $b^{\bar p_{\bar\kappa} \restriction \nu}$ we mean $(b^{\bar p_{\bar\kappa} \restriction \nu})^{L_{\bar\mu}[\bar A \cap \bar\kappa, \bar p_{\bar\kappa} \restriction \nu]}$.

(iv) Say for $\gamma \in \mathbf{Card}^{L_{\bar\mu}}$ below the Mahlo of $L_{\bar\mu}$ (if there is one), we have already constructed $\bar A \cap \gamma$ and $\bar p \restriction \gamma$ inside $L_{\bar\mu}[\bar A \cap \delta, \bar p_\delta]$. We claim:

$$\nu \in \bar A \cap [\gamma, \gamma^{+L_{\bar\mu}}) \iff$$
$$\{\xi \in [\gamma, \gamma^{+L_{\bar\mu}}) \mid \bar p_\gamma((\langle \xi, \nu \rangle)_0) = 1\} \text{ is unbounded below } \gamma^{+L_{\bar\mu}}.$$

The same holds for $\bar A_0$ if $\gamma = \bar\kappa$, the Mahlo of $L_{\bar\mu}$.

From these claims, the lemma follows by induction. In a sense the proof of these claims is again trivial, by definition of $\mathbf{D}$, i.e. by strategicity or (B). The claim in (i) is a straightforward consequence of elementarity: since $\bar p_\gamma \restriction \nu \in \bar X$ for $\nu < |\bar p_\gamma|$, $j(f^{\bar p_\gamma \restriction \nu}) = f^{p^\xi_{j(\gamma)}} \restriction j(\nu)$ for large enough $\xi < \rho$, and $X \models$"$p^\xi \restriction j(\gamma)$ exactly codes $p^\xi_{j(\gamma)}$. By the obvious continuity of singular limit coding, the claim in (i) follows.

Now for the claim in (ii). We restrict our attention to the exemplary case that $\gamma < \beta$ and is inaccessible in $L_{\bar\mu}$. Observe for the claim we can assume $\delta < \gamma$, but note for later that the present proof also works if $\delta = \gamma$ and the second alternative of (4.5) obtains.

Now say $\nu \in [\gamma, \gamma^{+L_{\bar\mu}})$ and show the claim in (ii). We have seen for $\gamma \in U^M(p^\xi, \{\bar w \restriction \xi+1, \vec x\})$ and $j(\nu) \in H_{j(\gamma)}(p^\xi, \{\bar w \restriction \xi+1, \vec x\})$ for large enough $\xi < \rho$ (we assume for now that $\nu > \delta$, but note for later reference the present proof also goes through if the second alternative of (4.5) obtains). Thus by (D 2b), $b^{p^\xi_{j(\gamma)} \restriction j(\nu)} = b^{p^\rho_{j(\gamma)} \restriction j(\nu)} \in (p^\xi)^*_{j(\gamma)}$ for large enough $\xi < \rho$, restraining extensions of p^ξ, i.e. making sure $p^\rho_{<j(\gamma)}((\zeta)_1) = 0$ for a tail of $\zeta \in b^{p^\rho_{j(\gamma)} \upharpoonright j(\nu)}$ if $p^\rho_{j(\gamma)}(j(\nu)) = 0$. Moreover, (D 2f) makes sure $p^\rho_{<j(\gamma)}((\zeta)_1) = 1$ for a tail of $\zeta \in b^{p^\xi_{j(\gamma)} \upharpoonright j(\nu)}$ if $p^\rho_{j(\gamma)}(j(\nu)) = 1$. Together, for large enough $\xi < \rho$ we have

$$p^\xi_{j(\gamma)}(j(\nu)) = 1 \iff X \models \text{“}p^\xi_{<j(\gamma)}((\zeta)_1) = 1 \text{ for unboundedly many } \zeta \in b^{p^\xi_{j(\gamma)} \upharpoonright j(\nu)}\text{”}$$

and since

$$\sigma(b^{p^\xi_{j(\gamma)} \upharpoonright j(\nu)}) = (b^{\bar{p}_\gamma \upharpoonright \nu})^{L_{\bar{\mu}}[\bar{A} \cap \gamma, \bar{p} \upharpoonright \nu]}$$

applying σ proves the claim.

The other case of the claim in (ii) is entirely analogous, substituting the use of (D 2f) by (D 2d) if $\gamma < \beta$ and using (C) if $\gamma = \beta$. Claim (iii) is proved in the same manner, once more using (D 2f). The claim in (iv) is proved using using (D 2c).

□

Lemma 4.31. *Provided δ satisfies* (4.5), $\bar{X} \in \mathcal{A}_0^{\bar{p}_\delta}$

Proof of lemma 4.31. It suffices to observe that $L_{\bar{\mu}}[A \cap \delta] \models |\bar{p}_\delta| = \pi^{-1}(\delta^+) \in$ **Card**. Thus by the definition of steering ordinal, $\bar{\mu} + \omega < \mu^{\bar{p}_\delta}$. Since by lemma 4.30, $\bar{A}$ and (in case $\beta < \infty$) $\bar{s}$ are definable over $L_{\bar{\mu}}[A \cap \delta, \bar{p}_\delta] \in L_{\mu^{\bar{p}_\delta}}[A \cap \delta, \bar{p}_\delta] = \mathcal{A}_0^{\bar{p}_\delta}$.

□

Lemma 4.32. *If $\delta = \bigcup_{\xi < \rho} \sup(\mathrm{supp}(p^\xi) \cap \delta)$,*

$$p \upharpoonright \delta \in \mathcal{A}_0^{\bar{p}_\delta}. \tag{4.7}$$

Proof of lemma 4.32. As $\pi(p^\xi) \upharpoonright \delta = p^\xi \upharpoonright \delta$, lemmas 4.26 and 4.31 together yield $p^\rho \upharpoonright \delta \in \mathcal{A}_0^{\bar{p}_\delta}$. □

We proceed to show (4.3) and (4.4). Let $\tilde{\delta} = \min \mathrm{On} \cap X \setminus \delta$, so that $\pi(\tilde{\delta}) = \delta$. First, say $\delta < \tilde{\delta}$. Then we must have $X \models \tilde{\delta} \in$ **Inacc** (δ is the critical point of π). Observe that for each $\xi < \rho$, by Easton support and since $\sup(\mathrm{supp}(p^\xi) \cap \tilde{\delta}) \in X$ we have $\sup(\mathrm{supp}(p^\xi) \cap \tilde{\delta}) < \delta$. Thus (4.3) is trivially satisfied as $p^\rho_\delta = \emptyset_\delta$. We may assume without loss of generality that

$$\delta = \sup_{\xi < \rho}(\mathrm{supp}(p^\xi) \cap \tilde{\delta}) \tag{4.8}$$

hold, for otherwise (4.4) is trivially satisfied and we are done. This means $\delta \in$ **Sing** ($\rho \leq \lambda$ and we must have $\lambda < \delta$ as the conditions grow below δ).

Fake inaccessible coding: We show that $\bar{p}_\delta$ is coded using fake inaccessible coding, i.e.

$$s(\bar{p}_{<\delta}) = \bar{p}_\delta \tag{4.9}$$

(of course $\bar{p}_{<\delta} = p_{<\delta}$). We adapt the proof of claim (ii), lemma 4.30.[18] Because of (D 2e), $p_{<\delta}$ does not immediately use singular coding. We remind the reader that the proof of the claim in (ii), lemma 4.30 also goes through if $\nu = \gamma = \delta$ because (4.8) holds. This means that for $\nu \in [\delta, \delta^{+L_{\bar{\mu}}})$ we have

$$\bar{p}_\delta(\nu) = 1 \iff \bar{p}_{<\delta}((\zeta)_1) \text{ for a tail of } \zeta \in (b^{\bar{p}_\delta \restriction \nu})^{\bar{X}}.$$

By (4.7) $b^{\bar{p}_\delta}$ is eventually disjoint from $\bar{p}_{<\delta}$. Moreover $p \restriction \delta \in \mathcal{A}_0^{\bar{p}_\delta}$ so the coding stops there. It remains to show that

$$\forall \nu \in [\delta, |\bar{p}_\delta|) \quad b^{\bar{p}\restriction\nu} \subseteq^* \mathrm{dom}(\bar{p}_{<\delta}). \tag{4.10}$$

Fixing ν as above, in the proof of lemma 4.30 we've seen $j(\delta) \in U^M(p^\xi, \{\bar{w} \restriction \xi + 1, \vec{x})$ and $j(\nu) \in H^M_{<\delta}(p^\xi, \{\bar{w} \restriction \xi + 1, \vec{x})$ for large enough $\xi < \rho$, as the second alternative of (4.5) obtains. So by (D 2g) (which we haven't used up to now) and (4.8) it follows that

$$b^{\bar{p}^\rho_{j(\delta)}\restriction j(\nu)} \cap [\eta, \delta) \subseteq \mathrm{dom}(p^\rho_{<\delta})$$

for some $\eta < \delta$. Now use elementarity and apply σ to get (4.10).

By (4.10) and (4.9), we have $\mathcal{A}_0^{\bar{p}_\delta} = \mathcal{B}(p^\rho_{<\delta})^{\emptyset_\delta}$, so (4.4) follows from (4.7) and we are done.

Now say $\delta \in X^\rho_\delta$. Since $\bar{p}_\delta = p^\rho_{\tilde{\delta}}$ and $\mathcal{A}_0^{p_\delta} \subseteq \mathcal{B}(p_{<\delta})^{p_\delta}$ if $\delta \in \mathbf{Sing}$, (4.4) immediately follows from (4.7). It remains to show (4.3) (definition 4.7).

We only need to check the requirement given in definition 4.7 for $\zeta = |p^\rho_\delta|$, for when $\zeta < |p^\rho_\delta|$, the requirement is met as $\zeta < |p^\xi_\delta|$ for some $\xi < \rho$ and $p^\xi_\delta \in S^*_\delta$. So let $N = L_\eta[A \cap \delta, p^\rho_\delta]$ be a test model, i.e. let $\eta, \bar{\kappa}$ be such that

$$N \vDash \mathrm{ZF}^- \text{and } |p^\rho_\delta| = \delta^+, \tag{4.11}$$

$$N \vDash \mathbf{Card} = \mathbf{Card}^{L[A\cap\omega]} \tag{4.12}$$

$$L_\eta \models \bar{\kappa} \text{ is the least Mahlo, } \bar{\kappa}^{++} \text{ exists.} \tag{4.13}$$

We must show that $L_\eta[\bar{A}^*] \models r$ is coded by branches, where we let $\bar{A}^* = A^*(A \cap \delta, p_\delta)^N$, i.e. the set obtained when running the decoding procedure (see 4.6) relative to N. By lemma 4.26 and 4.30 and as $\pi(p^\xi_\delta) = p^\xi_\delta$ for $\xi < \rho$ the sequence $(|p^\xi_\delta|)_{\xi<\rho}$ is definable over $L_{\bar{\mu}}[A \cap \delta, p^\rho_\delta]$. Thus (4.11) implies $\eta < \bar{\mu} = \bar{X} \cap \mathrm{On}$. Let $\alpha = \mathbf{Card}^{L_{\bar{\mu}}} \cap \eta$. Clearly $\eta \in [\alpha, \alpha^{+\bar{X}})$.

[18] At this point we need lemma 4.28 for the second alternative of (4.5).

Case 1: $\alpha \geq \bar{\kappa}^{+\bar{X}}$. Observe $\eta > \bar{\kappa}^{+N} = \bar{\kappa}^{+\bar{X}}$, where the first inequality holds by (4.13). Thus also $\bar{X} \vDash \bar{\kappa}$ is the least Mahlo and $j(\bar{\kappa}) = \kappa$. It also easily follows by elemenarity of $j\colon \bar{X} \to M$ that $\beta = \kappa$ or $\beta = \infty$: if $\beta < \kappa$, $M = \mathcal{A}^{q_{\beta+}} \vDash$ there is no Mahlo cardinal.

By lemma 4.30 and by (4.12), the decoding procedure of $\bar{p}_\delta$ over $L_\eta[A \cap \delta]$ run up to κ^{++N} yields $\bar{A}_0 \cap \bar{\kappa}^{++N}$, where of course $\bar{A}_0 = \bar{A} = \pi[A_0]$. By elementarity, $L_{\bar{\mu}}[\bar{A}_0] \models \Phi(r)$ and also $L_\eta[\bar{A}_0 \cap \bar{\kappa}^{++N}] \models \Phi(r)$ by the construction of $\bar{T}$.[19]

Case 2: Otherwise, assume $\alpha \leq \bar{\kappa}$ and if $\alpha = \bar{\kappa}$, then $\bar{\kappa}$ is Mahlo in $\bar{X}$.

Write $\bar{\zeta}$ for $\alpha^{+N} = \alpha^{+L_\eta}$. Observe that as in the previous case, by the proof of lemma 4.30 and as $\mathbf{Card}^N = \mathbf{Card}^{L_\eta}$, the decoding procedure of $\bar{p}_\delta \restriction \bar{\zeta}$ over $L_\eta[A \cap \delta]$ run up to α yields $\bar{A} \cap \bar{\zeta}$ and $\bar{p} \restriction \bar{\zeta}$. In the case $\alpha = \bar{\kappa}$, this uses the assumption that $\bar{\kappa}$ is Mahlo in $\bar{X}$ so that N contains enough of the coding apparatus of $\bar{X}$ to run the proof of lemma 4.30 inside N.

As $\bar{\zeta} < \alpha^{+\bar{X}} = \sup_{\xi<\rho} |p^\xi_\alpha|$, we can pick $\xi < \rho$ such that $\bar{\zeta} < |p^\xi_\alpha|$. By elementarity,

$$\tilde{N} = L_{j(\eta)}[A \cap j(\alpha), p^\xi_{j(\alpha)} \restriction j(\bar{\zeta})],$$

is a valid test-model for $\tilde{s} = p^\xi_{j(\alpha)} \restriction j(\bar{\zeta})$ and as $\tilde{s} \in S^*_{j(\alpha)}$, this string codes a predicate A^* over $L_{j(\eta)}[A \cap j(\alpha)]$ such that $L_{j(\eta)}[A^*] \models \Phi(r)$. By the way the coding works and using elementarity of π, $\bar{p}_\delta$ also codes $\bar{A}^* = \pi(A^*)$ and $L_\eta[\bar{A}^*] \models \Phi(r)$.

Case 3: It remains to deal with the case $\alpha = \bar{\kappa}$ when $\bar{\kappa}$ is not Mahlo in $\bar{X}$. Let's assume for simplicity that $\kappa \in X$, or equivalently, $\beta \geq \kappa$. The other case differs only in notation. Again using the proof of lemma 4.30 and as $\mathbf{Card}^N = \mathbf{Card}^{L_\eta[A\cap\omega]}$, the decoding procedure of p_δ over $L_\eta[A \cap \delta]$ run up to $\bar{\kappa}$ yields $\bar{p}_{<\pi(\kappa)} \restriction \bar{\kappa}$ (or if $\beta < \kappa$, the appropriate collapse of $p^0_{<\kappa} \restriction X$). As $\bar{\kappa} < \pi(\kappa)$, we can find $\xi < \rho$ such that $|p^\xi_{<\kappa}| > j(\bar{\kappa})$ (if $\beta < \kappa$, this holds for $\xi = 0$). Since N is a model of ZF$^-$, the construction can be carried out in N. By elementarity,

$$\tilde{N} = L_{j(\eta)}[A \cap j(\bar{\kappa}), p^\xi_{<\kappa} \restriction j(\bar{\kappa})],$$

is a valid test-model for $\tilde{s} = p^\xi_{<\kappa} \restriction j(\bar{\kappa})$ and as $\tilde{s} \in S^*_{<\kappa}$, this string codes a predicate A^* over $L_{j(\eta)}[A \cap j(\bar{\kappa})]$ such that $L_{j(\eta)}[A^*] \models \Phi(r)$. By the way

[19] In more detail: If If $\eta = \bar{\mu}$, this is obvious, so assume $\eta < \bar{\mu}$. Then $j(\bar{A}_0 \cap \bar{\kappa}^{++N}) = \bar{B}^- \restriction \tilde{\kappa}$, where $\tilde{\kappa} = j(\bar{\kappa}^{++N}) = \kappa^{++L_{j(\eta)}}$ and $L_{j(\eta)}$ is a model of ZF$^-$. Thus the trees defined in $L_{j(\eta)}$ are just initial segments of the real ones, and the respective components of $\bar{B} \restriction \tilde{\kappa}$ are branches through the local versions.

the coding works and using elementarity of π, $\bar{p}_\delta$ also codes $\bar{A}^* = \pi(A^*)$ and $L_\eta[\bar{A}^*] \models \Phi(r)$. Analogously for $\beta < \kappa$.

Finally, we prove $p^\rho_{<\kappa} \in S^*_{<\kappa}$, i.e. the requirement in definition 4.7 is met (to avoid trivialities, we assume $\lambda < \kappa$). So let N be a test-model and let $\bar{\kappa}$ be the least Mahlo in N. We assume $\bar{\kappa} = |p^\rho_{<\kappa}|$, as otherwise (as before) there is nothing to prove. Let

$$X = \bigcup_{\xi<\rho} H_{|p^\xi_{<\kappa}|},$$

let $\pi\colon X \to \bar{X}$ be the the collapsing map, $j = \pi^{-1}$, $\mu = X \cap \mathrm{On}$ and $\bar{\mu} = \bar{X} \cap \mathrm{On}$. By (D 2a), we have $X = H_{\bar{\kappa}}$, $\bar{\kappa} = |p^\rho_{<\kappa}|$ and $j(\bar{\kappa}) = \kappa$ (so $\mathrm{crit}(j) = \bar{\kappa}$).

As before, let $\eta = N \cap \mathrm{On}$. Since the sequence $\{|p^\xi_{<\kappa}|\}_{\xi<\rho}$ is definable over $\bar{X}$, we can argue as previously that $\eta < \mu$. Also, as before, $\bar{p}_{\bar{\kappa}} = \bigcup_{\xi<\rho} \pi(p^\rho_\kappa)$ is decoded from $p^\rho_{<\kappa}$ relative to $\bar{X}$, and $\bar{p}_{\bar{\kappa}} \restriction \bar{\kappa}^{+N}$ is decoded relative to N. Writing $\tilde{\eta} = j(\eta)$ it follows that the model $\tilde{N} = L_{\tilde{\eta}}[A \cap \kappa, p^\rho_\kappa \restriction j(\bar{\kappa}^{+N})]$ is a valid test-model for κ and so we finish the argument once more using that $p^\rho_\kappa \in S^*_\kappa$.

□

4.8 The strategic class D is dense

The next lemma shows density of **D**, i.e. (C I): in a sort of bootstrapping process using theorem 4.24 in its local version, we shall see that the auxiliary sets $\mathbf{D}^M_{[\lambda,\gamma)}(p, \vec{x}) \neq \emptyset$, for larger and larger γ. To be able to meet (D 2g), we have to strengthen our inductive hypothesis and assume that densely, any Easton set can be "covered" by the domain of a condition.

Lemma 4.33. *Let $\gamma \in$ **Card**. For any $q \in P$ and both $\beta \geq \gamma$ and M as in assumption 4.20, the following holds: For any $\lambda \in$ **Reg** and $p \in P$,*

1. *for any $\vec{x} \in M$, there is $p' \in P$ such that $p' \preccurlyeq^\lambda p$ and $p' \in \mathbf{D}^M_{[\lambda,\gamma)}(p, \vec{x})$*
2. *for any Easton set $B \subseteq [\lambda, \gamma)$ there is $p' \in P$ such that $p' \preccurlyeq^\lambda p$ and $B \subseteq \mathrm{dom}(p'_{<\gamma})$.*

In the proof, you will notice a somewhat unexpected role-reversal of γ and M: the induction is over γ while q, β and M vary freely. Also, note how $\mathbf{D}^M_{[\lambda,\gamma)}(p, \vec{x})$ has been decoupled from R; we talk about density in P and q is merely a parameter.

To make life a little easier, we could proove from (1) the seemingly stronger fact that for any $\bar\gamma \geq \gamma$ and $r \in P$ such that $r \leq p$, $\mathbf{D}^M_{[\lambda,\gamma)}(p,\vec{x})$ is $\preccurlyeq^\lambda$-dense in $P(r)^{\bar\gamma^+}$ (but we only use it in this proof, so we leave it implicit).

Corollary 4.34. *P is quasi-closed: Setting $\beta = \infty$, $M = L[A]$ and $\gamma = \kappa^{+3}$, it clearly follows that for any $p \in P$, $\lambda \in$ **Reg** and $\vec{x} \in L[A]$, there is $q \in \mathbf{D}(\lambda, p, \vec{x})$ such that $q \preccurlyeq^\lambda p$. This complements theorem 4.24 in its global form (corollary 4.25).*

Proof of the lemma. The proof is by induction on γ, so say both statements are true for all cardinals $\gamma' < \gamma \in$ **Card**. Fix q, M, $\lambda \in$ **Reg**, $\vec{x} \in M$ and an Easton set B as in the hypothesis, and let $p \in P$ be arbitrary. We shall find $p' \preccurlyeq^\lambda p$ satisfying both $p' \in \mathbf{D}^M_{[\lambda,\gamma)}(p,\vec{x})$ and $B \subseteq \operatorname{dom}(p'_{<\gamma})$.

For the succcessor case, assume $\gamma = \delta^+$, for $\delta \in$ **Card**. We need to take care of (D 1) and (D 2a) to (D 2g) (in definition 4.21, p. 46) for δ, and then we can finish this case quickly by induction. So let ι_0 be least above $|p_\delta| + \delta$ such that any pair $b, b' \in p^*_{\delta^+}$ is disjoint above ι_0 and let

$$\iota = \sup(H_\delta \cap \mathrm{On}) \cup \sup_{b \in p^*_{\delta^+}} (\min(b \setminus \iota_0))_1 \cup \sup B$$

Note that $\iota < \delta^+$ and let

$$p^1_\delta = (p^0_\delta \frown E) \cup Z$$

where $E \subseteq \delta$ codes the relation $\in \restriction \iota \times \iota$ in some recursive way and where

$$Z \colon [|p^0_\delta| + \delta, \iota) \to 2$$

$$Z(\zeta) = \begin{cases} 1 & \text{if } \zeta = (\langle \xi, |p^0_\delta| \rangle)_0 \text{ for some } \xi \in [\delta, |p^0_\delta|), \\ 1 & \text{if } \zeta = (\xi)_1 \text{ for some } \xi \in b \setminus \iota_0 \text{ and } b \in p^*_{\delta^+}, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, let $\eta = \rho(p^*_\delta) \cup \sup(U \cap \delta) \cup \|p^*_\delta\| \cup \lambda$ and let

$$(p^1)^*_\delta = (p^0)^*_\delta \cup \{ b^{p_\delta \restriction \nu} \setminus \eta \mid \nu \in H_{<\delta} \} \cup \lambda.$$

If $\delta \in$ **Sing**, we must find $p^1 \restriction \delta$ so that it exactly codes p^1_δ, using the extendibility lemma 4.14. Otherwise we can set $p^1 \restriction \delta = p^0 \restriction \delta$. It is clear that p^1 satisfies (D 2b) and also requirement (D 1) in definition of **D**. Also, by the choice of Z and since ι was chosen large enough, p^1 satisfies (D 2a), (D 2c) and (D 2d). It is easy to see that $p^1_\delta \in S^*_\delta$: by choice of E, p^1_δ collapses the size of ι to δ over every ZF^- model.[20] Thus any test model for p^1 is a test model for p^0 and we are done.

We must also arrange (D 2g) for δ. For this we made the inductive assumption that (2) holds below δ. Let $\gamma' = \sup(\mathrm{supp}(p) \cap \delta)$ and let

$$B' = \bigcup\{b \cap \gamma' \mid b \in p^*_\delta \setminus \mathrm{On}\}.$$

By (2) we can find $p^2 \preccurlyeq^\lambda p^1$ such that $B' \cup (B \cap \delta) \subseteq \mathrm{dom}(p^2_{<\delta})$.

By induction hypothesis, we can find $p' \preccurlyeq^\lambda p^2$ such that

$$p' \in \mathbf{D}^M_{[\lambda,\delta)}(p^2, \vec{x}) \subseteq \mathbf{D}^M_{[\lambda,\delta)}(p, \vec{x}).$$

It follows that $p' \in \mathbf{D}^M_{[\lambda,\gamma)}(p, \vec{x})$ and $B \subseteq \mathrm{dom}(p'_{<\gamma})$, finishing the successor case.

Let's say γ is a limit ordinal. We can assume $\gamma \in \mathbf{Sing}$, for otherwise $\mathrm{supp}(p) \cap \gamma$ and B are both bounded below γ and we can simply use the induction hypothesis. So let $(\beta'_\xi)_{\xi<\rho}$ be the increasing enumeration of a club in $\beta' = \gamma$, $\rho = \mathrm{cf}\,(\beta')$ and $\beta'_0 > \rho$.

We can assume without loss of generality that $|p_{<\kappa}| \geq \beta^+$ (by 4.12, extendibility for the Mahlo coding). Let $R' = P(p)^{\beta'^+}$, $M' = \mathcal{A}^{p_{\beta'^+}} = \mathcal{A}(A_p)^{p_{\beta'^+}}$ and let $p^0 = p \restriction \beta'^+ + 1$, observing $p^0 \in R'$. For $\xi < \rho$, let

$$D_\xi = \mathbf{D}^M_{[\lambda,\beta'_\xi)}(p, \vec{x}) \cap L_{\beta'^+_\xi}[A]$$

and let $\vec{x}' = \{\lambda, p^0, (\beta'_\xi)_{\xi<\rho)}, (D_\xi)_{\xi<\rho}, B\}$, noting that $\vec{x}' \in M'$. The point here is that for any $p' \in P$,

$$p' \in \mathbf{D}^M_{[\lambda,\beta'_\xi)}(p, \vec{x}) \iff p' \restriction \beta'_\xi \in D_\xi,$$

so that we can talk about $\mathbf{D}^M_{[\lambda,\beta'_\xi)}(p, \vec{x})$ inside M'.

We shall now use theorem 4.24 for M' and R' to construct a sequence whose greatest lower bound will be the desired condition. Let X^0 be least such that $X^0 \prec_1 M'$, $x' \in X^0$ and $X^0 \in M'$; we can find such X^0 by the fact that $\mathrm{cf}\,(M' \cap \mathrm{On}) = \beta'^+$.

We now find a sequence $\bar{w} = (w^\xi)_{\xi<\rho}$ such that for each $\xi < \rho$, $w^\xi = (X^\xi, p^\xi, \bar{w} \restriction \xi)$ and

$$w^\xi \in M'$$
$$X_\xi = h^{M'}_{\Sigma_1}(\{p^\xi, \bar{w} \restriction \xi\}),$$

[20] Of course, much less than ZF^- is needed here.

and if ξ is limit, then p^ξ is a greatest lower bound of $\bar{p} \restriction \xi + 1$ in R'. Note that this implies $(X^\nu)_{\nu<\xi} \in X_\xi$ (also note the sequence $(X_\xi)_{\xi<\rho}$ is not continuos). To complete the definition of $\bar{w}$, we must specify how to construct $p^{\xi+1}$ in the successor case.

By induction, assume we have built such a sequence $\bar{w} \restriction \xi + 1$. Now let $p^{\xi+1} \in R'$ be least such that

- $p^{\xi+1} \restriction \beta'_\xi \in D_\xi$—equivalently, $p^{\xi+1} \in \mathbf{D}^M_{[\lambda,\beta'_\xi)}(p, \vec{x})$,
- $p^{\xi+1} \in \mathbf{D}^{M'}_{[\lambda,\beta'_\xi)}(p^\xi, \{\vec{x}', \bar{w}\xi + 1\})$ (note this is $\Pi^T_1(\{\vec{x}', \bar{w}\xi + 1, p^\xi, \xi\})$ in M')
- $B \cap \beta'_\xi \subseteq \mathrm{dom}(p^{\xi+1})_{<\beta'^+_\xi}$,
- the clauses in (C) of theorem 4.24 hold for $p^{\xi+1}$ with β, β_ξ, M, $\vec{x}$ in (C) replaced by β', β'_ξ, M', and $\vec{x}'$ (note this is $\Pi^T_1(\{\vec{x}', \bar{w}\xi + 1, p^\xi, \xi\})$ in M' as well).

The last point can be arranged as in the first part of the argument in the successor case (using extendibility, 4.14). The first point can be arranged as the induction hypothesis implies that D_ξ is dense in R'.[21] If in doubt, here is a pedestrian proof: since $p^\xi \in R' = P(p)^{\beta'^+}$, we know $p^\xi \cdot p \neq 0$; so there is $p' \preccurlyeq^\lambda p^\xi \cdot p$ such that $p' \in \mathbf{D}^M_{[\lambda,\beta'_\xi)}(p^\xi \cdot p, \vec{x})$; But then since trivially $\mathbf{D}^M_{[\lambda,\beta'_\xi)}(p^\xi \cdot p, \vec{x}) \subseteq \mathbf{D}^M_{[\lambda,\beta'_\xi)}(p, \vec{x})$,

$$p' \restriction \beta'^+_\xi \cup p^\xi \restriction [\beta'^+_\xi, \beta') \in \mathbf{D}^M_{[\lambda,\beta'_\xi)}(p, \vec{x}) \cap R'.$$

Similar arguments work for the remaining two points. Note that the conjunction of theses items can be expressed by a $\Pi^T_1(\vec{x}' \cup \{p^\xi, \xi\})$ formula (say $\Phi(p^{\xi+1}, \vec{x}', \bar{w} \restriction \xi + 1)$) inside M'.

The sequence $\bar{w}$ is well defined since by the following fact and theorem 4.24, the sequence $\bar{p} \restriction \rho'$ has a greatest lower bound $p^{\rho'}$ when $\rho' \leq \rho$ is a limit ordinal.

Finally, we have that $p^\rho \in \bigcap_{\xi<\rho} D_\xi = \mathbf{D}^M_{[\lambda,\gamma)}(p, \vec{x})$ and $B \subseteq \mathrm{dom}(p^\rho_{<\gamma})$. Since also $p^\rho \in R' = P(p)^{\gamma^+}$, $p' = (p^\rho \cdot p)$ is the desired condition. It remains to proove:

Fact 4.35. The sequence $\bar{w} = (w^\xi)_{\xi<\rho'} = (X^\xi, p^\xi)_{\xi<\rho'}$ is a (λ, x')-canonical witness for $\bar{p} \restriction \rho' = (p^\xi)_{\xi<\rho'}$ inside M'

[21]Literally, the induction hypothesis talks about P, not R'. In fact, we have that $\mathbf{D}^M_{[\lambda,\gamma)}(p, \vec{x})$ is dense in P below p if and only if it has non-empty intersection with $P(q)^{\bar{\gamma}^+}$ for any $q \leq p$ and any $\bar{\gamma} \geq \gamma$.

Sketch of proof. The idea is that w^ξ is the unique $x = (X, p, \bar{w}')$ such that

$$X = h^{M'}_{\Sigma_1}(\{p, \bar{w}^*\}) \text{ and}$$
$$X \models p \text{ is } \leq_{M'}\text{-least such that } \Phi(p, \vec{x}', \bar{w}) \tag{4.14}$$

where Φ comes from the inductive definition of $\bar{p}$ discussed above, and $\bar{w}^* = \bar{w} \upharpoonright \xi$. We leave it to the reader to check that (4.14) can be expressed by a $\Pi^T_1(\vec{x}')$-formula $\Psi(X, p, \bar{w}')$.

Now it is easy to see the fact holds: $x = w^\xi$ if and only if $x = (X, p, \bar{w}^*)$, $\Psi(X, p, \bar{w}^*)$ holds and $\bar{w}^*$ is a sequence of length ξ such that for each $\nu < \xi$, we have $\Psi(\bar{w}^*(\nu)_0, \bar{w}^*(\nu)_1, \bar{w}^* \upharpoonright \nu)$. This is obviously $\Pi^T_1(\vec{x}')$ if Ψ is. See fact 3.15, especially lemma 3.14 for a similar (but harder) argument. □

□

4.9 Stratification of $P(A_0)$

We now define the stratification system for $P(A_0)$. Remember $\preccurlyeq^\lambda$ and $\mathbf{D}$ was defined in 4.21.

Definition 4.36. 1. Let $q \preccurlyeq^\lambda p$ just if $q \upharpoonright [\lambda, \infty) \leq p \upharpoonright [\lambda, \infty)$ in $P(A_0)$.

2. For any $p \in P(A_0)$, writing ρ for $\rho(p^*_\lambda) \cup \sup\operatorname{dom}(p_{<\lambda})$, let $\mathbf{C}^\lambda(p) = \{(p_{<\lambda}, \rho, B^\rho_p)\}$.

We remind the reader that we take the singleton on the right-hand side in the above equation only to satisfy the abstract definition of stratification, which allows for a "multifunction" $\mathbf{C}^\lambda$, as this is necessary for iterations.

A more straightforward definition would be $\mathbf{C}^\lambda(p) = \{p_{<\lambda}\}$ for all $p \in P(A_0)$. This would work with the more traditional definition of $\leq$ in [Fri00].

The problem is that we had to introduce $\rho(p^*_\lambda)$ for inaccessible λ. Recall that $\leq$ was defined so that "new restraints" (those in q^*_λ for $q \leq p$) may only differ from old ones (those already in p^*_λ) *above* $\rho(p^*_\lambda)$. This means that with the more straightforward definition, in (S III), when $\mathbf{C}^\lambda(p) \cap \mathbf{C}^\lambda(q) \neq \emptyset$ and $p \preccurlyeq^\lambda q$, and we want to form $q \cdot p$, the natural candidate (the point-wise union) might not actually lie below p and q: not if any "new" restraints disagree with old restraints below $\rho(p^*_\lambda)$ (or $\rho(p^*_\lambda)$, respectively). Thus, it is natural to make $\rho(p^*_\lambda)$ and the restraints below it part of $\mathbf{C}^\lambda$.

Having checked quasi-closure, we invite the reader to check the rest of stratification, all of which is outright trivial.

Chapter 5

Extension and iteration

The proof of the main result makes it necessary to consider iterations $\bar{Q}^\theta$ such that each initial segment P_ι is stratified on $[\lambda_\iota, \kappa]$, but it is not forced that $\dot{Q}_\iota$ be stratified for all $\iota < \theta$—amalgamation is one example; also, I don't see a proof that $P(A)$ "codes no unwanted branches" if we don't use forcings from an inner model, and these are not quasi-closed in the model where we force with them. We deal with this difficulty by introducing the concept of $(P_\iota, P_{\iota+1})$ being a *stratified extension*. With diagonal support, this ensures that the initial segments are sufficiently coherent so that we can conclude that P_θ is stratified. This coherency provided by extension is vital: e.g. an iteration whose proper initial segments are all σ-strategically closed can add a real (see [KS10]).

To further complicate things, λ_ι is not the same fixed cardinal throughout the iteration.

We treat quasi-closed and stratified extension separately (sections 5.1 and 5.2). Each axiom of stratified (or quasi-closed) extension corresponds to an axiom of stratification (or quasi-closure)—in fact, interestingly, P is stratified if and only if $(\{1_P\}, P)$ is a stratified extension. To prove the iteration theorem, we also have to add some additional axioms concerning the interplay of the pre-stratification (pre-closure) systems on P_ι and $P_{\iota+1}$; see definitions 5.1 and 5.18.

In section 5.3 we show products of stratified forcings are stratified extensions. Finally, we introduce *the stable meet operator* in section 5.4 and *remote sub-orders* in section 5.5. See the beginning of those sections for a motivating discussion.

5.1 Quasi-closed extension and iteration

In this section, we show that composition of quasi-closed forcing is a special case of quasi-closed extension. We give a sufficient condition which makes sure that if (P_0, P_1) is a quasi-closed extension, then P_1 is quasi-closed. We prove that the relation of being a quasi-closed extension is transitive. Finally, we formulate and prove an iteration theorem for quasi-closed forcing.

Let P_0 be a complete sub-order of P_1 and let $\pi\colon P_1 \to P_0$ be a strong projection. Moreover, assume we have a system $\mathbf{s}_i = (\mathbf{D}_i, c_i, \preccurlyeq_i^\lambda)_{\lambda\in I}$ for $i \in \{0,1\}$ such that $\mathbf{D}_i \subseteq I \times V \times (P_i)^2$ is a class definable with parameter c_i and for every $\lambda \in I$, $\preccurlyeq_i^\lambda$ is a binary relation on P_i.

Definition 5.1. We write $\mathbf{s}_0 \lhd \mathbf{s}_1$ to mean for every x and every $\lambda \in I$,

($\lhd_\mathrm{c}1$) For all $p, q \in P_0$, $p \preccurlyeq_0^\lambda q \Rightarrow p \preccurlyeq_1^\lambda q$.

($\lhd_\mathrm{c}2$) For all $p, q \in P_1$, $p \preccurlyeq_1^\lambda q \Rightarrow \pi(p) \preccurlyeq_0^\lambda \pi(q)$.

($\lhd_\mathrm{c}3$) For all $p, q \in P_1$, $p \in \mathbf{D}_1(\lambda, x, q) \Rightarrow \pi(p) \in \mathbf{D}_0(\lambda, x, \pi(q))$.

Observe that if $\mathbf{s}_0 \lhd \mathbf{s}_1$ we can drop the subscripts on $\preccurlyeq_0^\lambda$, $\preccurlyeq_1^\lambda$ and just write $\preccurlyeq^\lambda$ without causing confusion.

Definition 5.2. We say the pair (P_0, P_1) is a *quasi-closed extension on I, as witnessed by* $(\mathbf{s}_0, \mathbf{s}_1)$ if and only if $\mathbf{s}_0$ witnesses that P_0 is quasi-closed on I, $\mathbf{s}_1$ is a pre-closure system on P_1, $\mathbf{s}_0 \lhd \mathbf{s}_1$ and for $\lambda, \bar\lambda \in I$ such that $\lambda \leq \bar\lambda$, the following conditions hold:

($\mathrm{E_c I}$) If $p \in P_1$ and $q \in P_0$ is such that $q \preccurlyeq^\lambda \pi(p)$ and $q \in \mathbf{D}(\lambda, x, \pi(p))$, there is $r \preccurlyeq^\lambda p$ such that $r \in \mathbf{D}(\lambda, x, p)$, $r \preccurlyeq^\lambda p$ and $\pi(r) = q$. Moreover we can ask of r that for any $\lambda' \in I$ such that $p \preccurlyeq^{\lambda'} \pi(p)$ we also have $r \preccurlyeq^{\lambda'} \pi(r)$.

($\mathrm{E_c II}$) If $\bar p = (p_\xi)_{\xi<\rho}$ is a sequence of conditions in P_1 such that for some $q \in P_0$ and some $\bar w$

(a) q is a greatest lower bound of the sequence $(\pi(p_\xi))_{\xi<\rho}$ and for all $\xi < \rho$, $q \preccurlyeq^\lambda \pi(p_\xi)$,

(b) $\bar w$ is a (λ, x)-strategic guide and a $(\bar\lambda, x)$-canonical witness for $\bar p$,

(c) either $\lambda = \bar\lambda$ or $p_\xi \preccurlyeq_1^{\bar\lambda} \pi(p_\xi)$ for each $\xi < \rho$,

then $\bar p$ has a greatest lower bound p in P_1 such that for each $\xi < \rho$, $p \preccurlyeq^\lambda p_\xi$ and $\pi(p) = q$. Moreover, if $p_\xi \preccurlyeq_1^{\bar\lambda} \pi(p_\xi)$ for each $\xi < \rho$, then also $p \preccurlyeq_1^{\bar\lambda} \pi(p)$.

As before, if we say (P_0, P_1) is a quasi-closed extension and don't mention either of $\mathbf{s}_0$, $\mathbf{s}_1$ or I, that entity is either clear from the context or we are claiming that one can find such an entity.

We will grow tired of repeating all the conditions $\bar{p}$ has to satisfy in ($\mathrm{E_c}$II), so we issue the following definition:

Definition 5.3. We say $\bar{p}$ is $(\lambda, \bar{\lambda}, x)$-adequate if and only if $\lambda, \bar{\lambda} \in I$, $\lambda \leq \bar{\lambda}$ and $\bar{p}$ satisfies conditions ($\mathrm{E_c}$IIb) and ($\mathrm{E_c}$IIc) above. We say q is a π-bound if and only if ($\mathrm{E_c}$IIa) holds.

Of course, the obvious example for quasi-closed extension is provided by composition of forcing notions:

Lemma 5.4. *If P is quasi-closed on I and $\Vdash_P \dot{Q}$ is quasi-closed on I, then $(P, P * \dot{Q})$ is a quasi-closed extension on I.*

To be more precise, let $\mathbf{s}_0$ denote the pre-quasi-closure system witnessing that P is quasi-closed and let $\mathbf{s}_1 = (\bar{\mathbf{D}}, \bar{c}, \bar{\preccurlyeq}^{\lambda})_{\lambda \in I}$ be the pre-quasi-closure system constructed as in the proof of 3.17, where we showed that $P * \dot{Q}$ is stratified. Then $(\mathbf{s}_0, \mathbf{s}_1)$ witnesses that $(P, P * \dot{Q})$ is a quasi-closed extension on I.

We give the proof after we prove the following simple lemma, which will be useful in several contexts.

Lemma 5.5. *Say R carries a pre-closure system $\mathbf{s}$ on I and $\bar{p} = (p_\xi)_{\xi < \rho}$ has a (λ, x)-strategic guide $\bar{w}$ which is also a $(\bar{\lambda}, x)$-canonical witness. If for all $\xi < \rho$, $p_\xi \preccurlyeq^{\bar{\lambda}} 1_R$, then $\bar{p}$ is in fact $(\bar{\lambda}, x)$-adequate.*

Proof of lemma 5.5. For arbitrary $\xi < \bar{\xi} < \rho$, by 3.1(C 3), as $p_{\bar{\xi}} \leq p_\xi \leq 1_R$ and $p_{\bar{\xi}} \preccurlyeq^{\bar{\lambda}} 1_R$, we have $p_{\bar{\xi}} \preccurlyeq^{\bar{\lambda}} p_\xi$. Thus $\bar{w}$ is in fact a $(\bar{\lambda}, x)$-strategic guide for $\bar{p}$ and so $\bar{p}$ is $(\bar{\lambda}, x)$-adequate. □

Proof of lemma 5.4. Just by looking at the definition of $\bar{\preccurlyeq}^{\lambda}$ and $\bar{\mathbf{F}}$, it is immediate that $\mathbf{s}_0 \lhd \mathbf{s}_1$ and that $\mathbf{s}_1$ is a pre-closure system. To check that $\mathbf{s}_1$ is a pre-closure system, observe 3.1(C 3) has already been checked in the proof of theorem 3.17. The other conditions we leave to the reader.

Condition 5.2($\mathrm{E_c}$I) holds since P forces 3.4(C I) for $\dot{Q}$: say we have $(p, \dot{p}) \in P * \dot{Q}$ and $q \in P$ such that $q \in \mathbf{D}(\lambda, x, p)$. By fullness we may find $\dot{q}$ such that $p \Vdash \dot{q} \in \mathbf{D}(\lambda, x, \dot{q})$ and $p \Vdash \dot{q} \preccurlyeq^{\lambda} \dot{p}$. We have $(q, \dot{q}) \preccurlyeq^{\lambda} (p, \dot{p})$ and $(q, \dot{q}) \in \bar{\mathbf{D}}(\lambda, x, (p, \dot{p}))$. Moreover, by the last clause of 3.4(C I), we can demand that $p \Vdash$ for any $\lambda' \in I$, $\dot{p} \preccurlyeq^{\lambda'} 1_{\dot{Q}} \Rightarrow \dot{q} \preccurlyeq^{\lambda'} 1_{\dot{Q}}$. Thus, for any $\lambda' \in I$ such that $(p, \dot{p}) \bar{\preccurlyeq}^{\lambda'} (p, 1_{\dot{Q}})$ we also have $(q, \dot{q}) \bar{\preccurlyeq}^{\lambda'} (q, 1_{\dot{Q}})$, which proves the last statement of 5.2($\mathrm{E_c}$I).

Now to the main point, that is 5.2($\mathrm{E_c}$II): Say $\bar{p} = (p_\xi, \dot{p}_\xi)_{\xi<\rho}$ is sequence of conditions in P_1 which is $(\lambda, \bar{\lambda}, x)$-adequate. That is, we may fix $\bar{w}$ which is a (λ, x)-strategic guide and a $(\bar{\lambda}, x)$-canonical witness. We may also fix $q \in P_0$ which is a π-bound—i.e. ($\mathrm{E_c}$IIa) holds.

We have already seen that q forces that $\bar{w}$ is a $(\bar{\lambda}, x)$-canonical witness for $(\dot{p}_\xi)_{\xi<\rho}$ (this is the same argument as in the proof of theorem 3.17).

It is easy to check that q also forces that $\bar{w}$ is a (λ, x)-strategic guide for $(\dot{p}_\xi)_{\xi<\rho}$.

If $\lambda = \bar{\lambda}$, we conclude that q forces that $(\dot{p}_\xi)_{\xi<\rho}$ is λ-adequate in $\dot{Q}$. Thus we may pick a P-name $\dot{q}$ such that q forces $\dot{q} \in \dot{Q}$ is the greatest lower bound of $(\dot{p}_\xi)_{\xi<\rho}$ in $\dot{Q}$. Then $(q, \dot{q})$ is the greatest lower bound of $\bar{p}$ and we are done with the proof of 5.2($\mathrm{E_c}$II) in this case.

If on the other hand, $\lambda < \bar{\lambda}$, we have that for all $\xi < \rho$, $(p_\xi, \dot{p}_\xi) \preccurlyeq_1^{\bar{\lambda}} (p_\xi, 1_{\dot{Q}})$. Thus by lemma 5.5 we have that q forces $(\dot{p}_\xi)_{\xi<\rho}$ is $\bar{\lambda}$-adequate. Thus q also forces that this sequence has a lower bound, for which we may fix a name $\dot{q}$. By quasi-closure for $\dot{Q}$ in the extension and since for all $\xi < \rho$ we have

$$q \Vdash \dot{p}_\xi \dot{\preccurlyeq}^{\bar{\lambda}} 1_{\dot{Q}},$$

we conclude that for any $\xi < \rho$ we have

$$q \Vdash \dot{q} \dot{\preccurlyeq}^{\bar{\lambda}} \dot{q}_\xi.$$

Thus $(q, \dot{q})$ is a greatest lower bound of $\bar{p}$ and

$$(q, \dot{q}) \bar{\preccurlyeq}^{\bar{\lambda}} (q, 1_{\dot{Q}}).$$

□

We now embark on a series of lemmas culminating in the insight that the second forcing of a quasi-closed extension (P_0, P_1) is itself quasi-closed. Thus, we obtain a second proof that $P * \dot{Q}$ is quasi-closed (under the assumptions of the previous lemma). This makes use of the fact that the projection map $\pi_0 \colon P * \dot{Q} \to P$ is definable. In general, we shall see that we have to assume that the strong projection map from P_1 to P_0 is sufficiently definable.

Lemma 5.6. *Assume for $i \in \{0, 1\}$, P_i carries a pre-closure system $\mathbf{s}_i$ on I and $\mathbf{s}_0 \lhd \mathbf{s}_1$. If $\bar{p} = (p_\xi)_{\xi<\rho}$ is a sequence of conditions in P_1 and $\bar{w}$ is a (λ, x)-strategic guide with respect to $\mathbf{s}_1$, then $\bar{w}$ is also a (λ, x)-strategic guide for $(\pi(p_\xi))_{\xi<\rho}$ with respect to $\mathbf{s}_0$.*

Proof. Suppose we are given $\bar{p}$ and $\bar{w}$ as in the hypothesis. If $\xi < \bar{\xi} < \rho$, since $p_\xi \preccurlyeq_1^\lambda p_{\bar{\xi}}$, by 5.1($\lhd_c$2), $\pi(p_\xi) \preccurlyeq_0^\lambda \pi(p_{\bar{\xi}})$. Let $\xi < \rho$ be arbitrary. Fix a regular

λ' such that $p_{\xi+1} \preccurlyeq_1^{\lambda'} p_\xi$ and $p_{\xi+1} \in \mathbf{D}_1(\lambda', (x, \bar{w} \restriction \xi + 1), p_\xi)$ By 5.1($\lhd_c$2), $\pi(p_{\xi+1}) \preccurlyeq_0^{\lambda'} \pi(p_\xi)$ and by 5.1($\lhd_c$3), $\pi(p_{\xi+1}) \in \mathbf{D}_1(\lambda', (x, \bar{w} \restriction \xi + 1), , \pi(p_\xi))$, finishing the proof. □

Lemma 5.7. *Assume $(P_i, \mathbf{s}_i)$, $i \in \{0, 1\}$ are as in lemma 5.6. Further, assume that the strong projection map $\pi\colon P_1 \to P_0$ is $\Sigma_1^T(\lambda \cup \{x\})$. If $\bar{p} = (p_\xi)_{\xi<\rho}$ is a sequence of conditions in P_1 which is (λ, x)-adequate with respect to $\mathbf{s}_1$, then $(\pi(p_\xi))_{\xi<\rho}$ is (λ, x)-adequate with respect to $\mathbf{s}_0$.*

Proof. Fix $\bar{w}$ which is both a (λ, x)-strategic guide and a (λ, x)-canonical witness for $\bar{p}$. By the previous lemma, $\bar{w}$ is a (λ, x)-strategic guide for $(\pi(p_\xi))_{\xi<\rho}$. We may find a $\Sigma_1^T(\lambda \cup \{x\})$ function G such that $p_\xi = G(\bar{w} \restriction \xi + 1)$. As $\pi \circ G$ is also $\Sigma_1^T(\lambda \cup \{x\})$, $\bar{w}$ is also a (λ, x)-canonical witness for $(\pi(p_\xi))_{\xi<\rho}$. □

The following is useful e.g. when we show a condition has legal support. Here lies one of the reasons for asking (C 3).

Lemma 5.8. *Assume $(P_i, \mathbf{s}_i)$, for $i \in \{0, 1\}$ are as in lemma 5.6. For any $p \in P_1$ and any regular $\lambda \in I$ we have:*

$$(\exists q \in P_0 \quad p \preccurlyeq_1^\lambda q) \iff p \preccurlyeq_1^\lambda \pi(p) \tag{5.1}$$

Proof. One direction is clear, so say $p \preccurlyeq_1^\lambda q$ for some $q \in P_0$. Apply 3.1(C 3): As π is a strong projection, $p \leq \pi(p) \leq q$ and so $p \preccurlyeq_1^\lambda \pi(p)$. □

The intuition behind definition 5.2 is that P_0 and P_1 are both quasi-closed, not independently of each other, but in a very coherent way. That P_1 is quasi-closed is almost implicit in definition 5.2—it depends on a further assumption about the definability of π (this is responsible for the distinct flavor of quasi-closure, setting it apart from the other axioms of stratification):

Lemma 5.9. *If (P_0, P_1) is a quasi-closed extension on I and π is $\Sigma_1^T(\min I \cup \{c_1\})$, then P_1 is quasi-closed on I.*

Before we give the proof, note that this assumption on π is not entirely trivial: in an iteration, the canonical projection $\pi\colon P_\theta \to P_\iota$ is Δ_0 in the parameter ι; it is not in general $\Sigma_1^T(\min I)$. Also we would like to note in passing that in fact P is quasi-closed exactly if $(\{1_P\}, P)$ is a quasi-closed extension; the same will be true for stratified forcing.

Proof. First check 3.4(C I): Say $p \in P_1$, $\lambda \in I$ and x are given. Use (C I) for P_0 to find $q \in P_0$ such that $q \preccurlyeq^\lambda \pi(p)$ and $q \in \mathbf{D}_0(\lambda, x, \pi(p))$. Now apply ($\mathrm{E_c}$I) to get $p' \preccurlyeq^\lambda p$ such that $p' \in \mathbf{D}_0(\lambda, x, p)$ and $\pi(p') = q$.

For the last clause of 3.4(C I), we can assume that q has been chosen so that for any $\lambda' \in I$, if $\pi(p) \preccurlyeq_0^{\lambda'} 1_{P_0}$, then $q \preccurlyeq_0^{\lambda'} 1_{P_0}$. We can also assume that

p' has been chosen so that for any $\lambda' \in I$, if $p \preccurlyeq_1^{\lambda'} \pi(p)$, then $p' \preccurlyeq_1^{\lambda'} \pi(p')$. Thus, $p' \preccurlyeq_1^{\lambda'} \pi(p) \preccurlyeq_1^{\lambda'} 1_{P_1}$.

It remains to check 3.4(C II), so say $\bar{w}$ witnesses that $\bar{p} = (p_\xi)_{\xi<\rho}$ is (λ, x)-adequate for P_1. Observe we assume that x is a tuple with c_1 among its components. By assumption, π is $\Pi_1^T(\lambda \cup \{x\})$, so by lemma 5.7, $(\pi(p_\xi))_{\xi<\rho}$ is also (λ, x)-adequate. Since P_0 is quasi-closed, $(\pi(p_\xi))_{\xi<\rho}$ has a greatest lower bound q. Thus, applying 3.4(C II) for $\bar{\lambda} = \lambda$, we conclude that $\bar{p}$ has a greatest lower bound. □

The next lemma will be used in 5.12 when we show that if the initial segments of an iterations form a chain of quasi-closed extensions, then the limit is itself a quasi-closed extension. It says that the relation of being a quasi-closed extension is transitive. Let P_0, P_1 and P_2 be pre-orders such that for $i \in \{0, 1\}$, P_i is a strong sub-order of P_{i+1}.

Lemma 5.10. *Say $\pi_1 \colon P_2 \to P_1$ and $\pi_0 \colon P_2 \to P_0$ are strong projection maps and π_1 is $\Sigma_1^T(\min I \cup \{c_2\})$. If both (P_0, P_1) and (P_1, P_2) are quasi-closed extensions on I, then (P_0, P_2) is also a quasi-closed extension on I.*

Proof. Let $(\mathbf{s}_0, \mathbf{s}_1)$ and $(\mathbf{s}_1, \mathbf{s}_2)$ witness that (P_0, P_1) and (P_1, P_2) are quasi-closed extensions.

We now check all the conditions of 5.2 for (P_0, P_2) and $(\mathbf{s}_0, \mathbf{s}_2)$. That $\mathbf{s}_2$ is a pre-closure system holds by assumption, and that $\mathbf{s}_0 \lhd \mathbf{s}_2$ is obvious.

Observe that by 5.1($\lhd_c$1), we don't need to distinguish between $\preccurlyeq_0^\lambda$, $\preccurlyeq_1^\lambda$ and $\preccurlyeq_2^\lambda$ and therefore we drop the subscripts in what follows.

We check 5.2($\mathrm{E_c}$I): Say $p \in P_2$ and $q \in \mathbf{D}_0(\lambda, x, \pi_0(p))$. As in the previous proof, we can find $p' \in P_1$ such that $\pi_0(p') = q$, $p' \preccurlyeq_1^\lambda \pi_0(p)$ and $p' \in \mathbf{D}_1(\lambda, x, \pi_1(p))$ and then r such that $r \in \mathbf{D}_2(\lambda, x, p)$, with $\pi_1(r) = p'$ and $r \preccurlyeq_2^\lambda p$.

Now if $\lambda' \in I$ and $p \preccurlyeq^{\lambda'} \pi_0(p)$, we also have $p \preccurlyeq^{\lambda'} \pi_1(p)$ by lemma 5.8. This means we have both $p' \preccurlyeq^{\lambda'} \pi_0(p')$ and $r \preccurlyeq^{\lambda'} \pi_1(r)$. As $\pi_1(r) = p'$, it follows that $r \preccurlyeq^{\lambda'} \pi_0(r)$.

It remains to check 5.2($\mathrm{E_c}$II). So let $\bar{p} = (p_\xi)_{\xi<\rho}$ be a $(\lambda, \bar{\lambda}, x)$-adequate sequence of conditions in P_2 and let q_0 be a greatest lower bound of $(\pi_0(p_\xi))_{\xi<\rho}$ as in the hypothesis. Fix $\bar{w}$ which is a (λ, x)-strategic guide and a $(\bar{\lambda}, x)$-canonical witness for $\bar{p}$. Show exactly as in the proof of lemma 5.9 that $\bar{w}$ is both a $(\bar{\lambda}, x)$-canonical witness and a (λ, x)-strategic guide for $(\pi_1(p_\xi))_{\xi<\rho}$ in P_1. Denote this sequence by $\bar{q}$. Moreover, if it is the case that $\bar{\lambda} > \lambda$, then

$$\forall \xi < \rho \quad p_\xi \preccurlyeq^{\bar{\lambda}} \pi_0(p_\xi). \tag{5.2}$$

By 3.1($\lhd_c$2), we have that

$$\forall \xi < \rho \quad \pi_1(p_\xi) \preccurlyeq^{\bar{\lambda}} \pi_0(p_\xi). \tag{5.3}$$

Thus $\bar{q}$ satisfies the hypothesis of 5.2($\mathrm{E_c}$II) for (P_0, P_1) and we may find a greatest lower bound $q_1 \in P_1$ with $\pi_0(q_1) = q$. Now use 5.2($\mathrm{E_c}$II) for (P_1, P_2) to find a greatest lower bound q of $\bar{p}$ such that $\pi_1(q) = q_1$ and so $\pi_0(q) = q_0$.

If $\lambda < \bar{\lambda}$, lemma 5.8 and (5.2) yield

$$\forall \xi < \rho \quad p_\xi \preccurlyeq^{\bar{\lambda}} \pi_1(p_\xi). \tag{5.4}$$

So finally, as $q \preccurlyeq^{\lambda} \pi_1(q)$ by (5.4), and $\pi_1(q) = q_1 \preccurlyeq^{\bar{\lambda}} \pi_0(q_1)$ by (5.3), we conclude $q \preccurlyeq^{\bar{\lambda}} \pi_0(q)$. □

Definition 5.11. Say θ is a limit ordinal, and $\bar{Q}^\theta$ is an iteration such that for each $\iota < \theta$, P_ι carries a pre-closure system $\mathbf{s}_\iota$ on $\mathbf{Reg} \cap [\lambda_\iota, \lambda^*)$, where the sequence $\bar{\lambda} = (\lambda_\iota)_{\iota<\theta}$ is a non-decreasing sequence of regulars. All of the following definitions are relative to these pre-closure systems and to $\bar{\lambda}$.

1. For a thread p through $\bar{Q}^\theta$, let

$$\operatorname{supp}^\lambda(p) = \{\iota < \theta \mid \lambda_\iota \leq \lambda \text{ and } \pi_{\iota+1} \not\preccurlyeq^{\lambda}_{\iota+1} \pi_\iota(p)\},$$

   and let $\sigma^\lambda(p)$ be the least ordinal σ such that $\operatorname{supp}^\lambda(p) \subseteq \sigma$.

2. Let λ^* be regular such that $\lambda^* \geq \lambda_\iota$ for all $\iota < \theta$. We say P_θ is the *λ^*-diagonal support limit of $\bar{Q}^\theta$* if and only if P_θ consists of all threads p through $\bar{Q}^\theta$ such that for each regular $\lambda \geq \lambda^*$, $\operatorname{supp}^\lambda(p)$ has size less than λ and $\sigma^{\lambda^*}(p) < \theta$.

3. We define the *natural system of relations on P_θ* as follows
   (a) $p \preccurlyeq^\lambda_\theta q \iff \forall \iota < \theta \quad \pi_\iota(p) \preccurlyeq^\lambda_\iota \pi_\iota(q)$;
   (b) $p \in \mathbf{D}(\lambda, x, q) \iff \forall \iota < \theta \pi_\iota(q) \in \mathbf{D}(\lambda, x, \pi_\iota(q)$.
   (c) The parameter c_θ has as the first of its components a sequence $\bar{c} = (c_\iota)_{\iota<\theta}$, where c_ι is the parameter from $\mathbf{s}_\iota$.

   We shall see in the proof of theorem 5.12 that under natural assumptions the natural system of relations is a pre-closure system.

4. We say $\bar{Q}^\theta$ is a *$\bar{\lambda}$-diagonal support iteration* if and only if for any limit $\iota < \theta$, P_ι is the λ_ι-diagonal support limit of $\bar{Q}^\iota$.

Theorem 5.12. *Let $\bar{Q}^\theta$ be an iteration such that for each $\iota < \theta$, P_ι carries a pre-closure system $\mathbf{s}_\iota$ on $[\lambda_\iota, \kappa^*)$, where the sequence $\bar{\lambda} = (\lambda_\iota)_{\iota<\theta}$ is non-decreasing. Moreover, let $\lambda_\theta = \min(\mathbf{Reg} \setminus \sup_{\iota<\theta} \lambda_\iota)$ and assume*

1. *For all $\iota < \theta$, $(P_\iota, P_{\iota+1})$ is a quasi-closed extension on $[\lambda_\iota, \kappa^*)$.*

2. *If $\bar{\iota} < \theta$ is limit, $\mathbf{s}_{\bar{\iota}}$ is the natural system of relations on $P_{\bar{\iota}}$ on $[\lambda_\iota, \kappa^*)$ and $\bar{Q}^\iota$ is a $\bar{\lambda} \restriction \iota$-diagonal support iteration.*

3. *We have* $\mathrm{cf}\,(\theta) < \kappa$.

Let P_θ be the λ_θ-diagonal support limit of $\bar{Q}^\theta$. Then P_θ is quasi-closed on $I_\theta = [\lambda_\theta, \kappa^*)$, as witnessed by the natural system of relations $\mathbf{s}_\theta$.

In the proof of the theorem, we need the following lemmas 5.13–5.16, showing that the notion of λ-support behaves as we expect. So fix an iteration $\bar{Q}^{\theta+1}$ and pre-closure systems as in the hypothesis of the theorem. These lemmas are somewhat technical but straightforward to show.

Lemma 5.13. *For each $\lambda \in [\lambda_\theta, \kappa^*)$ and $p \in P_\theta$,*

$$\mathrm{supp}^\lambda(p) = \bigcup_{\iota<\theta} \mathrm{supp}^\lambda(\pi_\iota(p)).$$

Proof. First, prove $\supseteq$: Say ξ is a member of the set on the right. Thus there is some $\iota < \theta$ such that

$$\pi_{\xi+1}(\pi_\iota(p)) \not\preccurlyeq^\lambda \pi_\xi(\pi_\iota(p)). \tag{5.5}$$

We consider two cases: first, assume $\iota \le \xi$. Then $\pi_{\xi+1}(\pi_\iota(p)) = \pi_\iota(p) = \pi_\xi(\pi_\iota(p))$, and so as $\preccurlyeq^\lambda$ is a pre-order, (5.5) is false. Thus this case never occurs, and we can assume $\iota > \xi$. Then $\pi_{\xi+1}(\pi_\iota(p)) = \pi_{\xi+1}(p)$ and $\pi_\xi(\pi_\iota(p)) = \pi_\xi(p)$, so (5.5) is equivalent to $\pi_{\xi+1}(p) \not\preccurlyeq^\lambda \pi_\xi(p)$. We infer that $\xi \in \mathrm{supp}^\lambda(p)$. All of the above inferences can be reversed, so $\subseteq$ holds as well. □

Lemma 5.14. *If $\lambda, \bar{\lambda}$ are regular such that $\lambda_\theta \le \lambda \le \bar{\lambda} < \kappa^*$ and $p \preccurlyeq^{\bar{\lambda}} q$, then* $\mathrm{supp}^\lambda(p) \subseteq \mathrm{supp}^\lambda(q)$.

Proof. Left to the reader. □

Observe though that $\supseteq$ does not necessarily hold: in a two-step iteration $P * \dot{Q}$, we could have and $(p, 1_{\dot{Q}}) \bar{\preccurlyeq}^\lambda (q, \dot{q})$ but $q \nVdash_P \dot{q} \dot{\preccurlyeq}^\lambda 1_{\dot{Q}}$ (say e.g. $p \Vdash \dot{q} = 1_{\dot{Q}}$). In this example we have $\mathrm{supp}^\lambda(p, 1_{\dot{Q}}) = \{0\} \not\supseteq \mathrm{supp}^\lambda(q, \dot{q}) = \{0, 1\}$.

Lemma 5.15. *Fix $\bar{\iota} < \theta$. If $p \in P_\theta$ and $\lambda, \bar{\lambda}$ are regular such that $\lambda_\theta \le \lambda \le \bar{\lambda} < \kappa^*$ and $p \preccurlyeq^{\bar{\lambda}} \pi_{\bar{\iota}}(p)$, then*

$$\mathrm{supp}^\lambda(p) = \mathrm{supp}^\lambda(\pi_{\bar{\iota}}(p)).$$

Proof. A short proof: $\supseteq$ holds by lemma 5.13 and $\subseteq$ is a consequence of lemma 5.14. We also give a direct proof: If $\iota < \bar\iota$, $\pi_\iota(\pi_{\bar\iota}(p)) = \pi_\iota(p)$, so for such ι,

$$\iota \in \operatorname{supp}^\lambda(p) \iff \iota \in \operatorname{supp}^\lambda(\pi_{\bar\iota}(p)).$$

If $\iota \geq \bar\iota$, we have $\pi_{\iota+1}(p) \leq \pi_\iota(p) \leq \pi_{\bar\iota}(p)$. By assumption and by 5.1($\lhd_c 2$) for $(P_{\iota+1}, P_\theta)$, we have $\pi_{\iota+1}(p) \preccurlyeq^\lambda \pi_{\bar\iota}(p)$. So by lemma 5.8, $\pi_{\iota+1}(p) \preccurlyeq^\lambda \pi_\iota(p)$. We conclude that for $\iota \geq \bar\iota$, we have $\iota \notin \operatorname{supp}^\lambda(p)$. □

Lemma 5.16. *Let $\lambda_1 \in [\lambda_\theta, \kappa^*)$ such that $\lambda_1 \geq \operatorname{cf}(\theta)$. Say $q = (q^\iota)_{\iota<\theta}$ is a thread through $\bar Q^\theta$ (see definition 2.6, p. 9 for this terminology) and say there is $w \in P_\theta$ such that for all $\iota < \theta$, $q^\iota \preccurlyeq^{\lambda_1} w$. Then q has legal support, i.e. $q \in P_\theta$.*

Proof. Let λ be regular. First consider the case $\lambda_\theta \leq \lambda \leq \lambda_1$. As $q \preccurlyeq^{\lambda_1} w$, by lemma 5.14, $\operatorname{supp}^\lambda(q) \subseteq \operatorname{supp}^\lambda(w)$, which satisfies the requirement of diagonal support by assumption. Now say $\lambda > \lambda_1$ and fix a sequence $(\theta(\zeta))_{\zeta<\operatorname{cf}(\theta)}$ which is cofinal in θ. By lemma 5.13

$$\operatorname{supp}^\lambda(q) = \bigcup_{\zeta<\operatorname{cf}(\theta)} \operatorname{supp}^\lambda(q^{\theta(\zeta)}),$$

and by assumption the right hand side is a union over bounded subsets of λ. Thus $\operatorname{supp}^\lambda(q)$ is a bounded subset of λ. □

Proof of theorem 5.12: Observe we must make the assumption that x includes the parameters $L_\mu[A]$ and $(\pi_\iota)_{\iota\leq\theta}$, where μ is a cardinal and and $\mu > \theta$, to make sure we can talk about the least sequence witnessing the cofinality of θ in a $\Sigma^T_1(x)$ manner (alternatively, one could include this sequence in c_θ; also note that in practice, we can replace $(\pi_\iota)_{\iota\leq\theta}$ as a parameter by just θ since it and $(P_\iota)_{\iota\leq\theta}$ will be recursive in θ and $L_\mu[A]$ and we can usually also drop θ since it will be definable from any condition $p \in P_\theta$).

We will show by induction on θ that for each pair $\iota < \bar\iota \leq \theta$, $(P_\iota, P_{\bar\iota})$ is a quasi-closed extension. Thus (P_0, P_θ) is a quasi-closed extension and so by lemma 5.9, P_θ is quasi-closed. The inductive hypothesis thus says that for each pair $\iota < \bar\iota < \theta$, $(P_\iota, P_{\bar\iota})$ is a quasi-closed extension as witnessed by $(\mathbf{s}_\iota, \mathbf{s}_{\bar\iota})$. We may assume θ is limit: For if θ is a successor ordinal, $\pi^\theta_{\theta-1}$ is a Δ_1-definable function and thus by induction hypothesis and lemma 5.10, for any $\iota < \theta$, (P_ι, P_θ) is a quasi-closed extension.

So assume θ is limit and let $\mathbf{s}_\theta$ be the natural system of relations on the diagonal support limit P_θ. Fix an arbitrary $\iota^* < \theta$. We show that (P_{ι^*}, P_θ) is a quasi-closed extension witnessed by $(\mathbf{s}_{\iota^*}, \mathbf{s}_\theta)$. By definition of $\mathbf{s}_\theta$, we have $\mathbf{s}_{\iota^*} \lhd \mathbf{s}_\theta$. It is straightforward to show that $\mathbf{s}_\theta$ is a pre-closure system (as defined in 3.1, p. 12). It is obvious that $\mathbf{D}_\theta$ is $\Pi^T_1(c_\theta)$: Letting Φ be a universal Π^A_1 formula, by definition of the natural system of relations we can assume the first component $\bar{c} = (c_\iota)_{\iota<\theta}$ of c_θ is such that for each $\iota < \theta$ and each $p, q \in P_\iota$,

$$q \in \mathbf{D}_\iota(\lambda, x, p) \iff \Phi(c_\iota, q, \lambda, x, p).$$

Thus Φ and $\bar{c}$ witness that $\mathbf{D}_\theta$ is $\Pi^T_1(\{c_\theta\})$: for $q \in \mathbf{D}_\theta(\lambda, x, p)$ is equivalent to

$$\forall \iota \in \operatorname{dom}(p) \quad \Phi(c_\iota, \pi_\iota(q), \lambda, x, \pi_\iota(p)).$$

We finish the proof that $\mathbf{s}_\theta$ is a pre-closure system by proving 3.1(C 3), as the remaining conditions have similar proofs: Say $p \leq_\theta q \leq_\theta r$ and $p \preccurlyeq^\lambda_\theta r$. Fixing an arbitrary $\iota < \theta$, we have $\pi_\iota(p) \leq_\iota \pi_\iota(q) \leq_\iota \pi_\iota(r)$ and $\pi_\iota(p) \preccurlyeq^\lambda_\iota \pi_\iota(r)$. Thus, by 3.1(C 3) for P_ι, $\pi_\iota(p) \preccurlyeq^\lambda_\iota \pi_\iota(q)$. As $\iota < \theta$ was arbitrary, $p \preccurlyeq^\lambda_\theta q$ holds. So as mentioned earlier, the natural system of relations is a pre-closure system.

We check 5.2($\mathrm{E_cI}$). Say $p \in P_\theta$ and $q^* \in P_{\iota^*}$ are such that $q^* \preccurlyeq^\lambda_{\iota^*} \pi_{\iota^*}(p)$ and $q^* \in \mathbf{D}_{\iota^*}(\lambda, x, \pi_{\iota^*}(p))$. Assume first that $\mathrm{cf}\,(\theta) \in I$ and $\mathrm{cf}\,(\theta) > \lambda$. Let $\sigma = \sigma^{\mathrm{cf}\,(\theta)}(p)$ and observe that $\sigma < \mathrm{cf}\,(\theta) \leq \theta$ by diagonal support. If $\sigma \leq \iota^*$, set $r_0 = q^*$ and let $(\theta(\zeta))_{\zeta<\mathrm{cf}\,(\theta)}$ be the least normal sequence cofinal in θ such that $\theta(0) = \iota^*$. Otherwise, let $(\theta(\zeta))_{\zeta<\mathrm{cf}\,(\theta)}$ be the least normal sequence cofinal in θ such that $\theta(0) = \sigma$. Use 5.2($\mathrm{E_cI}$) for (P_{ι^*}, P_σ) to get $r_0 \in P_\sigma$ such that $r_0 \in \mathbf{D}(\lambda, x, \pi_\sigma(p))$, $r_0 \preccurlyeq^\lambda_\sigma \pi_\sigma(p)$ and $\pi_{\iota^*}(r_0) = q^*$. Observe that in either case, $p \preccurlyeq^{\mathrm{cf}\,(\theta)} \pi_{\theta(0)}(p)$.

Let $(\theta(\zeta))_{\zeta<\mathrm{cf}\,(\theta)}$ be the least normal sequence cofinal in θ such that $\theta(0) = \sigma$. Now construct, by induction on $\zeta < \mathrm{cf}\,(\theta)$, a thread $(r_\zeta)_{\zeta<\mathrm{cf}\,(\theta)}$: having $r_\zeta \in P_\zeta$, use 5.2($\mathrm{E_cI}$) for $(P_{\theta(\zeta)}, P_{\theta(\zeta+1)})$ to get $r_{\zeta+1} \in \mathbf{D}(\lambda, x, \pi_{\theta(\zeta+1)}(p))$ such that $r_{\zeta+1} \preccurlyeq^\lambda_{\theta(\zeta+1)} \pi_{\theta(\zeta+1)}(p))$, $\pi_{\theta(\zeta)}(r_{\zeta+1}) = r_\zeta$ and in addition, $r_{\zeta+1} \preccurlyeq^{\mathrm{cf}\,(\theta)}_{\theta(\zeta+1)} r_\zeta$. Let r be the unique condition in P_θ defined by the thread $(r_\zeta)_{\zeta<\mathrm{cf}\,(\theta)}$. Since $r \preccurlyeq^{\mathrm{cf}\,(\theta)} r_0$, lemma 5.16 allows us to conclude that r has legal support. By construction, $r \in \mathbf{D}_\theta(\lambda, x, p)$, $r \preccurlyeq^\lambda_\theta p$ and $\pi_{\iota^*}(r) = q^*$.

If $\mathrm{cf}\,(\theta) \leq \lambda$, we can skip the first step in the above: we just let $r_0 = q^*$. At the end, we use $r \preccurlyeq^\lambda p$ and lemma 5.16 allows us to conclude that r has legal support. We leave the rest of 5.2($\mathrm{E_cI}$) to the reader, as it is similar to previous arguments.

Main point of the argument: Now check 5.2($\mathrm{E_cII}$), the existence of greatest lower bounds. Say $\bar{p} = (p_\xi)_{\xi<\rho}$ is a $(\lambda, \bar{\lambda}, x)$-adequate sequence of conditions in P_θ; then λ and $\bar{\lambda}$ are both regular, $\lambda_\theta \leq \lambda \leq \bar{\lambda} < \kappa^*$. We may fix $\bar{w}$ which is both a (λ, x)-strategic guide and a $(\bar{\lambda}, x)$-canonical witness for $\bar{p}$. Moreover, let q^* be a greatest lower bound of the sequence $(\pi_{\iota^*}(p_\xi))_{\xi<\rho}$.

The construction of q is by induction on $\zeta < \mathrm{cf}\,(\theta)$, and we shall use a sequence $(\theta(\zeta))_{\zeta<\mathrm{cf}\,(\theta)}$. The construction is split in cases for the following reasons: When $\mathrm{cf}\,(\theta) > \bar\lambda$, the definability of each $\pi_{\theta(\zeta)}$ poses a problem, and so we first have to find a $\pi_{\mathrm{cf}\,(\theta)}$-bound. Moreover, the argument that supports are legal goes differently depending on wether $\mathrm{cf}\,(\theta) \le \lambda$ or not. There are several possibilities for distinguishing cases as the trick for dealing with $\mathrm{cf}\,(\theta) > \bar\lambda$ can be applied whenever $\mathrm{cf}\,(\theta) > \lambda_\theta$, but in my opinion this obscures the argument.

First, assume that $\mathrm{cf}\,(\theta) \le \bar\lambda$. Let $(\theta(\zeta))_{\zeta\le\mathrm{cf}\,(\theta)}$ be the $\prec$-least increasing continuous sequence such that $\theta(0) = \iota^*$ and $\theta(\mathrm{cf}\,(\zeta)) = \theta$. By induction on ζ, we now construct a lower bound $q^\zeta \in P_{\theta(\zeta)}$ of the sequence $(\pi_{\theta(\zeta)}(p_\xi))_{\xi<\rho}$ for each $\zeta \le \mathrm{cf}\,(\theta)$. Set $q^0 = q^*$. Now assume we have q^ζ and show how to find $q^{\zeta+1}$. It is easily checked that $(\pi_{\theta(\zeta+1)}(p_\xi))_{\xi<\rho}$ is $(\lambda, \bar\lambda, x)$-adequate in $(P_{\theta(\zeta+1)}, P_{\theta(\zeta)})$, because by the presence of $L_\mu[A]$ and θ in x we only need bounded quantifiers (which we may assume come after the unbounded ones) to be able to talk about the sequence $(\theta(\zeta))_\zeta$.[1] If θ is, as usual, equal to $\mathrm{dom}(p)$ for $p \in P_\theta$, we don't need θ in x. By 5.2($\mathrm{E_c}$II) we obtain a greatest lower bound $q^{\theta(\zeta+1)} \in P_{\theta(\zeta)}$.

Now let $\zeta \le \mathrm{cf}\,(\theta)$ be limit. By construction and by ($\mathrm{E_c}$II), the $q^{\zeta'}$, for $\zeta' < \zeta$ form a thread. Define $q^{\theta(\zeta)}$ to be this thread. To show it has legal support, first assume that $\mathrm{cf}\,(\theta) \le \lambda$. By construction and by ($\mathrm{E_c}$II), for each ζ, we have $q^\zeta \preccurlyeq^\lambda p_0$. As λ is greater than the maximum of λ_θ and $\mathrm{cf}\,(\theta)$, lemma 5.16 allows us to infer that q^ζ has legal support, and thus is a condition in $P_{\theta(\zeta)}$ and a $\pi_{\theta(\zeta)}$-bound of $\bar p$ (in the sense of definition 5.3). The final condition $q^{\mathrm{cf}\,(\theta)}$ is a greatest lower bound of $(p_\xi)_{\xi<\rho}$ and for all $\xi < \rho$, $q^{\mathrm{cf}\,(\theta)} \preccurlyeq^\lambda p_\xi$.

To finish the case where $\mathrm{cf}\,(\theta) \le \bar\lambda$, we have to consider the sub-case where $\mathrm{cf}\,(\theta) \le \lambda$ fails, i.e. we assume $\lambda < \bar\lambda$ and $\mathrm{cf}\,(\theta) \in (\lambda, \bar\lambda]$. In this case we have $p_\xi \preccurlyeq^{\bar\lambda} \pi_{\iota^*}(p_\xi)$ for all $\xi < \rho$. By 5.2($\mathrm{E_c}$II) and by induction, we have $q^\zeta \preccurlyeq^{\bar\lambda} q^{\iota^*}$ for each $\zeta < \mathrm{cf}\,(\theta)$, and so $q \preccurlyeq^{\bar\lambda} q^{\iota^*}$. Moreover, as $\bar\lambda$ is greater than both $\mathrm{cf}\,(\theta)$ and λ_θ, q^ζ has legal support by lemma 5.16. Thus we are finished with the case $\mathrm{cf}\,(\theta) \le \bar\lambda$.

Now assume $\mathrm{cf}\,(\theta) > \bar\lambda$. We will now find ι' and q' such that q' is a $\pi_{\iota'}$-bound of $\bar p$ and for each $\xi < \rho$,

$$p_\xi \preccurlyeq^{\mathrm{cf}\,(\theta)} \pi_{\iota'}(p_\xi). \tag{5.6}$$

As $\rho \le \lambda < \mathrm{cf}\,(\theta)$ and $\sigma^{\mathrm{cf}\,(\theta)}(p_\xi) < \mathrm{cf}\,(\theta)$ for each $\xi < \rho$, letting

$$\iota' = \sup_{\xi<\rho} \sigma^{\mathrm{cf}\,(\theta)}(p_\xi),$$

[1] In fact, again, it seems we could have done with a more restricted class of functions G in definition 3.3 (of canoncal witnesses).

we have $\iota' < \mathrm{cf}\,(\theta)$ and so $\iota' < \theta$.

Fix a $\nu < \rho$ and let $\sigma(\nu) = \sigma^{\mathrm{cf}\,(\theta)}(p_\nu)$. Let $\bar{q}^\nu$ be the sequence defined by $q^\nu_\xi = \pi_{\sigma(\nu)}(p_\xi)$ for $\xi \geq \nu$. For $\xi < \nu$, the value of q^ν_ξ is arbitrary as long as $\bar{q}^\nu$ has $\bar{w}$ as a canonical witness (e.g. set $q^\nu_\xi = \pi_1(p_0)$). It is straightforward to check that $\bar{w}$ is a $(\bar{\lambda}, x)$-canonical witness and a (λ, x)-strategic guide for each sequence $\bar{q}^\nu$, for $\nu < \xi$ — this is why we only make demands on a tail in the definition of a strategic guide. It is also clear that we can build a thread $(q_\nu)_{\nu<\rho}$ using ($\mathrm{E_c}$II), where q_ν is a greatest lower bound of $\bar{q}^\nu$. Let q' be this thread. We invite the reader to check that q' is a greatest lower bound of the sequence $(\pi_{\iota'}(p_\xi))_{\xi<\rho}$ and that $q' \preccurlyeq^\lambda_{\iota'} \pi_{\iota'}(p_0)$. Thus, q' has legal support as $\lambda \geq \mathrm{cf}\,(\rho)$ and by lemma 5.16. By choice of ι', (5.6) holds. Observe that in the case $\lambda < \bar{\lambda}$, ($\mathrm{E_c}$II) also entails $q' \preccurlyeq^\lambda \pi_{\iota^*}(q') = q^*$.

Now we argue exactly as in the case where $\mathrm{cf}\,(\theta) \leq \lambda$, but this time setting $q^0 = q'$ and $\theta(0) = \iota'$. Again we build a sequence $(q^\zeta)_{\zeta<\mathrm{cf}\,(\theta)}$ by induction. At each succesor step, $\{\pi_{\theta(\zeta+1)}(p_\xi) \mid \xi < \rho\}$ is $(\lambda, \mathrm{cf}\,(\theta), x)$-adequate in $(P_{\theta(\zeta)}, P_{\theta(\zeta+1)})$. By 5.2($\mathrm{E_c}$II) we obtain a greatest lower bound $q^{\zeta+1} \in P_{\theta(\zeta)}$ such that $q^{\zeta+1} \preccurlyeq^{\mathrm{cf}\,(\theta)} q^\zeta$. By induction, this entails $q^{\iota'} \preccurlyeq^{\mathrm{cf}\,(\theta)} q^0 = q'$. Thus at each limit stage $\zeta \leq \mathrm{cf}\,(\theta)$, $q^\zeta \preccurlyeq^{\mathrm{cf}\,(\theta)} q'$ and lemma 5.16 allows us to conclude that q^ζ has legal support. Lastly, if $\lambda < \bar{\lambda}$, as $q^\zeta \preccurlyeq^{\mathrm{cf}\,(\theta)} q' \preccurlyeq^\lambda q^*$, we also have $q^\zeta \preccurlyeq^\lambda q^* = \pi_{\iota^*}(q^\zeta)$. □

We conclude this section with an observation about the support of a greatest lower bound of an adequate sequence.

Lemma 5.17. *Say $\bar{p} = (p_\xi)_{\xi<\rho}$ is a (λ, x)-adequate sequence with greatest lower bound p. Then for any regular $\bar{\lambda}$,*

$$\mathrm{supp}^{\bar{\lambda}}(p) \subseteq \bigcup_{\xi<\rho} \mathrm{supp}^{\bar{\lambda}}(p_\xi).$$

Proof. Assume $\iota < \theta$ and $\iota \notin \bigcup_{\xi<\rho} \mathrm{supp}^{\bar{\lambda}}(p_\xi)$. We may assume $\iota < \bar{\lambda}$ (since p has diagonal support). Then as $\pi_{\iota+1}$ is $\Pi^T_1(\bar{\lambda})$, the sequence $(\pi_{\iota+1}(p_\xi))_{\xi<\rho}$ is $\Pi^T_1(\bar{\lambda} \cup \{x\})$-definable, and for all $\xi < \rho$, $\pi_{\iota+1}(p_\xi) \preccurlyeq^{\bar{\lambda}} \pi_\iota(p_\xi)$. Therefore we can apply 5.2($\mathrm{E_c}$II) for $(P_\iota, P_{\iota+1})$ (see 5.2, p. 62). We conclude that $\pi_{\iota+1}(p) \preccurlyeq^{\bar{\lambda}} \pi_\iota(p)$ and so $\iota \notin \mathrm{supp}^{\bar{\lambda}}(p)$. □

5.2 Stratified extension and iteration

In this section, we show that composition of stratified forcing is a special case of stratified extension. We show that the second forcing in a stratified extension is stratified. Finally we prove an iteration theorem for stratified forcing.

Let P_0 be a complete sub-order of P_1 and let $\pi\colon P_1 \to P_0$ be a strong projection and let I be an intervall of regular cardinals. Moreover, assume for $i \in \{0,1\}$, we have a system

$$\mathbf{S}_i = (\mathbf{D}_i, c_i, \preccurlyeq_i^\lambda, \precsim_i^\lambda, \mathbf{C}_i^\lambda)_{\lambda\in I}$$

such that $\mathbf{D} \subseteq I \times V \times (P_i)^2$ is a class which is definable with parameter c_i, and for every $\lambda \in I$, $\preccurlyeq_i^\lambda$ and $\precsim_i^\lambda$ are binary relations on P_i and $\mathbf{C}_i^\lambda \subseteq P_i \times \lambda$.

Definition 5.18. We write $\mathbf{S}_0 \lhd \mathbf{S}_1$ if and only if in addition to ($\lhd_\mathrm{c}1$), ($\lhd_\mathrm{c}2$) and ($\lhd_\mathrm{c}3$) (see 5.1, p. 62), the following hold:

($\lhd_\mathrm{s}1$) If $q, q' \in P_0$ and $p, p' \in P_1$ are such that $q' \preccurlyeq_0^\lambda q \leq \pi(p')$ and $p' \preccurlyeq_1^\lambda p$, then $q' \cdot p' \preccurlyeq_1^\lambda q \cdot p$.

($\lhd_\mathrm{s}2$) For all $p, q \in P_0$, $p \precsim_0^\lambda q \Rightarrow p \precsim_1^\lambda q$.

($\lhd_\mathrm{s}3$) For all $p, q \in P_1$, $p \precsim_1^\lambda q \Rightarrow \pi(p) \precsim_0^\lambda \pi(q)$.

($\lhd_\mathrm{s}4$) If $w \leq \pi(d), \pi(r)$ and $d \precsim^\lambda r$ then $w \cdot d \precsim^\lambda w \cdot r$.

($\lhd_\mathrm{s}5$) If $\mathbf{C}_1^\lambda(p) \cap \mathbf{C}_1^\lambda(q) \neq 0$ then $\mathbf{C}_0^\lambda(\pi(p)) \cap \mathbf{C}_0^\lambda(\pi(q)) \neq 0$.

Observe that if $\mathbf{S}_0 \lhd \mathbf{S}_1$, we can drop the subscripts on $\precsim_0^\lambda$, $\precsim_1^\lambda$ and just write $\precsim^\lambda$ without causing confusion. Observe also that by corollary 3.10, we can assume that $r \preccurlyeq^{|P_1|} p$ holds exactly if $p = r$. This implies[2]

$$\forall p \in P_1 \big(p \preccurlyeq^{|P_1|} \pi(p) \iff p \in P_0\big). \tag{5.7}$$

We could assume that $\mathbf{C}_0^\lambda = \mathbf{C}_1^\lambda \cap P_0 \times \lambda$. For if not, simply replace $\mathbf{C}_1^\lambda$ by the following relation $\mathbf{C}_*^\lambda$: $s \in \mathbf{C}_*^\lambda(p)$ if and only if $s \in {}^{\leq 2}\lambda$ such that $s(0) \in \mathbf{C}_0^\lambda(\pi(p))$ and if $p \notin P_0$ then $1 \in \mathrm{dom}(s)$ and $s(1) \in \mathbf{C}_1^\lambda(p)$ (now in fact we get $\mathbf{C}_0^\lambda(p) = \{s \upharpoonright 1 \mid s \in \mathbf{C}_1^\lambda(p)\}$ for $p \in P_0$). To sum up, we could in principle completely eliminate any mention of $\mathbf{S}_0$ from the definition of stratified extension.

Replacing ($\lhd_\mathrm{s}1$)by the following two conditions yields an equivalent version of the above definition:

($\lhd_\mathrm{s}$A) $w \preccurlyeq_0^\lambda \pi(p) \Rightarrow w \cdot p \preccurlyeq_1^\lambda p$.

($\lhd_\mathrm{s}$B) If $w \leq \pi(p')$ and $p' \preccurlyeq^\lambda p$ then $w \cdot p' \preccurlyeq^\lambda w \cdot p$.

[2]Interestingly, (5.7) also follows just from the assumption that for any $r, p \in P_1$, $r \precsim^{|P_1|} p$, together with (E$_\mathrm{s}$I) *coherent expansion*

Sometimes it is more convenient to check both of these rather than $(\lhd_{\mathrm{s}}1)$, which is concise but cumbersome to show. Further notice that $(\lhd_{\mathrm{s}}\mathrm{B})$ implies

$(\lhd_{\mathrm{s}}\mathrm{b})$ If $w \leq \pi(p)$ and $p \preccurlyeq^\lambda \pi(p)$ then $w \cdot p \preccurlyeq^\lambda w$.

This is weaker than $(\lhd_{\mathrm{s}}\mathrm{B})$. We note in passing that we could do entirely with $(\lhd_{\mathrm{s}}\mathrm{A})$ and $(\lhd_{\mathrm{s}}\mathrm{b})$ and without $(\lhd_{\mathrm{s}}\mathrm{B})$. Neither condition $(\lhd_{\mathrm{s}}1)$ nor any of its variants were included in 3.1, the definition of a pre-closure system simply because they are not needed to preserve quasi-closure in iterations—rather we need $(\lhd_{\mathrm{s}}\mathrm{b})$ to preserve coherent centering, and $(\lhd_{\mathrm{s}}\mathrm{A})$ helps to preserve density at limits; see below.

We fix some convenient notation: If $d \leq r$, we say p *λ-interpolates d and r* to mean that $p \precsim^\lambda d$ and $p \preccurlyeq^\lambda r$. We say $p \preccurlyeq^{<\lambda} q$ to mean that for all $\lambda' \in I \cap \lambda$, we have $p \preccurlyeq^{\lambda'} q$.

Definition 5.19. Let I be an intervall of regular cardinals. We say the pair (P_0, P_1) is a *stratified extension on I, as witnessed by* $(\mathbf{S}_0, \mathbf{S}_1)$ if and only if $\mathbf{S}_0$ witnesses that P_0 is stratified on I, $\mathbf{S}_1$ is a pre-stratification system on P_1 and $\mathbf{S}_0 \lhd \mathbf{S}_1$; Moreover, for all $\lambda \in I$ we have that $(\mathrm{E_cI})$, $(\mathrm{E_cII})$ and all of the following conditions hold:

$(\mathrm{E_sI})$ *Coherent Expansion*: For $p, d \in P_1$, if $p \precsim^\lambda d$, $d \preccurlyeq^\lambda \pi(d)$ and $\pi(p) \leq \pi(d)$, we have that $p \leq d$.

$(\mathrm{E_sII})$ *Coherent Interpolation*: Given $d, r \in P_1$ such that $d \leq r$ and $p_0 \in P_0$ such that p_0 λ-interpolates $\pi(d)$ and $\pi(r)$ we can find $p \in P_1$ which λ-interpolates d and r such that $\pi(p) = p_0$. If moreover $d \preccurlyeq^{<\bar{\lambda}} \pi(d)$, we can in addition assume $p \preccurlyeq^{<\bar{\lambda}} \pi(p) \cdot r$.

$(\mathrm{E_sIII})$ *Coherent Centering*: Say $d, p \in P_1$, $d \precsim^\lambda p$ and $\mathbf{C}_1^\lambda(d) \cap \mathbf{C}_1^\lambda(p) \neq \emptyset$. Given $w_0 \in P_0$ such that both $w_0 \preccurlyeq^{<\lambda} \pi(d)$ and $w_0 \preccurlyeq^{<\lambda} \pi(p)$, we can find $w \in P_1$ such that $w \preccurlyeq^{<\lambda} p, d$ and $\pi(w) = w_0$.

We find it relieving to notice that P is stratified exactly if $(\{1_P\}, P)$ is a stratified extension. Again, if we don't mention $\mathbf{S}_0$, $\mathbf{S}_1$ or I we are either claiming that they can be appropriately defined or they can be inferred from the context.

Lemma 5.20. *If P is stratified on I and $\Vdash_P \dot{Q}$ is stratified on I, then $(P, P * \dot{Q})$ is a stratified extension on I.*

To be more precise, let $\mathbf{S}_0$ denote the pre-stratification system witnessing that P is stratified and let $\mathbf{S}_1 = (\bar{\mathbf{D}}, \bar{c}, \bar{\preccurlyeq}^\lambda, \bar{\precsim}^\lambda, \bar{\mathbf{C}}^\lambda)_{\lambda \in I}$ be the pre-stratification system constructed as in the proof of 3.17, where we showed that $P * \dot{Q}$ is stratified. Then $(\mathbf{S}_0, \mathbf{S}_1)$ witnesses that $(P, P * \dot{Q})$ is a stratified extension on I.

Proof. We have already checked ($\lhd_c$1), ($\lhd_c$2), ($\lhd_c$3), (E_cI) and (E_cII)—i.e. that $(P, P * \dot{Q})$ is a *quasi-closed extension*—in lemma 5.4. We showed that $\mathbf{S}_1$ is a pre-stratification system when we proved theorem 3.17. It's technical but straightforward to check that $\mathbf{S}_0 \lhd \mathbf{S}_1$ (see definition 5.18, p. 73):

Fix $\bar{p} = (p, \dot{p}) \in P * \dot{Q}$ and $w \in P$, $w \leq p$. For ($\lhd_s$A), say $w \preccurlyeq^\lambda p$. Then as $p \Vdash \dot{p} \dot{\preccurlyeq}^\lambda p$, we have $(w, p) \bar{\preccurlyeq}^\lambda \bar{p}$. For ($\lhd_s$B), fix another condition $\bar{q} = (q, \dot{q}) \in P * \dot{Q}$ such that $\bar{p} \bar{\preccurlyeq}^\lambda \bar{q}$. Then $p \Vdash \dot{p} \dot{\preccurlyeq}^\lambda \dot{q}$, whence $w \Vdash \dot{p} \dot{\preccurlyeq}^\lambda \dot{q}$ and so $(w, p) \bar{\preccurlyeq}^\lambda (w, 1_{\dot{Q}})$, done. For ($E_s$I) and ($\lhd_s$4), let $\bar{r} = (r, \dot{r}) \in P * \dot{Q}$ and say $\bar{p} \bar{\curlyeqprec}^\lambda \bar{r}$, i.e. $p \curlyeqprec^\lambda r$ and if $p \cdot r > 0$ then $p \cdot q \Vdash \dot{p} \dot{\curlyeqprec}^\lambda \dot{r}$. To check ($E_s$I) *coherent expansion*, assume $\bar{r} \bar{\preccurlyeq}^\lambda (r, 1_{\dot{Q}})$ and $p \leq r$. Then $p \Vdash \dot{p} \dot{\curlyeqprec}^\lambda \dot{r}$. As P forces *expansion* for $\dot{Q}$, $p \Vdash \dot{p} \leq \dot{q}$ and we are done with (E_sI). To check ($\lhd_s$4), say $w \leq p$. Then $w \cdot r \leq p \cdot r$, and so if $w \cdot r > 0$, it forces $\dot{p} \dot{\curlyeqprec}^\lambda \dot{r}$. Since $w \curlyeqprec^\lambda w$, we infer that $(w, \dot{p}) \bar{\curlyeqprec}^\lambda \bar{r}$. The remaining ($\lhd_s$2), ($\lhd_s$3) and ($\lhd_s$5) are immediate by the definition.

Now we check the conditions of 5.19 (see p. 74). For (E_sII) *coherent interpolation*, just look at how we found an interpolant in the proof of theorem 3.17. Do the same for (E_sIII) *coherent centering*.

□

The following is the analogue of 5.9 for quasi-closed extension:

Lemma 5.21. *If (P_0, P_1) is a stratified extension on I and π is $\Sigma^T_1(\min I \cup \{c_1\})$, then P_1 is stratified on I.*

Proof. The proof is a straightforward consequence of the definition and lemma 5.9. We leave it to the reader. □

Definition 5.22. Say $\bar{Q}^\theta$ is an iteration such that each initial segment P_ι carries a pre-stratification system $\mathbf{S}_\iota$ on $I_\iota = \mathbf{Reg} \cap [\lambda_\iota, \kappa)$, where the sequence $\bar{\lambda} = (\lambda_\iota)_{\iota \leq \theta}$ is a non-decreasing sequence of regulars. Let P_θ be its λ_θ-diagonal support limit. We now add to the definition of the *natural system of relations on P_θ*. Let $\lambda \in I_\theta$, where we set $I_\theta = [\lambda_\theta, \kappa) \cap \mathbf{Reg}$. The relations $\preccurlyeq^\lambda$ and $\mathbf{D}$ are defined as in 5.11, p. 67. Let

1. $p \curlyeqprec^\lambda_\theta q \iff \forall \iota < \theta \quad \pi_\iota(p) \curlyeqprec^\lambda_\iota \pi_\iota(q)$;

2. $p \in \mathrm{dom}(\mathbf{C}^\lambda)$ if and only if for all $\iota < \sigma^\lambda(p)$, $\pi_\iota(p) \in \mathrm{dom}(\mathbf{C}^\lambda_\iota)$;

3. $s \in \mathbf{C}^\lambda_\theta(p)$ if and only if $s \colon \sigma^\lambda(p) \to \lambda$ and for all $\iota < \mathrm{dom}(s)$, we have $s(\iota) \in \mathbf{C}^\lambda_\iota(\pi_\iota(p))$.

As before, the above yields a pre-stratification system under natural assumptions, as we shall see in the proof of theorem 5.23.

Theorem 5.23. *Let $\bar{Q}^\theta$ be an iteration such that for each $\iota < \theta$, P_ι carries a pre-stratification system $\mathbf{S}_\iota$ on $I_\iota = \mathbf{Reg} \cap [\lambda_\iota, \kappa)$, where the sequence $\bar{\lambda} = (\lambda_\iota)_{\iota<\theta}$ is a non-decreasing sequence of regulars. Moreover, let $\lambda_\theta = \min(\mathbf{Reg} \setminus \sup_{\iota<\theta} \lambda_\iota)$, let $I_\theta = [\lambda_\theta, \kappa)$ and assume*

1. *For all $\iota < \theta$, $(P_\iota, P_{\iota+1})$ is a stratified extension on I_ι.*
2. *If $\bar{\iota} < \theta$ is limit, $\mathbf{S}_{\bar{\iota}}$ is the natural system of relations on $P_{\bar{\iota}}$ and $P_{\bar{\iota}}$ is the $\lambda_{\bar{\iota}}$-diagonal support limit of $\bar{Q}^{\bar{\iota}}$.*
3. *For each regular $\lambda \in [\lambda_\theta, \lambda^*)$ there is $\iota < \lambda^+$ such that for all $p \in P_\theta$ we have $\operatorname{supp}^\lambda(p) \subseteq \iota$.*

Let P_θ be the λ_θ-diagonal support limit of $\bar{Q}^\theta$. Then P_θ is stratified on I_θ.

Remark 5.24. In our particular application we will have that for each regular $\lambda < \kappa$, there is $\iota < \lambda^+$ such that $\lambda < \lambda_\iota$. Observe that by the definition of $\operatorname{supp}^\lambda(p)$, this implies that the last clause of the above is satisfied.

Of course, the following proof can be easily adapted to show that under the same hypothesis, for every $\iota < \theta$, (P_ι, P_θ) is a stratified extension on I_θ; while this approach facilitated the inductive proof in the case of quasi-closure, it would serve no purpose in the present context.

Proof of theorem 5.23. By lemma 5.21, we may assume θ is limit. That P_θ is stratified on I_θ is witnessed by the natural system of relations $\mathbf{S}_\theta$, as defined in 5.22. The proof of the following lemma is a straightforward induction, which we leave to the reader:

Lemma 5.25. *For any $\iota < \bar{\iota} \leq \theta$, $\mathbf{S}_\iota \lhd \mathbf{S}_{\bar{\iota}}$.*

Next, we check that $\mathbf{S}_\theta$ is a pre-stratification system (see 3.7 p. 15): Conditions (S 1), (S 2) and (S 3) are immediate by the definition of $\preceq_\theta^\lambda$ and the fact that for each $\iota < \theta$, $\mathbf{S}_\iota$ is a pre-stratification system. The proofs resemble that of (S I), see below.

The non-trivial condition is 3.7(S 4), *Density*. First we must check that $\operatorname{ran}(\mathbf{C}_\theta^\lambda)$ has size at most λ: this is because by the last assumption of the theorem and by diagonal support, $\operatorname{supp}^\lambda(p) \in [\iota]^{<\lambda}$ for some $\iota < \lambda^+$.

For the more interesting part of the argument, we use density and continuity for the initial segments P_ι, $\iota < \theta$ together with quasi-closure. Observe that by theorem 5.12, for any $\iota < \bar{\iota} \leq \theta$, (P_ι, P_θ) is a quasi-closed extension on I_θ. Say we are given $p \in P_\theta$. Let $\sigma = \sigma^\lambda(p)$. We may assume that $\sigma = \theta$, for otherwise we can use induction and *Density* for P_σ and are done. Thus we can assume $\lambda_\theta < \lambda$, for otherwise, since P_θ is a diagonal support limit, $\mathrm{supp}^\lambda(p)$ is bounded below θ.

So say we are given $\lambda' \in [\lambda_\theta, \lambda)$. We must find $q \preccurlyeq^{\lambda'} p$ such that $q \in \mathrm{dom}(\mathbf{C}^\lambda)$. Let $\delta = \mathrm{cf}\ (\theta)$ and assume without loss of generality $\lambda' \geq \delta$ (otherwise we may increase λ'). Fix a normal sequence $(\sigma(\xi))_{\xi\leq\delta}$ such that $\sigma(\delta) = \theta$. We inductively construct a δ-adequate sequence $(p_\xi)_{\xi<\delta}$ such that $p_0 = p$ and for any ν, ξ such that $\nu < \xi < \delta$,

$$\pi_{\sigma(\nu)}(p_\xi) \in \mathrm{dom}(\mathbf{C}^\lambda_{\sigma(\nu)}). \tag{5.8}$$

Fix appropriate x so that the following sequence can be built in a (δ, x)-adequate fashion and such that $(\sigma(\xi))_{\xi\leq\delta}$ is a component of x. Let $p_0 = p$. Assuming we have p_ξ and $\bar{w} \restriction \xi + 1$, find $p_{\xi+1}$ as follows.

We may find $q \in P_\theta$ such that $q \in \mathbf{D}_\theta(\lambda', (x, \bar{w} \restriction \xi + 1), p_\xi)$ and $q \preccurlyeq^{\lambda'} p_\xi$. Also, there is $q' \preccurlyeq^{\lambda'} \pi_{\sigma(\xi)}(q)$ such that $q \in \mathrm{dom}(\mathbf{C}^\lambda_{\sigma(\xi)})$. Let $p_{\xi+1} = q' \cdot q$. Since $\mathbf{S}_{\sigma(\xi)} \lhd \mathbf{S}_\theta$, by 5.18($\lhd_s$A), $p_{\xi+1} \in \mathbf{D}_\theta(\lambda', x, p_\xi)$, and also $p_{\xi+1} \preccurlyeq^{\lambda'} p_\xi$. Moreover, by 5.18($\lhd_s$5), for any $\nu \leq \xi$,

$$\pi_{\sigma(\nu)}(p_{\xi+1}) \in \mathrm{dom}(\mathbf{C}^\lambda_{\sigma(\nu)}).$$

Apply the usual trick (see lemma 3.14) to find $w_{\xi+1}$ and $p_{\xi+1}$ as above so as to obtain a (δ, x)-adequate sequence. At limit stages $\bar{\xi} \leq \delta$, $p_{\bar{\xi}}$ is a greatest lower bound in P_θ of the sequence constructed so far. It exists by quasi-closure for P_θ. We show

$$\pi_{\sigma(\bar{\xi})}(p_{\bar{\xi}}) \in \mathrm{dom}(\mathbf{C}^\lambda_{\sigma(\bar{\xi})}). \tag{5.9}$$

Let $\nu < \bar{\xi}$ be arbitrary. As $(P_{\sigma(\nu)}, P_\theta)$ satisfies (C II),

$$\pi_{\sigma(\nu)}(p_{\bar{\xi}}) = \prod_{\xi\in(\nu,\bar{\xi})} \pi_{\sigma(\nu)}(p_\xi). \tag{5.10}$$

We want to apply (S 5) for $P_{\sigma(\nu)}$. By choice of x, $(\pi_{\sigma(\nu)}(p_\xi))_{\xi\in(\nu,\bar{\xi})}$ is a λ'-adequate sequence; and so we may use continuity for $P_{\sigma(\nu)}$. Now by induction hypothesis, (5.8) holds for all $\xi \in (\nu, \bar{\xi})$ and so by (S 5) we have $\pi_{\sigma(\nu)}(p_{\bar{\xi}}) \in \mathrm{dom}(\mathbf{C}^\lambda_{\sigma(\nu)})$. As $\nu < \bar{\xi}$ was arbitrary and by definition of $\mathbf{C}^\lambda_{\sigma(\bar{\xi})}$ we conclude that (5.9) holds. In particular, for the last stage of our construction, we set $\bar{\xi} = \delta$ in (5.9) and conclude $p_\delta \in \mathrm{dom}(\mathbf{C}^\lambda_\theta)$, finishing the proof of *Density*. So $\mathbf{S}_\theta$ is a pre-stratification system.

Quasi-closure was shown in lemma 5.12. First we check conditions (S I)–(S III) of 3.8, stratification (see p. 16). *Expansion* (S I) is trivial: If $d \preccurlyeq^\lambda_\theta r$ and $r \preccurlyeq^\lambda_\theta 1$, then for all $\iota < \theta$, $\pi_\iota(d) \preccurlyeq^\lambda_\iota \pi_\iota(r)$ and $\pi_\iota(r) \preccurlyeq^\lambda_\iota 1$. By induction, we may assume *expansion* holds for each P_ι, $\iota < \theta$. Thus $d \leq r$.

We show *interpolation* (S II) holds. So fix $d, r \in P_\theta$ such that $d \leq r$ holds. We construct the interpolant p by induction on its initial segments $p \restriction \iota$, for $\iota < \theta$. Say we have already constructed $p \restriction \iota$. Use coherent interpolation for $(P_\iota, P_{\iota+1})$ to obtain $p \restriction \iota + 1$ interpolating $\pi_{\iota+1}(d)$ and $\pi_{\iota+1}(r)$: demand that

$$p \restriction \iota + 1 \preccurlyeq^{<\bar{\lambda}(\iota)} \pi_{\iota+1}(r) \cdot p \restriction \iota, \tag{5.11}$$

where $\bar{\lambda}(\iota)$ is the maximal $\bar{\lambda}$ with the property that $\pi_{\iota+1}(d) \preccurlyeq^{<\bar{\lambda}} \pi_\iota(d)$.[3] We claim that for any $\gamma \in I_\theta$,

$$\iota \notin \mathrm{supp}^\gamma(d) \cup \mathrm{supp}^\gamma(r) \Rightarrow p \restriction \iota + 1 \preccurlyeq^\gamma p \restriction \iota. \tag{5.12}$$

So fix $\gamma \in \mathbf{Reg}$ and assume the hypothesis of (5.12). As $d \restriction \iota + 1 \preccurlyeq^\gamma d \restriction \iota$, by 3.1(C 4) and by definition of $\bar{\lambda}(\iota)$, we have $\gamma < \bar{\lambda}(\iota)$. Thus, (5.11) yields

$$p \restriction \iota + 1 \preccurlyeq^\gamma \pi_{\iota+1}(r) \cdot p \restriction \iota. \tag{5.13}$$

Since $r \restriction \iota + 1 \preccurlyeq^\gamma r \restriction \iota$, by 5.18($\lhd_{\mathrm{s}}$b) we infer

$$\pi_{\iota+1}(r) \cdot p \restriction \iota \preccurlyeq^\gamma p \restriction \iota. \tag{5.14}$$

From (5.13) and (5.14) we get $p \restriction \iota + 1 \preccurlyeq^\gamma p \restriction \iota$.

At limit stages $\bar{\iota} \leq \theta$ of the construction of the interpolant p, (5.12) holds for all $\iota < \bar{\iota}$, and so $p \restriction \bar{\iota}$ satisfies the support requirement. This completes the proof of interpolation.

Now for *centering* (S III). Say $p \preccurlyeq^\lambda d$ and fix $s \in \mathbf{C}^\lambda_\theta(p) \cap \mathbf{C}^\lambda_\theta(d)$.

Write σ for $\mathrm{dom}(s)$. By definition of $\mathbf{C}^\lambda_\theta$, $\sigma = \sigma^\lambda(p) = \sigma^\lambda(d)$. First, assume $\sigma = \theta$. In this case, we have $\lambda > \lambda_\theta$ by definition of diagonal support. We construct w by induction on its initial segments $w \restriction \iota$, for $\iota < \sigma$. To start, use *centering* for P_1 to obtain $w \restriction 1$. Assume we have $w \restriction \iota$; just use *coherent centering* for $(P_\iota, P_{\iota+1})$ to obtain $w \restriction \iota + 1$. At limits $\iota \leq \sigma$, use lemma 5.16 and the fact that $\mathrm{cf}\,(\sigma) < \lambda$ and so $w_0 \restriction \iota \preccurlyeq^{\mathrm{cf}\,(\sigma)} \pi_\iota(d)$.

Secondly, if $\sigma < \theta$, we can use *centering* for P_σ to obtain a lower bound w_0 of $\pi_\sigma(p)$ and $\pi_\sigma(d)$ with the desired properties. We claim that $w = w_0 \cdot d$ is the desired condition, i.e. $w \preccurlyeq^{<\lambda} p, d$. The proof is of course by induction on $\iota \leq \theta$. For limit ι, just use the induction hypothesis and the definition of $\preccurlyeq^{<\gamma}_\iota$. For the successor case, write

$$d^* = \pi_{\iota+1}(d),$$
$$p^* = \pi_{\iota+1}(p),$$
$$w_0^* = w_0 \cdot \pi_\iota(d)$$

[3] actually, it would suffice to demand this whenever $\iota \geq \sigma^\lambda(d)$

and let π denote π_ι. We may assume by induction that $w_0^* \preccurlyeq^{<\lambda} \pi(d^*), \pi(p^*)$. In the following, use that $\mathbf{S}_{\iota+1}$ is a pre-stratification system, $\mathbf{S}_\iota \lhd \mathbf{S}_{\iota+1}$ and 5.18($\mathrm{E_s}$I), *coherent expansion*.

Firstly, since $d^* \preccurlyeq^\lambda \pi(d^*)$ and $w_0^* \leq \pi(d^*)$, by 5.18($\lhd_\mathrm{s}$b), we have

$$w_0^* \cdot d^* \preccurlyeq^\lambda w_0^*. \tag{5.15}$$

In the same way, we can argue that

$$w_0^* \cdot p^* \preccurlyeq^\lambda w_0^*. \tag{5.16}$$

Equation (5.15) and $w_0^* \preccurlyeq^{<\lambda} \pi(d^*)$ give us $w_0^* \cdot d^* \preccurlyeq^{<\lambda} \pi(d^*)$, and together with $w_0^* \cdot d^* \leq d^* \leq \pi(d^*)$ and 3.1(C 3) we infer that $w_0^* \cdot d^* \preccurlyeq^{<\lambda} d^*$.

Since $d^* \preccurlyeq\!\!\!\!\mid^\lambda p^*$ and $w_0^* \leq \pi(d^*), \pi(p^*)$, we may conclude by $\mathbf{S}_\iota \lhd \mathbf{S}_{\iota+1}$ and 5.18($\lhd_\mathrm{s}$4) that

$$w_0^* \cdot d^* \preccurlyeq\!\!\!\!\mid^\lambda w_0^* \cdot p^*.$$

This together with (5.16), by *coherent expansion* 5.18($\mathrm{E_s}$I) yields

$$w_0^* \cdot d^* \leq w_0^* \cdot p^*.$$

Thus $w_0^* \cdot d^* \leq p^* \leq \pi(p^*)$ while at the same time $w_0^* \cdot d^* \preccurlyeq^\lambda w_0^* \preccurlyeq^{<\lambda} \pi(p^*)$. Another application of 3.1(C 3) yields $w_0^* \cdot d^* \preccurlyeq^{<\lambda} p^*$. This ends the successor step of the inductive proof that $w \preccurlyeq^{<\lambda} p, d$, and we are done with coherent centering.

Finally, check (S 5) *Continuity*: Fix $\lambda^*, \lambda \in I_\theta$ such that $\lambda^* < \lambda$. Say $\bar{p}$ and $\bar{q}$ are (λ^*, x)-adequate sequences of length ρ with greatest lower bound p and q respectively, and for each $\xi < \rho$, $\mathbf{C}^\lambda_\theta(p_\xi) \cap \mathbf{C}^\lambda_\theta(q_\xi) \neq \emptyset$. We show that $p, q \in \mathrm{dom}(\mathbf{C}^\lambda_\theta)$ and $\mathbf{C}^\lambda_\theta(p) \cap \mathbf{C}^\lambda_\theta(q) \neq \emptyset$.

For $\nu < \rho$, let $\sigma(\nu) = \sigma^\lambda(p_\nu)$. Look at the sequence $\bar{p}^\nu = (p^\nu_\xi)_{\xi<\rho}$ of conditions in $P_{\sigma(\nu)}$, defined by $p^\nu_\xi = 1_{P_{\sigma(\nu)}}$ for $\xi < \nu$ and $p^\nu_\xi = \pi_{\sigma(\nu)}(p_\xi)$ for $\xi \in [\nu, \rho)$. As in the proof of theorem 5.12, it is easy to see $p^\nu_\xi = G(\bar{w} {\restriction} \xi{+}1)$ for some $\Sigma^T_1(\lambda^* \cup \{x\})$ function G, where $\bar{w}$ is a canonical witness and strategic guide for $\bar{p}$. Also as in the proof of theorem 5.12, $\bar{p}^\nu$ is (λ^*, x)-adequate. Its greatest lower bound is $\pi_{\sigma(\nu)}(p)$. Thus by continuity for $P_{\sigma(\nu)}$, $\pi_{\sigma(\nu)}(p) \in \mathrm{dom}\, \mathbf{C}^\lambda_{\sigma(\nu)}$.

Observe that by definition of $\mathbf{C}^\lambda_\theta$, we have $\sigma^\lambda(p_\xi) = \sigma^\lambda(q_\xi)$ for all $\xi < \rho$. Analogously, define adequate sequences $\bar{q}^\nu$ in $P_{\sigma(\nu)}$ with greatest lower bound $\pi_{\sigma(\nu)}(q)$. By continuity for $P_{\sigma(\nu)}$, we infer $\mathbf{C}^\lambda_{\sigma(\nu)}(\pi_{\sigma(\nu)}(p)) \cap \mathbf{C}^\lambda_{\sigma(\nu)}(\pi_{\sigma(\nu)}(q)) \neq \emptyset$ for each $\nu < \rho$. Letting $\sigma = \sup_{\nu<\rho} \sigma(\nu)$, we infer $\mathbf{C}^\lambda_\sigma(\pi_\sigma(p)) \cap \mathbf{C}^\lambda_\sigma(\pi_\sigma(q)) \neq \emptyset$, by the definition of $\mathbf{C}^\lambda_\sigma$. As $\sigma^\lambda(p), \sigma^\lambda(q) \leq \sigma$ by lemma 5.17, this means $\mathbf{C}^\lambda_\theta(p) \cap \mathbf{C}^\lambda_\theta(q) \neq \emptyset$, by the definition of $\mathbf{C}^\lambda_\theta$. We are done with the proof of *continuity*.

□

Corollary 5.26. *Theorem 3.20 holds, that is, iterations with stratified components and diagonal support are stratified.*

Proof. By lemma 5.20, composition is an example of stratified extension. By theorem 5.23, since the iteration has diagonal support and its initial segments form a sequence of stratified extensions, the whole iteration is stratified. □

5.3 Products

So far, stratified extension has only given us an overly complicated proof that iterations with stratified components are stratified. Here is a first non-trivial application: as a consequence of the next lemma, one can mix composition and products of stratified forcing freely in iterations with diagonal support, and the resulting iteration will be stratified.

Lemma 5.27. *If P and Q are stratified on I, $(P, P \times Q)$ is a stratified extension (on I).*

Proof. The proof is entirely as you expect. Fix pre-stratification systems $\mathbf{S}_P = (\mathbf{D}_P, c_P, \preccurlyeq^\lambda_P, \curlyeqprec^\lambda_P, \mathbf{C}^\lambda_P)_{\lambda\in I}$ and $\mathbf{S}_Q = (\mathbf{D}_Q, c_Q, \preccurlyeq^\lambda_Q, \curlyeqprec^\lambda_Q, \mathbf{C}^\lambda_Q)_{\lambda\in I}$. We now define a stratification system $\bar{\mathbf{S}} = (\bar{\mathbf{D}}, \bar{c}, \bar{\preccurlyeq}^\lambda, \bar{\curlyeqprec}^\lambda, \bar{\mathbf{C}}^\lambda)_{\lambda\in I}$ on $P\times Q$ in the most natural way: let $\bar{\mathbf{D}}(\lambda, x, (p, q)) = \mathbf{D}_P(\lambda, x, p) \times \mathbf{D}_Q(\lambda, x, q)$ and let

$$
\begin{aligned}
(p, q) \mathrel{\bar{\preccurlyeq}^\lambda} (\bar{p}, \bar{q}) &\iff p \preccurlyeq^\lambda_P \bar{p} \text{ and } q \preccurlyeq^\lambda_P \bar{q}\\
(p, q) \mathrel{\bar{\curlyeqprec}^\lambda} (\bar{p}, \bar{q}) &\iff p \curlyeqprec^\lambda_P \bar{p} \text{ and } q \curlyeqprec^\lambda_P \bar{q}\\
s \in \bar{\mathbf{C}}^\lambda(p, q) &\iff \Big[s \in \mathbf{C}^\lambda_P(p) \text{ and } q \preccurlyeq^\lambda 1_Q\Big] \text{ or } \Big[s = (\chi, \zeta)\\
&\qquad\qquad \text{where } \chi \in \mathbf{C}^\lambda_P(p) \text{ and } \zeta \in \mathbf{C}^\lambda_Q(q)\Big].
\end{aligned}
$$

That $\bar{\mathbf{S}}$ is a pre-stratification system requires but a glance at the definitions (see 3.1, p. 12 and 3.7, p. 15). For example, *Continuity*, (S 5) is a straightforward application of continuity for both P and Q.

The same holds for $(\lhd_c 1)$, $(\lhd_c 2)$ and $(\lhd_c 3)$ (see p. 73 for the definition of $\mathbf{S}_P \lhd \bar{\mathbf{S}}$, and see p. 62 for $(\lhd_c 1)$, $(\lhd_c 2)$ and $(\lhd_c 3)$. For the following, let $(p, q) \in P \times Q$, $w \in P$. For your entertainment, we check 5.18($\lhd_s$A) (see page 73). Say $w \preccurlyeq^\lambda_p p$. Then clearly $(w, q) \mathrel{\bar{\preccurlyeq}^\lambda} (p, q)$, done. Now 5.18($\lhd_s$b): say $w \leq p$ and $(p, q) \mathrel{\bar{\preccurlyeq}^\lambda} (p, 1_Q)$. This means $q \preccurlyeq^\lambda_Q 1_Q$ and so $(w, q) \mathrel{\bar{\preccurlyeq}^\lambda} (w, 1_Q)$, which is what we wanted to prove.

For the next two conditions, let $\bar{d} = (d, d^*), \bar{r} = (r, r^*) \in P \times Q$ satisfy $\bar{d} \precapprox^{\lambda} \bar{r}$. We jump ahead and check ($\mathrm{E_s}$I) of 5.19 (see p. 74): say $d \leq r$ and $\bar{r} \precapprox^{\bar{\lambda}} (r, 1_Q)$. Then $r^* \preccurlyeq^{\lambda}_{Q} 1_Q$ and $d^* \precapprox^{\lambda}_{Q} r^*$ by assumption, so by 3.7(S I) for Q, $d^* \leq r^*$ and thus $\bar{d} \leq \bar{r}$.

Let's check ($\lhd_{\mathrm{s}}$4)). Say $w \leq d$ and $w \leq r$. By 3.7(S 1) for P, $w \precapprox^{\lambda}_{P} w$ and so $(w, d^*) \precapprox^{\lambda} (w, r^*)$. We omit the rest of 5.18 and conclude that $\mathbf{S}_P \lhd \bar{\mathbf{S}}$.

The most interesting part of the present proof is that of *quasi-closed extension* (definition 5.2, see p. 14), of which we check ($\mathrm{E_c}$II) , leaving ($\mathrm{E_c}$I) to the reader. So say $(p_\xi, q_\xi)_{\xi<\rho}$ is (λ, x)-strategic and $\Pi^T_1(\bar{\lambda} \cup \{x\})$, and $(p_\xi)_{\xi<\rho}$ has a greatest lower bound p. Firstly, since we can assume x contains a parameter X such that $P \subseteq X$, we conclude that $\bar{q} = (q_\xi)_{\xi<\rho}$ is $\Pi^T_1(\bar{\lambda} \cup \{x\})$ (by lemma 5.7). If $\lambda = \bar{\lambda}$, we are done as $\bar{q}$ is λ-adequate and Q is quasi-closed. If on the other hand, $\lambda < \bar{\lambda}$, we have that for all $\xi < \rho$, $q_\xi \preccurlyeq^{\bar{\lambda}}_{Q} 1_Q$. By lemma 5.5, $\bar{q}$ is $\bar{\lambda}$-adequate. Moreover, if q is a greatest lower bound of $\bar{q}$, by quasi-closure for Q, we have $q \preccurlyeq^{\bar{\lambda}}_{Q} 1_Q$. So $(p, q) \precapprox^{\bar{\lambda}} (p, 1_Q)$ and we are done.

To conclude that $(P, P \times Q)$ is a stratified extension, we check the remaining conditions of 5.19 (see p. 74). *Coherent interpolation*, 5.19($\mathrm{E_s}$II) and *Coherent centering*, 5.19($\mathrm{E_s}$III) are identical to interpolation and centering for Q in this context. □

5.4 Stable meets for strong sub-orders

In the next section, we introduce the operation of amalgamation and show that the amalgamation of a stratified forcing P is a stratified extension of P. In that proof, we must show that a certain dense subset of P is closed under taking meets with conditions from an "initial segment" (or rather, a strong sub-order) Q (see lemma 6.15, p. 98). This will be facilitated by the so-called Q-stable meet operation $p \wedge_Q r$, which we introduce in the present section. In a standard iteration this is a simple operation: $p \wedge_Q r$ "starts like p on Q" and then continues "like r" (as closely as possible) on $P : Q$. What makes it useful is the following: if r is "a direct extension on the tail P : Q" of a condition p, then $p \wedge_Q r$ is a *de-iure* direct extension of p, and moreover r can be obtained straightforwardly from $p \wedge_Q r$.

We now give a formal definition of such an operation, and then show that we can always define an operation $\wedge$ on products and compositions. Then we show how to define $\wedge$ for infinite iterations. In the next section we shall see we also have a stable meet operator for amalgamation. We take this formal, inductive approach (rather than defining $\wedge$ directly on the iteration used in the main theorem) since amalgamation necessarily introduces an element of recursion into the definition of this operation.

Let Q be a strong sub-order of P, and let $\pi\colon P \to Q$ be the strong projection. Say $\mathbf{S} = (\ldots, \preccurlyeq^\lambda, \ldots)_{\lambda\in I}$ is a pre-stratification system on P.

Definition 5.28. We call $\wedge$ a *Q-stable meet operator on P* with respect to $\mathbf{S}$ or a *stable meet on* (Q, P) if and only if

1. $\wedge\colon (p, r) \mapsto p \wedge r$ is a function with $\operatorname{dom}(\wedge) \subseteq P^2$ and $\operatorname{ran}(\wedge) \subseteq P$.
2. $\operatorname{dom}(\wedge)$ is the set of pairs $(p, r) \in P^2$ such that $r \leq p$ and
$$\exists\lambda \in I \quad r \preccurlyeq^\lambda \pi(r) \cdot p \tag{5.17}$$
3. Whenever $r \leq p$ and $r \preccurlyeq^\lambda \pi(r) \cdot p$, the following hold:
$$p \wedge r \preccurlyeq^\lambda p \tag{5.18}$$
$$\pi(p \wedge r) = \pi(p) \tag{5.19}$$
$$\pi(r) \cdot (p \wedge r) \approx r \tag{5.20}$$

As usual, we don't mention $\mathbf{S}$ when context permits.

A few remarks are in order to clarify this definition.

- We certainly don't have $p \wedge r = r \wedge p$.
- The gist of (5.17) is that we try to express that $\pi(r)$ forces that in $P : Q$, the "tail" of r is a direct extension (in the sense of $\preccurlyeq^\lambda$) of p; (5.17) captures the essence of this even when $P : Q$ is not stratified.
- Observe that $r \leq p$ implies $\pi(r) \leq \pi(p)$ and so $\pi(r) \cdot p \in P$; thus (5.17) makes sense.
- By $\pi(r) \cdot (p \wedge r) \approx r$ we mean that $\pi(r) \cdot (p \wedge r) \leq r$ and $\pi(r) \cdot (p \wedge r) \geq r$. Admittedly, we are very careful here.
- Observe that there could be more than one map $\wedge$ satisfying the definition. Intuitively, this is because (5.18) is not strong enough to fully determine $p \wedge r$ on $\pi(p) - \pi(r)$. If we add to the above the requirement that $-\pi(r) \cdot (p \wedge r) = p$ hold in r.o.(P), this uniquely determines $\wedge$. In fact this entails
$$p \wedge r = r + (p - \pi(r)) \tag{5.21}$$
in r.o.(P). [4] For our purposes, this point is moot.

[4] In all the applications we have in mind, the natural definition of $\wedge$ satisfies (5.21)—provided we work with the *separative quotient* of P.

To understand the concept of stable meet operators, it is best to consider an instance of such an operator.

Lemma 5.29. *Say $\bar{P} = Q_0 \times Q_1$, and say for each $\lambda \in I$, $\bar{\preccurlyeq}^\lambda$ is obtained from $\preccurlyeq_0^\lambda$ and $\preccurlyeq_1^\lambda$ as in the proof of 5.27 (where of course $\preccurlyeq_i^\lambda \subseteq (Q_i)^2$). Then there is a stable meet operator on $(Q_0, \bar{P})$ with respect to $\bar{\preccurlyeq}^\lambda$.*

Proof. Let π denote the projection to the first coordinate. Define dom($\wedge$) to be the set of pairs prescribed in definition 5.28. Say $r = (r_0, r_1) \in Q_0 \times Q_1$ and $p = (p_0, p_1) \in Q_0 \times Q_1$ are such that $(r, p) \in \mathrm{dom}(\wedge)$. Define

$$(p_0, p_1) \wedge (r_0, r_1) = (p_0, r_1).$$

As $(r, p) \in \mathrm{dom}(\wedge)$, we can fix λ such that $(r_0, r_1) \bar{\preccurlyeq}^\lambda \pi(r) \cdot p = (r_0, p_1)$, and so $r_1 \preccurlyeq_1 p_1$. Thus $(p_0, r_1) \bar{\preccurlyeq}^\lambda (p_0, p_1)$. To check the other properties is left to the reader. □

Lemma 5.30. *Say $\bar{P} = Q * \dot{R}$, and say for each $\lambda \in I$, $\bar{\preccurlyeq}^\lambda$ is obtained from $\preccurlyeq^\lambda$ and $\dot{\preccurlyeq}^\lambda$ as in the proof of 3.17. Then there is a stable meet $\wedge$ on (Q_0, P) with respect to $\bar{\preccurlyeq}^\lambda$.*

Proof. Let π denote the projection to the first coordinate. Again, define dom($\wedge$) to be the set of pairs prescribed in definition 5.28. Say $\bar{r} = (r, \dot{r})$ and $\bar{p} = (p, \dot{p})$ are such that $(\bar{r}, \bar{p}) \in \mathrm{dom}(\wedge)$. Define $\bar{p} \wedge \bar{r} = (p, \dot{r}^*)$, where $\dot{r}^*$ is such that $r \Vdash \dot{r}^* = \dot{r}$ and $-r \Vdash \dot{r}^* = \dot{p}$. Fixing a λ witnessing that $(\bar{r}, \bar{p}) \in \mathrm{dom}(\wedge)$, so that we have $(r, \dot{r}) \bar{\preccurlyeq}^\lambda \pi(\bar{r}) \cdot \bar{p} = (r, \dot{p})$, and so $r \Vdash \dot{r} \dot{\preccurlyeq}^\lambda \dot{p}$. Then $(p, \dot{r}^*) \bar{\preccurlyeq}^\lambda (p, \dot{p})$, since $r \Vdash \dot{r}^* = \dot{r} \dot{\preccurlyeq}^\lambda \dot{p}$ and $p - r \Vdash \dot{r}^* = \dot{p} \dot{\preccurlyeq}^\lambda \dot{p}$. To check the other properties is left to the reader. □

The stable meet operator behaves very nicely in iterations:

Lemma 5.31. *Let $\bar{Q}^{\theta+1}$ be an iteration with diagonal support and say for each $\iota < \theta$, P_ι carries a pre-stratification system $\mathbf{S}_\iota$ on I and*

1. *For all $\iota < \theta$, we have $\mathbf{S}_\iota \lhd \mathbf{S}_{\iota+1}$.*

2. *If $\bar{\iota} \leq \theta$ is limit, $\mathbf{S}_{\bar{\iota}}$ is the natural system of relations on $P_{\bar{\iota}}$.*

Moreover, say for each $\iota < \theta$, there is a stable meet operator $\wedge_\iota^{\iota+1}$ on $(P_\iota, P_{\iota+1})$ with respect to $\mathbf{S}_{\iota+1}$. Then for each $\iota < \theta$ such that $\iota > 0$ there is a P_ι-stable meet operator on P_θ.

Proof. By induction on θ, we show that for each pair ι, η such that $0 < \iota < \eta \leq \theta$, there is a stable meet operator $\wedge_\iota^\eta$ for (P_ι, P_η). For ι, η as above and for p, $r \in P$ such that $r \leq p$ and (5.17) hold, define

$$p \wedge_\iota^\eta r = \prod_{\iota \leq \nu < \eta} \pi_{\nu+1}(p) \wedge_\nu^{\nu+1} \pi_{\nu+1}(r). \tag{5.22}$$

We prove by induction on θ that

1. For ι, η such that $0 < \iota < \eta \leq \theta$ and for $(p, r) \in \mathrm{dom}(\wedge_\iota^\theta)$,

$$\pi_\eta(p \wedge_\iota^\theta r) = \pi_\eta(p) \wedge_\iota^\eta \pi_\eta(r). \tag{5.23}$$

   The sequence of $\pi_\eta(p) \wedge_\iota^\eta \pi_\nu(r)$, for $\eta \in (\iota, \theta]$ determines a thread in P_θ, in the sense of definition 2.6.

2. For ι and η as above, $\wedge_\iota^\eta$ is a stable meet operator on (P_ι, P_η).

Fix $\iota < \theta$. Let $(p, r) \in \mathrm{dom}(\wedge_\iota^\theta)$ be arbitrary and let λ be an arbitrary witness to (5.17). For the rest of the proof let t_ι^η denote $\pi_\eta(p) \wedge_\iota^\eta \pi_\eta(r)$, for $0 < \iota < \eta \leq \theta$.

First assume θ is limit. By induction hypothesis, $(t_\iota^\eta)_{\eta \in (\iota, \theta)}$ is a thread through $\bar{Q}^\theta$; by definition (5.22), this thread is $p \wedge_\iota^\theta r = t_\iota^\theta$. We must show that t_ι^θ has legal support. It suffices to show that for each $\gamma \in I$, $\mathrm{supp}^\gamma(t_\iota^\theta) \subseteq \mathrm{supp}^\gamma(p) \cup \mathrm{supp}^\gamma(r)$. So fix such a γ and a $\xi < \theta$ such that we have

$$\pi_{\xi+1}(p) \preccurlyeq^\gamma \pi_\xi(p), \tag{5.24}$$
$$\pi_{\xi+1}(r) \preccurlyeq^\gamma \pi_\xi(r). \tag{5.25}$$

We have $\pi_{\xi+1}(r) \leq \pi_\xi(r) \cdot \pi_{\xi+1}(p) \leq \pi_\xi(r)$ (simply because $r \leq p$) and so by (C 3) and (5.25), we have $\pi_{\xi+1}(r) \preccurlyeq^\lambda \pi_\xi(r) \cdot \pi_{\xi+1}(p)$. Since $\wedge_\xi^{\xi+1}$ is a stable meet operator, and by (5.24) we have

$$\pi_{\xi+1}(p) \wedge_\xi^{\xi+1} \pi_{\xi+1}(r) \preccurlyeq^\gamma \pi_\xi(p).$$

In other words, $t_\xi^{\xi+1} \preccurlyeq^\gamma \pi_\xi(p)$ and thus, taking the boolean meet with t_ι^ξ on both sides,

$$t_\iota^{\xi+1} = t_\xi^{\xi+1} \cdot t_\iota^\xi \preccurlyeq^\gamma \pi_\xi(p) \cdot t_\iota^\xi = t_\iota^\xi,$$

where the last equation holds since $\wedge_\iota^\xi$ is a stable meet operator by induction. So we have $t_\iota^{\xi+1} \preccurlyeq^\gamma t_\iota^\xi \in P_\xi$. We conclude by lemma 5.8 that $\xi \notin \mathrm{supp}^\gamma(t_\iota^\theta)$, finishing the proof that t_ι^θ has legal support.

It is straightforward to prove equations (5.18), (5.19) and (5.20) for $t_\iota^\theta = p \wedge_\iota^\theta r$, assuming by induction that for each $\eta < \theta$, $\wedge_\iota^\eta$ is a stable meet operator

(t_ι^θ is a thread whose initial segments satisfy these equations). We leave this to the reader.

Now let $\theta = \eta + 1$. To see that $(t_\iota^\nu)_{\nu \leq \theta}$ is a thread, it suffices to show that $\pi_\eta(t_\iota^\theta) = t_\iota^\eta$. In order to show this, observe

$$\pi_\eta(t_\iota^\theta) = t_\iota^\eta \cdot \pi_\eta(t_\eta^\theta) = t_\iota^\eta \cdot \pi_\eta(p) = t_\iota^\eta,$$

where the last equation holds since by induction, $t_\iota^\eta \leq \pi_\eta(p)$. It follows by the induction hypothesis that $(t_\iota^\nu)_{\nu \leq \theta}$ is a thread.

It remains to show that $\wedge_\iota^\theta$ is a P_ι-stable meet on P_θ, i.e we must show (5.18), (5.19) and (5.20). Firstly, by induction,

$$t_\iota^\eta \preccurlyeq^\lambda \pi_\eta(p),$$

and as $\wedge_\eta^\theta$ is a P_η-stable meet on P_θ,

$$\pi_\eta(p) = \pi_\eta(t_\eta^{\eta+1}).$$

By 5.18($\lhd_{\mathrm{s}}$A), this entails

$$t_\iota^\eta \cdot t_\eta^{\eta+1} \preccurlyeq^\lambda t_\eta^{\eta+1}.$$

As $\wedge_\eta^\theta$ is a P_η-stable meet on P_θ, we have $t_\eta^{\eta+1} \preccurlyeq^\lambda p$, whence $t_\iota^\theta = t_\iota^\eta \cdot t_\eta^{\eta+1} \preccurlyeq^\lambda p$, proving (5.18). Secondly,

$$\pi_\iota(t_\iota^\theta) = \pi_\iota(t_\iota^\eta \cdot \pi_\eta(t_\eta^{\eta+1})) = \pi_\iota(t_\iota^\eta) = \pi_\iota(p).$$

The first equality here is trivial. The second holds since $t_\iota^\eta \leq \pi_\eta(p)$ by induction hypothesis and since by the assumption that $\wedge_\eta^{\eta+1}$ is a P_η-stable meet, we have $\pi_\eta(p) = \pi_\eta(t_\eta^{\eta+1})$. The last equality of holds by induction. Finally, we prove (5.20). We have

$$\pi_\iota(r) \cdot t_\iota^\theta = \pi_\iota(r) \cdot t_\iota^\eta \cdot t_\eta^{\eta+1} = \pi_\eta(r) \cdot t_\eta^{\eta+1} = r,$$

where the first equation holds by definition, the second by induction hypothesis, and the last one since $\wedge_\eta^{\eta+1}$ is a P_η-stable meet. We are done with the successor case of the induction, and thus with the inductive proof of the lemma. □

By the lemma, if $\bar{Q}^\theta$ is an iteration as in the hypothesis of the lemma and $\iota < \eta < \theta$, the map $\wedge_\iota^\eta$ is the same as $\wedge_\iota^\theta \restriction (P_\eta)^2$. So as we do for strong projections, we just write $\wedge_\iota$ and we speak of the P_ι-stable meet operator (without specifying the domain). Moreover, we can formally set $p \wedge_0 r = r$ and $p \wedge_\iota r = p$ for $\iota \geq \theta$.

5.5 Remoteness: preserving strong sub-orders

Let C, Q be complete sub-orders of P, and say $\pi_C \colon P \to C$ and $\pi_Q \colon P \to C$ are strong projections. We want to find a sufficient condition to ensure that C is a complete sub-order of $P : Q$, after forcing with Q. In our application C will just be κ-Cohen forcing of L, for κ the least Mahlo. Our iteration will be of the form $P = Q * (\dot{Q}_0 \times C) * \dot{Q}_1$, so after forcing with Q, C is a complete sub-order of $P : Q = (\dot{Q}_0 \times C) * \dot{Q}_1$. We want the same to hold for $\Phi[C]$ (where Φ is a member of a particular family of automorphisms of P which we construct using the technique of amalgamation); this helps to ensure "coding areas" don't get mixed up by the automorphisms, see lemma 7.4 and lemma 8.2. So we have to introduce a property sufficient for C to be a complete sub-order of $P : Q$, in such a way that this condition is inherited by $\Phi[C]$. For this, we use of course the stratification of P. This is necessary since forming $P : Q$ will not only "take away an initial segment" and leave $\Phi[C]$ in the tail in same obvious fashion as for C; instead forming $P : Q$ will also "take away" a small sub-algebra of P (a copy of the random algebra).

Fix a pre-order P which is stratified on I. The following definition is, as usual, relative to a particular pre-stratification system.

Definition 5.32. We say C is *remote in P over Q (up to height κ)* if and only if for all $c \in C$ and $p \in P$ such that $c \leq \pi_C(p)$, we have

1. $p \cdot c \preccurlyeq^\lambda p$ for every $\lambda \in I \cap \kappa$);
2. $\pi_Q(p \cdot c) = \pi_Q(p)$.

Observe that if we drop the first clause, this just says that C is independent in P over Q (see definition 2.4).

For a P-name $\dot{C}$, we say $\dot{C}$ is remote in P over Q if and only if it is a name for a generic of a remote complete sub-order of P; i.e. there is a complete sub-order R_C of P (with a strong projection $\pi_C \colon P \to R_C$) such that R_C is dense in $\langle \dot{C} \rangle^{\mathrm{r.o.}(P)}$ and R_C is remote in P over Q.

Lemma 5.33. *If $\dot{C}$ is a P-name which is remote over Q, then $\dot{C}$ is not in V^Q.*

Proof. An immediate consequence of lemma 2.5 □

Chapter 6

Amalgamation

Amalgamation is a technique to build iterations which admit a homomorphism. We need two types of amalgamations: using type-1 amalgamation, we make sure a stage of our iteration has an automorphism extending an isomorphism of two complete sub-algebras B_0, B_1 of the previous stage of the iteration. Using type-2 amalgamation, we take care that we can extend automorphisms of initial segments (e.g. those created by type-1 amalgamation). The technique presented here differs substantially from that of [She84] (described also in [JR93]) in two important (and related) aspects: firstly, it has a "full support" flavour rather than a "finite support" flavour; secondly, additional fine tuning was needed to allow for amalgamation to preserve stratification (most instances are discussed in detail below).

In 6.1, we define the forcing $P_f^{\mathbb{Z}}$ which will be put to use when we define either type of amalgamation. Before issuing this definition, we pause to analyze $P_f^{\mathbb{Z}}$ and find that it can be decomposed as a product after forcing with B_0 (section 6.2; this will be put to use in lemmas 6.8 and 7.4 to show we can close off Γ^0 under automorphisms). In section 6.3 we define type-1 amalgamation (denoted by $\mathbf{Am}_1$) and show it is a stratified extension (by finding a dense set where "boolean values are stable"; two small details here are the use of "reduced pairs of random reals" in lemma 6.12, and the stable meet operator in lemma 6.14 to preserve that Q is a strong sub-order), and in section 6.4 we do the same for the simpler type-2 amalgamation (denoted by $\mathbf{Am}_2$). In the last section, we construct a stable meet operator for amalgamation and discuss remote sub-orders. There, we prove lemma 6.30 which helps to ensure "coding areas" don't get mixed up by the automorphisms (it will be put to use in lemmas 7.4 and 8.2). Also, we show that indeed there is a stable meet for amalgamation, which completes the inductive proof that there is a stable meet for each stage of the iteration.

6.1 Basic amalgamation

Let P be a forcing, Q a complete sub-order of P such that $\pi\colon P \to Q$ is a strong projection (see 2.3, p. 8 and the preceding discussion). For $i \in \{0,1\}$, let $\dot{B}_i$ be a Q-name such that $\Vdash_Q \dot{B}_i$ is a complete sub-algebra of $P : Q$. Moreover, say we have a Q-name $\dot{f}$ such that $\Vdash_Q \dot{f}\colon \dot{B}_0 \to \dot{B}_1$ is an isomorphism of Boolean algebras.

Our task is to find P' containing P as a complete sub-order, carrying an automorphism $\Phi\colon P' \to P'$ which extends the isomorphism of $\dot{B}_0$ and $\dot{B}_1$ (in the extension by Q) and which is trivial on Q. Moreover, we want to preserve stratification: say (Q,P) is a stratified extension on $I = [\lambda_0, \kappa) \cap \mathbf{Reg}$, where we allow $\kappa = \infty$. We want $\lambda_1 < \kappa$ (possibly strictly) greater than λ_0 such that (P', P) is a stratified extension above λ_1. We shall assume for this purpose that κ is a limit cardinal (or ∞).

We first make some observations: Let r.o.$(Q) * \dot{B}_i$ be denoted by B_i. This is a complete sub-algebra of $B = \text{r.o.}(P)$, consisting Q-names (or if you prefer, r.o.(Q)-names) $\dot{b}$ such that $1_Q \Vdash_Q \dot{b} \in \dot{B}_i$. Keep in mind that we can canonically identify the partial order $Q*(\dot{B}_i \setminus \{0\})$ with the set of $b \in B_0$ such that $\pi_Q(b) \in Q$. Also, don't confuse this with the set of $b \in B_0$ such that $\pi_Q(b) = 1$—or, equivalently, $1_Q \Vdash_Q [b]_{\dot{G}} > 0$, which is called the term-forcing, usually denoted by $(\dot{B}_i \setminus \{0\})^Q$.

Let π_i denote the canonical projection from P to B_i. Then π_i coincides with π on Q (by 2.2). Moreover, $\dot{f}$ can be viewed as an isomorphism f of B_0 and B_1 (mapping names to names). We have

$$\pi \circ f = f \circ \pi = \pi. \tag{6.1}$$

In fact, for any pair of sub-algebras B_0, B_1 of r.o.(P) such that $Q \subseteq B_0 \cap B_1$ and an isomorphism $f\colon B_0 \to B_1$, equation (6.1) holds if and only if f generates an isomorphism of the pair $B_i : Q$, $i \in \{0,1\}$ in any Q-generic extension. Thus instead of starting with $\dot{f}$ and $\dot{B}_0$, $\dot{B}_1$ as in the first paragraph, we could also have started with f, B_0 and B_1 as above, satisfying (6.1).

In a first step, we define $P^{\mathbb{Z}}_f$, the amalgamation of P over f. $P^{\mathbb{Z}}_f$ contains P as a complete sub-order and has an automorphism Φ extending f.

Remark 6.1. If we want to preserve stratification of P, we have to be more careful: we must carefully pick a dense subset $\mathbf{D}$ of P, such that $P' = \mathbf{D}^{\mathbb{Z}}_f$ is stratified. The partial order $\mathbf{D}^{\mathbb{Z}}_f$ is in general not equivalent to $P^{\mathbb{Z}}_f$, but solves the problem described in the first paragraph. Finally, we will define a forcing $\mathbf{Am}_1$ which is equivalent to $\mathbf{D}^{\mathbb{Z}}_f$, and moreover $(P, \mathbf{Am}_1)$ is a stratified extension. Let's postpone these complications, and first look at $P^{\mathbb{Z}}_f$.

Amalgamation is not a canonical operation. Firstly, if D is a dense subset of P, we cannot infer that $D^{\mathbb{Z}}_f$ is dense in $P^{\mathbb{Z}}_f$. This in combination with the fact that stratification is also not canonical is the main obstacle in this proof. Secondly, even the weaker statement fails: if $\text{r.o.}(P) = \text{r.o.}(R)$, we cannot conclude $\text{r.o.}(P^{\mathbb{Z}}_f) = \text{r.o.}(R^{\mathbb{Z}}_f)$.

Without precautions, we cannot even preclude $P^{\mathbb{Z}}_f = \{1_P\}$, although this pathology does not arise if we ask $B_0 \cup B_1 \subseteq P$. On the other hand, we cannot simply work with $\text{r.o.}(P)$; for although $\text{r.o.}(P)$ has a dense stratified subset (namely P), this doesn't mean that $\text{r.o.}(P)^{\mathbb{Z}}_f$ will have a dense stratified subset. Therefore, we want to stick as closely to P as possible, but still have $B_0, B_1 \subseteq P$, so we define a "hybrid":

Definition 6.2. Consider the set $P \times B_0 \times B_1$, i.e. the set of triples $(p, \dot{b}^0, \dot{b}^1)$ where $p \in P$ and $\Vdash_Q \dot{b}^i \in \dot{B}_i$ for $i \leq 2$. Order this set by $(p, \dot{b}^0, \dot{b}^1) \leq (q, \dot{d}^0, \dot{d}^0)$ *if and only if* $p \leq q$ and $p \cdot \dot{b}^0 \cdot \dot{b}^1 \leq q \cdot \dot{d}^0 \cdot \dot{d}^1$ in $\text{r.o.}(P)$. This makes sense since we can canonically identify $\dot{b}^j, \dot{d}^j$ with elements of B_j. We call $\widehat{P} = \widehat{P}(Q, f)$ the set of $(p, \dot{b}^0, \dot{b}^1) \in P \times B_0 \times B_1$ such that

$$\pi(p) \Vdash p \cdot \dot{b}^0 \cdot \dot{b}^1 \neq 0, \tag{6.2}$$

or equivalently,

$$\pi(p \cdot \dot{b}^0 \cdot \dot{b}^1) = \pi(p). \tag{6.3}$$

For $\hat{p} \in \widehat{P}$, when we refer to the components of $\hat{p}$, we use the notation $\hat{p} = (\hat{p}^P, \hat{p}^0, \hat{p}^1)$. When appropriate, we identify $\hat{p}$ with $\hat{p}^P \cdot \hat{p}^0 \cdot \hat{p}^1$, i.e. the meet of the components in $\text{r.o.}(P)$. In particular, if g is a function such that $\text{dom}(g) = \text{r.o.}(P)$, we write $g(\hat{p})$ for $g(\hat{p}^P \cdot \hat{p}^0 \cdot \hat{p}^1)$.

Clearly, P is isomorphic to the subset of $\widehat{P}$ where the two latter components are equal to $1_{\text{r.o.}(P)}$, and this set is in turn dense in $\widehat{P}$. So P can be considered a dense subset of $\widehat{P}$. Thus, the separative quotient of $\widehat{P}$ is the completion under $\cdot$ of $P \cup B_0 \cup B_1$ in $\text{r.o.}(P)$ (leaving aside the 0 element). Observe, moreover, that if $D \subseteq P$ is dense in P, then $\{\hat{p} \in \widehat{P} \mid \hat{p}^P \in D\}$ is the same as $\widehat{D}$, and we shall often use this fact tacitly. Lastly, observe that

$$\hat{p} \leq \hat{q} \iff \left[\ \hat{p}^P \leq \hat{q}^P \text{ and } \pi_j(\hat{p}) \leq \pi_j(\hat{q}) \text{ for } j \in \{0,1\}\ \right] \tag{6.4}$$

and $\hat{p} \approx (\hat{p}^P, \pi_0(\hat{p}), \pi_1(\hat{p}))$.[1] These two observations together would make for an equivalent, more strict definition of $\widehat{P}$, yielding separative $\widehat{P}$ provided P is separative. Notwithstanding, we find the current definition more

[1] We may regard $(\hat{p}^P, \pi_0(\hat{p}), \pi_1(\hat{p}))$ the canonical representative of $\hat{p}$ if P is separative.

convenient—if less elegant. In the following, we identify P with $\{\hat{p} \in \widehat{P} \mid \hat{p}^0 = \hat{p}^1 = 1\}$.

Much of the following would work if we replace (6.3) by the weaker $p \cdot \dot{b}_0 \cdot \dot{b}_1 \neq 0$. The advantage of asking (6.3) is that it makes the projection $\bar{\pi}\colon P_f^{\mathbb{Z}} \to P$ take a simple form.

Definition 6.3. We define $P_f^{\mathbb{Z}}$ to consist of all sequences $\bar{p} : \mathbb{Z} \to \widehat{P}$ such that for all but finitely many k we have $\bar{p}(k)^P \preccurlyeq^{\lambda_0} \pi(\bar{p}(k)^P)$ and for all k we have

$$f(\pi_0(\bar{p}(k+1)^P \cdot \bar{p}(k+1)^0 \cdot \bar{p}(k+1)^1)) = \pi_1(\bar{p}(k)^P \cdot \bar{p}(k)^0 \cdot \bar{p}(k)^1),$$

or, simply

$$f(\pi_0(\bar{p}(k+1))) = \pi_1(\bar{p}(k)). \tag{6.5}$$

The ordering on $P_f^{\mathbb{Z}}$ is given by $\bar{r} \leq \bar{p}$ if and only if for all k, $\bar{r}(k) \leq \bar{p}(k)$ in $\widehat{P}$. We define a map $\Phi\colon P_f^{\mathbb{Z}} \to P_f^{\mathbb{Z}}$ by:

$$\Phi(\bar{p})(i) = \bar{p}(i+1) \text{ for } i \in \mathbb{Z}.$$

Obviously, Φ is one-to-one and onto, and $\Phi(\bar{p}) \leq \Phi(\bar{q}) \iff \bar{p} \leq \bar{q}$.

Observe that (6.1) together with (6.5) and (6.3) imply that for all $i \in \mathbb{Z}$,

$$\pi(\bar{p}(i)) = \pi(\bar{p}(0)) = \pi(\bar{p}(0)^P). \tag{6.6}$$

Let $F\colon \widehat{P} \to B_1$ be defined by $F(x) = f(\pi_0(x))$ and let $G\colon \widehat{P} \to B_0$ be defined by $G(x) = f^{-1}(\pi_1(x))$.

It may seem more natural to replace (6.5) by the weaker requirement that $f(\pi_0(p(k+1)))$ and $\pi_1(p(k))$ be compatible; however, I'm not sure how to show P is a complete sub-order in this case. Moreover, (6.6) simplifies the proof that **D** remains dense in a stratified extension, as it allows to build conditions in **D** in a coherent way, that is, without changing initial segments over and over again (see (E$_\mathrm{c}$I)).

We now define a complete embedding $e\colon \widehat{P} \to P_f^{\mathbb{Z}}$ and a strong projection $\bar{\pi}\colon P_f^{\mathbb{Z}} \to \widehat{P}$. For $\hat{u} \in \widehat{P}$ define $e(\hat{u})\colon \mathbb{Z} \to \widehat{P}$ by

$$e(\hat{u})(i) = \begin{cases} (\pi(\hat{u}^P), G^i(\hat{u}), 1) & \text{for } i > 0, \\ \hat{u} & \text{for } i = 0, \\ (\pi(\hat{u}^P), 1, F^i(\hat{u})) & \text{for } i < 0. \end{cases}$$

For $\bar{p} \in P_f^{\mathbb{Z}}$, define $\bar{\pi}(\bar{p}) \in \widehat{P}$ by $\bar{\pi}(\bar{p}) = \bar{p}(0)$.

Lemma 6.4. *The map $\bar{\pi}$ is a strong projection, that is: if $\hat{w} \leq \bar{\pi}(\bar{q})$ in $\widehat{P}$, we may find $e(\hat{w}) \cdot \bar{q} \in P_f^{\mathbb{Z}}$.*

Proof. Let $\hat{w} \leq \bar{\pi}(\bar{p})$. We define $\bar{w}$ by induction, as follows:

$$\bar{w}(0) = \hat{w}$$

Assume $\bar{w}(i) \in \widehat{P}$ has already been defined. We know $\pi(\bar{w}(i)) = \pi(\bar{w}(i)^P)$. Assume by induction that $\pi(\bar{w}(i)^P) = \pi(\hat{w}^P)$. Also, assume by induction that $\bar{w}(i) \leq \bar{p}(i)$ and $\bar{w}(i) \leq e(\hat{w})(i)$ in $\widehat{P}$. To inductively define $\bar{w}$ on the positive integers, assume $i \geq 0$ and define:

$$\bar{w}(i+1) = (\pi(\hat{w}) \cdot \bar{p}(i+1)^P, \bar{p}(i+1)^0, \bar{p}(i+1)^1 \cdot F(\bar{w}(i))).$$

The definition of $\bar{w}$ on the negative integers is also by induction. Assuming $i \leq 0$, we set:

$$\bar{w}(i-1) = (\pi(\hat{w}) \cdot \bar{p}(i-1)^P, \bar{p}(i-1)^0 \cdot G(\bar{w}(i)), \bar{p}(i-1)^1)$$

For $i \geq 0$, as $\bar{w}(i) \leq \bar{p}(i)$, we have

$$\begin{aligned} f(\pi_0(\bar{w}(i))) = F(\bar{w}(i)) \cdot f(\pi_0(\bar{p}(i))) \\ = F(\bar{w}(i)) \cdot \pi_1(\bar{p}(i+1)) \\ = \pi_1(\pi(\hat{w}^P) \cdot \bar{p}(i+1) \cdot F(\bar{w}(i)) \end{aligned}$$

where the second equation holds as (6.5) holds for $\bar{p}$, and the last equation follows from $F(\bar{w}(i)) \leq \pi(\bar{w}(i)) = \pi(\hat{w}^P)$. We conclude, by definition of $\bar{w}(i+1)$, that

$$f(\pi_0(\bar{w}(i))) = \pi_1(\bar{w}(i+1)). \tag{6.7}$$

Applying π to (6.7), we see $\pi(\bar{w}(i+1)) = \pi(\bar{w}(i))$, and so

$$\pi(\bar{w}(i+1)) = \pi(\hat{w}^P) = \\ \pi(\pi(\hat{w}^P) \cdot \bar{p}(i+1)^P) = \pi(\bar{w}(i+1)^P),$$

where the first equation follows from the induction hypothesis and the second follows from

$$\pi(\hat{w}^P) \leq \pi(\bar{p}(0)^P) = \pi(\bar{p}(i+1)^P).$$

Thus, $\bar{w}(i+1) \in \widehat{P}$, $\pi(\bar{w}(i)) = \pi(\hat{w}^P)$ and by construction, both $\bar{w}(i+1) \leq \bar{p}(i+1)$ and $\bar{w}(i+1) \leq e(\hat{w}^P)(i+1)$ hold.

Replacing F by G in the above, we obtain a similar argument for the inductive step from $i \leq 0$ to $i-1$; we leave the details to the reader. Finally we have that $\bar{w}(i) \in \widehat{P}$ and (6.7) holds for all $i \in \mathbb{Z}$, whence $\bar{w} \in P_f^{\mathbb{Z}}$. We have already shown $\bar{w} \leq \bar{p}$ and $\bar{w} \leq e(\hat{w})$.

We now show $\bar{w} \geq e(\hat{w}) \cdot \bar{p}$: Say $\bar{r} \in P_f^{\mathbb{Z}}$ such that $\bar{r} \leq e(\hat{w}) \cdot \bar{p}$. Clearly $\bar{r}(0) \leq \bar{w}(0) = w$. Now assume by induction that $\bar{r}(i) \leq \bar{w}(i)$. Then by (6.5),

$$\bar{r}(i+1) \leq \pi_1(\bar{r}(i+1)) \leq F(\bar{w}(i))$$

so as $\bar{r}(i+1) \leq \bar{p}(i+1)$, we have $\bar{r}(i+1) \leq \bar{w}(i+1)$.

A similar argument shows $\bar{r}(i-1) \leq \bar{w}(i-1)$, so we we've shown by induction that $\bar{r} \leq \bar{w}$. So finally, $\bar{w} = e(\hat{w}) \cdot \bar{p}$. □

For $i \in \mathbb{Z}$, we write e_i for $\Phi^i \circ e$ and $\bar{\pi}_i$ for $\bar{\pi} \circ \Phi^i$.

Corollary 6.5. *For each $i \in \mathbb{Z}$, the map e_i is a complete embedding of $\widehat{P}$ into $P_f^{\mathbb{Z}}$. It is well-defined and injective on the separative quotient of $\widehat{P}$. The map $\bar{\pi}_i \colon P_f^{\mathbb{Z}} \to \widehat{P}$ is a strong projection. The map $e_i \restriction P$ is a complete embedding of P into $P_f^{\mathbb{Z}}$. Letting $R = \{\bar{p} \in P_f^{\mathbb{Z}} \mid \bar{p}(i)^0 = \bar{p}(i)^1 = 1\}$, R is dense in $P_f^{\mathbb{Z}}$, we have $e_i[P] \subseteq R$ and $\bar{\pi}_i \restriction R \colon R \to P$ is a strong projection.*

Proof. The first claim is an obvious corollary of the lemma. The rest follows straightforwardly from elementary properties of e and $\bar{\pi}$. □

From now on, we identify $\widehat{P}$ with $e[\widehat{P}]$ and accordingly P with $\{e(p,1,1) \mid p \in P\}$.

Corollary 6.6. *Φ is an automorphism of $P_f^{\mathbb{Z}}$ extending f.*

Proof. Let $b \in B_0$. We may assume $\pi(b) \in Q$ (this holds for a dense set of conditions in B_0). Thus $b \in \widehat{P}$ (to be precise, we should write $(\pi(b), b, 1)$ instead of b). Now as $F^n(f(b)) = F^{n+1}(b)$ and $G^{n+1}(f(b)) = G^n(b)$,

$$\Phi(e(b)) = \Phi((\ldots, G^2(b), G(b), \overset{\overset{0}{\downarrow}}{b}, f(b), F^2(b), \ldots)) =$$
$$(\ldots, G^2(b), G(b), b, \overset{\overset{0}{\downarrow}}{f(b)}, F^2(b), \ldots) = e(f(b))$$

So since Φ and f agree on a dense set of conditions in B_0, they are equal on B_0. □

6.2 Factoring the amalgamation

Interestingly, we can factor the amalgamation over a generic for B_0. We will put this to use when we investigate the tail $\mathbf{Am}_1 \colon P$. In particular, it enables us to show that if $\dot{r}$ is a P-name which is unbounded over V^Q, $\Phi(\dot{r})$ will be unbounded not just over V^Q but over V^P. This will play a crucial role in the proof of the main theorem, ensuring that when we make the set without the Baire property definable, the coding (ensuring its definability) doesn't conflict with the homogeneity afforded by the automorphisms. The main point of the present section is lemma 6.8; it is used in section 7.3 on p. 123, to prove lemma 7.4. This is in turn used in section 8 to prove the crucial lemma 8.2.

For an interval $I \subseteq \mathbb{Z}$, let P_f^I be the set of $\bar{p}\colon I \to \widehat{P}$ such that whenever both $k \in I$ and $k+1 \in I$, (6.5) holds. In other words

$$P_f^I = \{\bar{p} \restriction I \mid \bar{p} \in P_f^{\mathbb{Z}}\}.$$

It is clear that for each $k \in I$, the map $e_k^I\colon \widehat{P} \to P_f^I$, defined by $e_k^I(p) = e_k(p) \restriction I$ is a complete embedding. Similarly, there is a strong projection $\pi_k^I\colon P_f^I \to \widehat{P}$.

Lemma 6.7. *Let $G_0 = G_Q * H_0$ be $Q * \dot{B}_0$-generic. Then in $V[G_0]$, there is a dense embedding of $P_f^{\mathbb{Z}} : G_0$ into*

$$\left[P_f^{(-\infty,0]} : e_0[G_Q * H_0]\right] \times \left[P_f^{[1,\infty)} : e_1[G_Q * f[H_0]]\right]$$

and another one into

$$\left[P_f^{(-\infty,-1]} : e_{-1}[G_Q * H_0]\right] \times \left[P_f^{[0,\infty)} : e_0[G_Q * f[H_0]]\right].$$

Proof. We only show how to construct the first embedding; the second part of the proof is only different in notation. Let R_0 denote $P_f^{(-\infty,0]}$ and R_1 denote $P_f^{[1,-\infty)}$, let $H_1 = f[H_0]$ and $G_1 = G_Q * H_1$. In V, let S denote the obvious map $S\colon P_f^{\mathbb{Z}} \to R_0 \times R_1$: $S(\bar{p})_0 = \bar{p} \restriction (-\infty, 0]$ and $S(\bar{p})_1 = \bar{p} \restriction [1,\infty)$.

Let $S^* = S \restriction (P_f^{\mathbb{Z}} : G_0)$. We show that the range of S^* is dense in $(R_0 : G_0) \times (R_1 : G_1)$. Since $S(\bar{p}) \leq S(\bar{q}) \iff \bar{p} \leq \bar{q}$, this implies that S^* is injective on the separative quotient of its domain and thus is a dense embedding.

To show that $\operatorname{ran}(S^*)$ is dense, let $\bar{p}_0$, $\bar{p}_1$ be given such that $\bar{p}_i \in R_i : G_i$, for $i \in \{0,1\}$. Fix $i \in \{0,1\}$ for the moment. Without loss of generality, $\bar{p}_i(i) \in P$ (and not just in $\widehat{P}$). Let $b_i = \pi_i(\bar{p}_i(i)) \in \dot{B}_i^{G_Q}$. Then as $\bar{p}_i \in R_i : G_i$, $b_i \in H_i$. Find $q \in G_Q$ and a Q-name $\dot{b}$ such that $q \leq \pi_Q(\bar{p}_0), \pi_Q(\bar{p}_1)$ and q forces that

$$\dot{b} = b_0 \cdot f^{-1}(b_1) > 0 \quad \text{in } (\dot{B}_0)^{G_Q}. \tag{6.8}$$

We have that $q \Vdash \dot{b} \cdot \bar{p}_0(0) \neq 0$ and $f(\dot{b}) \cdot \bar{p}_1(1) \neq 0$, or in other words, $q \cdot \dot{b} \in \widehat{P}$, $q \cdot \dot{b} \leq \bar{\pi}(\bar{p}_0)$ and $q \cdot f(\dot{b}) \leq \bar{\pi}(\bar{p}_1)$. So we can define $\bar{p}_0^* = e(q \cdot \dot{b}) \cdot \bar{p}_0$ and $\bar{p}_1^* = e_1(q \cdot f(\dot{b})) \cdot \bar{p}_1$. As

$$q \in G_Q \text{ and } \dot{b}^{G_Q} \in H_0, \tag{6.9}$$

we have $p_0^* \in R_0 : G_0$ and $p_1^* \in R_1 : G_1$. Define $\bar{p}^*$:

$$\bar{p}^* = (\dots, \bar{p}_0^*(-1), \bar{p}_0^*(0), \bar{p}_1^*(1), \bar{p}_1^*(2), \dots)$$

Then $\pi_Q(\bar{p}^*) = q$ and by (6.8), q forces that the following hold in $(\dot{B}_0)^{G_Q}$:

$$f(\pi_0(\bar{p}_0^*(0))) = f(\pi_0(q \cdot \bar{p}_0(0) \cdot \dot{b})) = f(\dot{b})$$
$$\pi_1(\bar{p}_1^*(1)) = \pi_1(q \cdot \bar{p}_1(1) \cdot f(\dot{b})) = f(\dot{b}).$$

Thus $\bar{p}^* \in D_f^{\mathbb{Z}}$, and again by (6.9), $\bar{p}^* \in D_f^{\mathbb{Z}} : G_0$. As $S^*(\bar{p}^*) = (\bar{p}_0^*, \bar{p}_1^*) \leq (\bar{p}_0, \bar{p}_1)$, we are done. □

Let $\dot{R}_i$ be a $Q * \dot{B}_0$-name for R_i, for each $i \in \{0, 1\}$. We just showed that $Q * \dot{B}_0$ forces that there is a dense embedding from $P_f^{\mathbb{Z}} : G_0$ into $\dot{R}_0 \times \dot{R}_1$. So there is a dense embedding of $P_f^{\mathbb{Z}}$ into $Q * \dot{B}_0 * (\dot{R}_0 \times \dot{R}_1)$. Since the latter is equivalent to $P_f^{(-\infty,0]} * \dot{R}$ for some $\dot{R}$, we find that $P_f^{(-\infty,0]}$ is a complete sub-order of $P_f^{\mathbb{Z}}$. The same is true for $P_f^{[0,\infty)}$ (or more generally, for P_f^I, where I is any interval in $\mathbb{Z}$). In fact, it's easy to show that the natural embedding and projection witness this.

The previous lemma affords insight concerning the action of the automorphism Φ. E.g. it enables us to show that if $\dot{x}$ is a P-name which is not in V^{B_0} (and hence also not in V^{B_1}), then for all $i \in \mathbb{Z} \setminus \{0\}$, $\Phi(\dot{x}) \notin V^P$. In fact, for the proof of the main theorem, we shall need something a bit more specific:

Lemma 6.8. *Assume that $\dot{r}_0, \dot{r}_1$ be P-names for reals random over V^Q, and assume $\Vdash_Q \dot{B}_i = \langle \dot{r}_i \rangle^{P:Q}$ (as is the case in our application). If $\dot{r}$ is a P-name for a real such that $\dot{r}$ is unbounded over V^Q, then for any $i \in \mathbb{Z} \setminus \{0\}$, $\Phi^i(\dot{r})$ unbounded over V^P.*

Proof. Firstly, $\dot{r}$ is unbounded over V^{B_i}, for each $i \in \{0, 1\}$, since the random algebra does not add unbounded reals. For a start, let's assume $i = 1$.

Let $G_1 = G_Q * f[H_0]$ be $Q * \dot{B}_1$-generic and work in $W = V[G_1]$. We have that $\dot{r}$ is a $P : G_1$ name for a real which is unbounded over W in the sense of definition 2.8—in any $P : G_1$-generic extension of $V[G_1]$, the interpretation of $\dot{r}$ will be unbounded over $V[G_1]$. Let R_0, R_1 be defined as in the previous proof, i.e.

$$R_1 = P_f^{[1,\infty)} : G_1,$$
$$R_0 = P_f^{(-\infty,0]} : (G_Q * H_0),$$

let $\dot{R}_i$ be a $Q * \dot{B}_0$-name for R_i, for each $i \in \{0, 1\}$, and let $I = [1, \infty)$. As $P : G_1$ is a complete sub-order of R_1, $e_1^I(\dot{r})$ is an R_1-name which is unbounded over W. By lemma 2.9, viewing $e_1^I(\dot{r})$ as a $R_0 \times R_1$-name, it is unbounded over W^{R_0}. As G_1 was arbitrary, $e_1^I(\dot{r})$ is a $Q * \dot{B}_0 * (\dot{R}_1 \times \dot{R}_0)$-name unbounded over $V^{Q*\dot{B}_0*\dot{R}_0}$. By the previous theorem this means that $e_1(\dot{r})$ is a $P_f^{\mathbb{Z}}$-name unbounded over $V^{Q*\dot{B}_0*\dot{R}_0} = V^{P_f^{(-\infty,0]}}$ and hence over V^P, since $S \circ e_0 = e_0^I$ shows that P is a complete sub-order of $Q * \dot{B}_0 * \dot{R}_0$.

For arbitrary $i \in \mathbb{Z}$ such that $i > 0$: We just showed that $e_1(\dot{r})$ is a $P_f^{\mathbb{Z}}$-name unbounded over $V^{P_f^{(-\infty,0]}}$. Since $e_{-i+1}^I[P]$ is a complete sub-order of $P_f^{(-\infty,0]}$, we know $e_1(\dot{r})$ is unbounded over $V^{e_{-i+1}[P]}$. Apply Φ^{i-1} to see $\Phi^i(\dot{r})$ is unbounded over $V^{e_0[P]}$, as $e_0 = \Phi^{i-1} \circ e_{-i+1}$. For $i < 0$, argue exactly as above but use the second dense embedding mentioned in lemma 6.7. □

6.3 Stratified type-1 amalgamation

We now turn to the matter of stratification. Assume (Q, P) is a stratified extension on $I = [\lambda_0, \kappa) \cap \mathbf{Reg}$, as witnessed by

$$\mathbf{S}_Q = (\mathbf{D}_Q, c_Q, \preccurlyeq_Q^\lambda, \preccurlyeq\!\!\!\!\cdot_Q^\lambda, \mathbf{C}_Q^\lambda)_{\lambda \in I}$$

and $\mathbf{S}_P = (\mathbf{D}_P, c_P, \preccurlyeq^\lambda, \preccurlyeq\!\!\!\!\cdot^\lambda, \mathbf{C}^\lambda)_{\lambda \in I}$. As mentioned, we allow κ to be ∞, as well. We never need to mention $\preccurlyeq_Q^\lambda$, $\preccurlyeq\!\!\!\!\cdot_Q^\lambda$, $\mathbf{C}_Q^\lambda$ and $\mathbf{D}_Q$ as we can always use the corresponding relation from $\mathbf{S}_P$ (see the remark following definition 5.18, p. 73). Moreover, assume $\Vdash_Q |\dot{B}_0| \leq \lambda_0$.

The main problem with stratification and amalgamation is quasi-closure: Consider two sequences $(p_\xi)_{\xi<\rho}$ and $(q_\xi)_{\xi<\rho}$ such that p_ξ and q_ξ are compatible for every $\xi < \rho$, with greatest lower bounds p and q respectively. In general, p and q don't have to be compatible. A similar problem occurs with regard to the defining equation (6.5) of amalgamation: say we have a sequence of conditions $\bar{p}_\xi \in \mathbf{Am}_1$ and for each $i \in \mathbb{Z}$, $\bar{p}(i)$ is a greatest lower bound of $(\bar{p}_\xi(i))_\xi$. Even though (6.5) holds for every $\bar{p}_\xi$, it could fail for $\bar{p}$.

The solution to this problem is to thin out to a dense subset of P where π_i is stable with respect to "direct extension", before we amalgamate. That is, on this dense subset, π_i doesn't change (in a strong sense) when conditions are extended in the sense of $\preccurlyeq^\lambda$, for $\lambda \in I$.

Definition 6.9. Let $\mathcal{D} = \mathcal{D}(Q, P, f, \lambda_0)$ be the set of $p \in P$ such that for all $q \in P$, if $q \preccurlyeq^{\lambda_0} p$ we have

$$\forall (b_0, b_1) \in B_0 \times B_1 \quad \big(\pi(q) \cdot p \cdot b_0 \cdot b_1 \neq 0\big) \Rightarrow \big(q \cdot b_0 \cdot b_1 \neq 0\big) \tag{6.10}$$

Observe that (6.10) is equivalent to:

$$\forall j \in \{0,1\} \quad \pi(q) \Vdash_Q \forall b \in \dot{B}_{1-j} \quad \pi_j(q \cdot b) = \pi_j(p \cdot b), \tag{6.11}$$

and also to the following:

$$\forall j \in \{0,1\} \quad \forall b \in B_{1-j} \quad \pi_j(q \cdot b) = \pi(q) \cdot \pi_j(p \cdot b). \tag{6.12}$$

Lemma 6.10. *$\mathcal{D}$ is open dense in $\langle P, \preccurlyeq^{\lambda_0} \rangle$.*

Proof. Let p_0 be given. We inductively construct an adequate sequence of p_ξ, $0 < \xi \leq \lambda_0$ with $p_{\lambda_0} \in \mathcal{D}$. First fix x such that the following definition is Π^T_1 in parameters from x. Fix Q-names $\dot{b}_j$ such that $\Vdash_Q \dot{b}_j \colon \lambda_0 \to \dot{B}_j$ is onto, for $j = 0, 1$, and let $\xi \mapsto (\alpha_\xi, \beta_\xi, \zeta_\xi)$ be a surjection from λ_0 onto $(\lambda_0)^3$.

For limit ξ, let p_ξ be the greatest lower bound of the sequence constructed so far. Say we have constructed p_ξ, we shall define $p_{\xi+1}$. Let's first assume there are $p^*, \bar{p}$ such that $\bar{p} \preccurlyeq^{\lambda_0} p_\xi$, $\bar{p} \in \mathbf{D}(\lambda_0, x, p_\xi)$, $p^* \leq \bar{p}$ and

1. $\pi(p^*) \Vdash \bar{p} \cdot \dot{b}_0(\alpha_\xi) \cdot \dot{b}_1(\beta_\xi) = 0$,
2. $\zeta_\xi \in \mathbf{C}^{\lambda_0}(p^*)$.

In this case pick $p_{\xi+1}$ such that $p_{\xi+1} \preccurlyeq^{\lambda_0} \bar{p}$ and $p_{\xi+1} \mathrel{\dot{\preccurlyeq}}^{\lambda_0} p^*$ (using interpolation). If, on the other hand, no such $\bar{p}, p^*$ exist, just pick $p_{\xi+1} \in \mathbf{D}(\lambda_0, x, p_\xi)$.

We now show (6.11) holds for the final condition p_{λ_0}: say, to the contrary, we can find $j \in \{0,1\}$ and $\dot{b} \in B_{1-j}$ together with $\bar{q} \preccurlyeq^{\lambda_0} p_{\lambda_0}$ such that

$$\pi(\bar{q}) \nVdash_Q \pi_j(\bar{q} \cdot \dot{b}) = \pi_j(p_{\lambda_0} \cdot \dot{b}).$$

Without loss of generality say $j = 0$. We can find $q^* \leq \bar{q}$ such that for some $\alpha, \beta < \lambda_0$

(i) $\pi(q^*) \Vdash \pi_0(p_{\lambda_0} \cdot \dot{b}) - \pi_0(\bar{q} \cdot \dot{b}) = \dot{b}_0(\alpha) \neq 0$,

(ii) $\pi(q^*) \Vdash \dot{b} = \dot{b}_1(\beta)$,

(iii) $q^* \in \mathrm{dom}(\mathbf{C}^{\lambda_0})$.

Find $\xi < \lambda_0$ so that $\alpha = \alpha_\xi$, $\beta = \beta_\xi$ and $\zeta_\xi \in \mathbf{C}^{\lambda_0}(q^*)$. By construction, at stage ξ of our construction we had $\bar{p}$ and p^* satisfying (1) and (2). As $\mathbf{C}^{\lambda_0}(p^*) \cap \mathbf{C}^{\lambda_0}(q^*) \neq 0$ and $q^* \leq p_{\xi+1} \mathrel{\dot{\preccurlyeq}}^{\lambda_0} p^*$, we can find $w \leq p^*, q^*$. But by ((i)) and ((ii)), $\pi(w) \Vdash p_{\lambda_0} \cdot \dot{b}_0(\alpha) \cdot \dot{b}_1(\beta) \neq 0$. But since $w \leq p^*$, $\pi(w) \Vdash \bar{p} \cdot \dot{b}_0(\alpha) \cdot \dot{b}_1(\beta) = 0$ and so also $\pi(w) \Vdash p_{\lambda_0} \cdot \dot{b}_0(\alpha) \cdot \dot{b}_1(\beta) = 0$, contradiction.

Now we show $\mathcal{D}$ is open: For any $r \preccurlyeq^{\lambda_0} q$, $j = 0, 1$ and $\dot{b} \in B_{1-j}$, since $r \preccurlyeq^{\lambda_0} p$, we have $\pi(r) \Vdash \pi_j(r \cdot \dot{b}) = \pi_j(p \cdot \dot{b})$. Since $\pi(q) \Vdash \pi_j(q \cdot \dot{b}) = \pi_j(p \cdot \dot{b})$ and $r \leq q$, $\pi(r) \Vdash \pi_j(r \cdot \dot{b}) = \pi_j(q \cdot \dot{b})$. So $q \in \mathcal{D}$. ∎

Having $Q \subseteq \mathcal{D}$ helps in many circumstances, in particular we like to have $1_P \in \mathcal{D}$. To this end we introduce the notion of B_0, B_1 being λ_0-reduced.

Definition 6.11. We say the pair B_0, B_1 is λ-reduced over Q if and only if whenever $p \in P$, $p \preccurlyeq^\lambda q$ for some $q \in Q$ and $b \in B_j$ for $j = 0$ or $j = 1$, we have

$$\pi_{1-j}(p \cdot b) = \pi(p) \cdot \pi(b).$$

Henceforth assume B_0, B_1 is a λ_0-reduced pair. We will later see that this is a very mild assumption, see lemmas 6.13 and 7.3.

Lemma 6.12. *If $p \preccurlyeq^{\lambda_0} q$ for some $q \in Q$ and $j \in \{0,1\}$ we have*

$$\pi(p) \Vdash \forall b \in \dot{B}_{1-j} \setminus \{0\} \quad \pi_j(p \cdot b) = 1,$$

and moreover, $p \in \mathcal{D}$. In particular, we have $Q \subseteq \mathcal{D}$.

Proof. Fix p as in the hypothesis. Say $r \in Q$, $r \leq \pi(p)$ and $b \in B_0$ such that $r \Vdash b \in \dot{B}_0 \setminus \{0\}$. Then $r \leq \pi(b)$. So as B_0, B_1 is λ_0-reduced, $r \leq \pi_1(p \cdot b)$, whence $r \Vdash \pi_1(p \cdot b) = 1$. This proves the first statement for $j = 1$, and in the other case the proof is the same.

We now show $p \in \mathcal{D}$: Say $p' \preccurlyeq^{\lambda_0} p$. Since also $p' \preccurlyeq^{\lambda_0} q$, we have

$$\pi(p') \Vdash \forall b \in \dot{B}_{1-j} \setminus \{0\} \quad \pi_j(p' \cdot b) = 1 = \pi_j(p \cdot b),$$

and thus $p \in \mathcal{D}$. □

In fact, the first statement of lemma 6.12 is equivalent to B_0, B_1 being a reduced pair (this is really just a slight variation of lemma 6.13).

The following provides a hint as to how we can assume that B_0, B_1 is λ_0-reduced:

Fact 6.13. Assume that $\dot{r}_0, \dot{r}_1$ are P-names for reals random over V^Q, and assume $\Vdash_Q \dot{B}_i = \langle \dot{r}_i \rangle^{P:Q}$ (as is the case in our application). Say $j = 0$ or $j = 1$. The following are equivalent (interestingly, in (2), there is no mention of j):

1. Whenever $p \in P$, $p \preccurlyeq^\lambda q$ for some $q \in Q$ and $b \in B_j$, we have

$$\pi_{1-j}(p \cdot b) = \pi(p) \cdot \pi(b).$$

2. Whenever $p \in P$, $p \preccurlyeq^\lambda q$ for some $q \in Q$ and b_0, b_1 are Q-names for Borel sets such that for some $w \leq \pi(p)$, $w \Vdash_Q$" both b_0 and b_1 are not null", there is $p' \leq p$ such that $p' \Vdash_P \dot{r}_0 \in b_0$ and $\dot{r}_1 \in b_1$.

Proof. First, assume (2). We carry out the proof for $j = 0$ (the other case is exactly the same). Let $p \in P$ such that for some $q \in Q$, $p \preccurlyeq^{\lambda_0} q$ and let $b_0 \in B_0$. As for any $r \in \text{r.o.}(P)$, $r \leq \pi_j(r) \leq \pi(r)$ holds, we have $\pi_j(p \cdot b_0) \leq \pi(p) \cdot \pi(b_0)$. We now show $\pi_j(p \cdot b_0) \geq \pi(p) \cdot \pi(b_0)$. It suffices to show that whenever $b_1 \in B_1$ is compatible with $\pi(p) \cdot \pi(b_0)$, it is compatible with $p \cdot b_0$. So fix $b_1 \in B_1$. We have $\pi(b_1) \cdot \pi(b_0) \cdot \pi(p) \neq 0$, so we may pick $w \leq \pi(b_1) \cdot \pi(b_0) \cdot \pi(p)$. For $j = 0, 1$, let $\dot{b}_j$ be a Q-name for a Borel set such that $b_j = \|\dot{r}_j \in \dot{b}_j\|^{\text{r.o.}(P)}$. The last inequality means $w \Vdash \dot{b}_0$ and $\dot{b}_1$ are not null. So by assumption, we can find p' forcing $\dot{r}_j \in \dot{b}_j$ for both $j = 0, 1$. In other words, $p' \leq p \cdot b_0 \cdot b_1$, whence b_1 is compatible with $p \cdot b_0$.

For the other direction, assume (1) and again assume $j = 0$, fix p as above, and say $\dot{b}_0, \dot{b}_1$ are Q-names such that $w \Vdash \dot{b}_0, \dot{b}_1 \in \mathbf{Borel}^+$ for some $w \leq \pi(p)$. Let $b_j = \|\dot{r}_j \in \dot{b}_j\|^{\text{r.o.}(P)}$. As $\pi(b_0) \cdot \pi(b_1) \cdot \pi(p) \neq 0$, b_1 is compatible with $\pi(b_0) \cdot \pi(p) = \pi_1(p \cdot b_0)$. Thus b_1 is compatible with $p \cdot b_1$. So we may pick $p' \in P$, $p' \leq p \cdot b_0 \cdot b_1$. □

Definition 6.14. Under the assumptions of the previous lemma, we also say the pair $\dot{r}_0$, $\dot{r}_1$ is λ-reduced.

We shall need the next lemma to show that P completely embeds into $\mathbf{Am}_1$ (see 6.17). Observe that the next lemma does not make the assumption that B_0, B_1 is a λ_0-reduced pair obsolete, i.e. by itself the lemma does not imply $Q \subseteq \mathcal{D}$.

Lemma 6.15. *Assume that there exists a Q-stable meet operator $\wedge_Q$ on P with respect to $\mathbf{S}$. Then $Q \cdot \mathcal{D} \subseteq \mathcal{D}$. More precisely, if $p \in \mathcal{D}$ and $q \in Q$ are such that $q \leq \pi(p)$, we have $q \cdot p \in \mathcal{D}$. Moreover, if $(p, r) \in \text{dom}(\wedge_Q)$ and $p \in \mathcal{D}$, for any $j \in \{0, 1\}$ and $b \in B_{1-j}$ we have $\pi_j((p \wedge_Q r) \cdot b) = \pi_j(p \cdot b)$.*

Proof. Let $p \in P$, $q \in Q$ and $q \leq \pi(p)$. We check that $q \cdot p \in \mathcal{D}$. So let

$$r \preccurlyeq^{\lambda_0} q \cdot p, \tag{6.13}$$

and fix $j \in \{0, 1\}$ and $b \in B_{1-j}$. To prove that $q \cdot p \in \mathcal{D}$, it suffices to show

$$\pi_j(r \cdot b) = \pi(r) \cdot \pi_j(q \cdot p \cdot b). \tag{6.14}$$

Observe that (6.13) implies that $r \preccurlyeq^{\lambda_0} \pi(r) \cdot p$—for by 5.1($\lhd_c$2), $\pi(r) \preccurlyeq^{\lambda} \pi(q \cdot r) = q$; now use 5.18($\lhd_s$A). Thus $(p, r) \in \text{dom}(\wedge_Q)$ and $p \wedge_Q r \preccurlyeq^{\lambda_0} p$. Thus $p \wedge_Q r \in \mathcal{D}$ and

$$\pi_j((p \wedge_Q r) \cdot b) = \pi(p \wedge_Q r) \cdot \pi_j(p \cdot b) = \pi_j(p \cdot b),$$

where the last equation holds because $\pi(p \wedge_Q r) = \pi(p) \geq \pi_j(p \cdot b)$. Note in passing that this proves of the "moreover" clause of the lemma. We continue with the proof of the remaining part of the lemma. By the previous, as $r = \pi(r) \cdot (p \wedge_Q r)$,

$$\pi_j(r \cdot b) = \pi(r) \cdot \pi_j((p \wedge_Q r) \cdot b) = \pi(r) \cdot \pi_j(p \cdot b) = \pi(r) \cdot \pi_j(q \cdot p \cdot b).$$

The last equation holds as $\pi(r) \leq q$. This finishes the proof of the lemma. □

From now on, assume we have a Q-stable meet $\wedge_Q$ on P.

While it is true that $(\widehat{\mathcal{D}}, \mathcal{D}_f^{\mathbb{Z}})$ is a stratified extension, this is not quite the partial order we use in the main theorem: for this construction would require to repeatedly thin out to a dense set. As a consequence, we would need the main iteration theorem 5.23 not just for iterations but rather for sequences $(D_\xi)_{\xi<\theta}$ where $(D_\xi, D_{\xi+1})$ is a stratified extension, but we do not have strong projections from $D_{\bar{\xi}}$ to D_ξ for $\xi < \bar{\xi} \leq \theta$. Moreover, we would need to prove that the limits in this directed system of partial orders are what we expect them to be (in particular, that each D_ξ is embedded in this limit as a complete sub-order).[2] Instead, we have a much simpler solution.

Definition 6.16 (Type-1 amalgamation)**.** Let $\mathbf{Am}_1 = \mathbf{Am}_1(Q, P, f, \lambda)$ be the set of $\bar{p} \colon \mathbb{Z} \to \widehat{P}$ such that the following conditions are met.

1. For all $i \in \mathbb{Z}$, $\pi(\bar{p}(i)^P) = \pi(\bar{p}(0)^P)$.
2. For all but finitely many $i \in \mathbb{Z}$, $\bar{p}(k)^P \preccurlyeq^{\lambda_0} \pi(\bar{p}(k)^P)$.
3. For all $i \in \mathbb{Z} \setminus \{-1, 0\}$, $f(\pi_0(\bar{p}(i))) = \pi_1(\bar{p}(i+1))$ — that is, (6.5) holds.
4. $\bar{p}(0) \in P$, i.e. $\bar{p}^0(0) = \bar{p}^1(0) = 1$ and
$$f(\pi_0(\bar{p}(-1))) \geq \pi_1(\bar{p}(0)), \tag{6.15}$$
$$f(\pi_0(\bar{p}(0)) \leq \pi_1(\bar{p}(1)). \tag{6.16}$$
5. For $i \in \mathbb{Z} \setminus \{0\}$, $\bar{p}(i)^P \in \mathcal{D}(Q, P, f, \lambda)$.

Observe we can replace (6.15) and (6.16) by

$$\bar{p}(0)^P \leq f(\pi_0(\bar{p}(-1))) \cdot f^{-1}(\pi_1(\bar{p}(1))). \tag{6.17}$$

and obtain an equivalent definition. Thus, $\bar{p} \in \mathbf{Am}_1$ if and only if the following conditions are met:

1. $\bar{p}(0) \in P$,

[2]It seems plausible that theorem 5.23 would go through in this broader case. This provided, it is possible, but lengthy to show that limits contain the D_ξ's.

2. $\bar{p} \restriction [1, \infty) \in \mathcal{D}_f^{[1,\infty)}$ and $\bar{p} \restriction (-\infty, -1] \in \mathcal{D}_f^{(-\infty,-1]}$

3. for both $j \in \{-1, 1\}$ we have $\pi(\bar{p}(0)) = \pi(\bar{p}(j))$ and (6.17) holds.

Let $a\colon P \to \mathbf{Am}_1$ be defined by $a(p)(0) = (p, 1, 1)$ and $a(p)(i) = (\pi(p), 1, 1)$ for all $i \in \mathbb{Z} \setminus \{0\}$. As before, let $\bar{\pi}(\bar{p}) = \bar{p}(0)^P$ (we see no problem in using the same designation as for the projection from $\mathcal{D}_f^{\mathbb{Z}}$ to $\mathcal{D}$—see the remark after the next lemma).

Lemma 6.17. *The map $a\colon P \to \mathbf{Am}_1$ is a complete embedding and*

$$\bar{\pi}\colon \mathbf{Am}_1 \to P$$

is a strong projection.

Proof. Let $\bar{p} \in \mathbf{Am}_1$, $w \in P$, and $w \le \bar{p}(0)$. Define $\bar{p}'$ by

$$\bar{p}'(i) = \begin{cases} w & \text{for } i = 0, \\ (\pi(w) \cdot \bar{p}(i)^P, \bar{p}(i)^0, \bar{p}(i)^1) & \text{for } i \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

Clearly $\bar{p}' \in \mathbf{Am}_1$, $\bar{p}' \le a(w)$ and $\bar{p}' \le \bar{p}$. Moreover, for arbitrary $\bar{q} \in \mathbf{Am}_1$, if $\bar{q} \le a(w)$ and $\bar{q} \le \bar{p}$, clearly $\bar{q} \le \bar{p}'$; so $\bar{p}' = a(w) \cdot \bar{p}$. This shows that $\bar{\pi}$ is a strong projection and accordingly, a is a complete embedding. □

In what follows, we identify P and $a[P]$—except when we feel this would hide the point of the argument. Next we show that in fact, $\mathbf{Am}_1$ and $\mathcal{D}_f^{\mathbb{Z}}$ are presentations of the same forcing.

Lemma 6.18. *The set $D^* = \{\bar{p} \in \mathcal{D}_f^{\mathbb{Z}} \mid \bar{p}(0)^0 = \bar{p}(0)^1 = 1\}$ is dense in both $\mathcal{D}_f^{\mathbb{Z}}$ and $\mathbf{Am}_1$.*

Proof. First, we notice that $D^* \subseteq \mathbf{Am}_1$ and that the ordering of $\mathcal{D}_f^{\mathbb{Z}}$ and that of $\mathbf{Am}_1$ coincide on D^*. Given $\bar{p} \in \mathcal{D}_f^{\mathbb{Z}}$, find $d \in \mathcal{D}$ such that $d \le \bar{p}(0)^P \cdot p(0)^0 \cdot p(0)^1$; clearly, $d \cdot \bar{p} \in D^*$.

Now let $\bar{p} \in \mathbf{Am}_1$. We find $\bar{w} \le \bar{p}$, such that $\bar{w} \in D^*$. Find $d \in \mathcal{D}$ such that $d \le \bar{p}(0)$. First let $I = (-\infty, 0]$ and construct $\bar{w}^- = \bar{w} \restriction I$. Let $b_0 = f(\pi_0(\bar{p}(-1)))$ and define $\bar{p}^- \in \mathcal{D}_f^I$ by

$$\bar{p}^- = (\dots, \bar{p}(i), \dots, \bar{p}(-1), b_0),$$

where of course we identify b_0 and $(1_P, \pi(b_0), b_0) \in \widehat{\mathcal{D}}$. Since $d \le \bar{p}(0) \le b_0$ and $b_0 = \bar{\pi}_0^I(\bar{p}^-)$, we can let $\bar{w}^- = d \cdot \bar{p}^- \in \mathcal{D}_f^I$. Observe that $\bar{\pi}_0^I(\bar{w}^-) = d$.

Now let $I = [0, \infty)$. In an analogous fashion, define $\bar{w}^+ \in \mathcal{D}_f^I$ such that $\bar{w}^+ \leq \bar{p} \restriction I$ and $\pi_0^I(\bar{w}^+) = d$. Letting

$$\bar{w}(i) = \begin{cases} \bar{w}^-(i) & \text{for } i < 0, \\ \bar{w}^+(i) & \text{for } i \geq 0, \end{cases}$$

we conclude $\bar{w} \in \mathbf{Am}_1$. Moreover, $\bar{\pi}(\bar{w}) = d \in \mathcal{D}$ whence $\bar{w} \in D^*$, and $\bar{w} \leq \bar{p}$ in $\mathbf{Am}_1$. □

Thus, although Φ is not an automorphism of $\mathbf{Am}_1$, since it is an automorphism of $\mathcal{D}_f^{\mathbb{Z}}$, it gives rise to an automorphism of the associated Boolean algebra. We call Φ the *automorphism resulting from the amalgamation,* and we refer to Q as the *base of the amalgamation* or, interchangeably, *the base of* Φ.

That r.o.$(\mathbf{Am}_1)$ = r.o.$(\mathcal{D}_f^{\mathbb{Z}})$ justifies that we use the same notation for the strong projections $\bar{\pi} \colon \mathbf{Am}_1 \to P$ and $\bar{\pi} \colon \mathcal{D}_f^{\mathbb{Z}} \to \mathcal{D}$—as we know a strong projection coincides with the canonical projection on (the separative quotient of) its domain. The next lemma clarifies the role of $\mathcal{D}$.

Lemma 6.19. *Let $\bar{p} \in \mathbf{Am}_1$ and say $\bar{q} \colon \mathbb{Z} \to P \times B_0 \times B_1$ satisfies the following conditions:*

1. *for each $i \in \mathbb{Z}$, $\pi(\bar{q}(i)^P) = \pi(\bar{q}(0)^P)$.*
2. *$\bar{q}(0)^0 = \bar{q}(0)^1 = 1$.*
3. *$\forall i \in \mathbb{Z} \setminus \{0\} \quad \bar{q}(i)^P \preccurlyeq^{\lambda_0} \bar{p}(i)^P$.*
4. *$\forall i \in \mathbb{Z} \setminus \{0\} \quad \pi_j(\bar{q}(i)) = \pi(\bar{q}(i)^P) \cdot \pi_j(\bar{p}(i))$*

Then $\bar{q} \in \mathbf{Am}_1$.

Proof. First, let $I = [1, \infty)$ and show $\bar{q} \restriction I \in \mathcal{D}_f^I$. Let $i \in I$ be arbitrary. By 4 above, we have

$$\pi_j(\bar{q}(i)) = \pi(\bar{q}(i)^P) \cdot \pi_j(\bar{p}(i)) \tag{6.18}$$

for $j \in \{0, 1\}$. Since by 1 we have $\pi(\bar{q}(i)^P) = \pi(\bar{q}(0)^P) \leq \pi(\bar{p}(0)^P) = \pi(\bar{p}(i))$, applying π to (6.18) yields

$$\pi(\bar{q}(i)) = \pi(\bar{q}(i)^P) \cdot \pi(\bar{p}(i)) = \pi(\bar{q}(i)^P), \tag{6.19}$$

which means

$$\bar{q}(i) \in \widehat{P}. \tag{6.20}$$

Since $\bar{p} \in \mathbf{Am}_1$ and since (6.18) holds, we have

$$f(\pi_0(\bar{q}(i))) = \pi(\bar{q}(0)) \cdot f(\pi_0(\bar{p}(i))) = \pi(\bar{q}(0)) \cdot \pi_1(\bar{p}(i+1)) = \pi_1(\bar{q}(i))$$

Thus $\bar{q} \restriction I \in \mathcal{D}_f^I$. Repeat the argument above to show $\bar{p} \restriction (-\infty, -1] \in \mathcal{D}_f^{(-\infty,-1]}$. As $\bar{q}(0)^0 = \bar{q}(0)^1 = 1$ by assumption, (6.20) holds for $i = 0$. Let $b = f(\pi_0(\bar{p}(-1))) \cdot f^{-1}(\pi_1(\bar{p}(1)))$. As $\bar{q}(0) \leq \bar{p}(0) \leq b$, clearly

$$\bar{q}(0) \leq \bar{\pi}(\bar{q}(0)) \cdot b = f(\pi_0(\bar{q}(-1))) \cdot f^{-1}(\pi_1(\bar{q}(1))).$$

Thus, finally $\bar{q} \in \mathbf{Am}_1$. □

Finally, we are ready to state and prove the main theorem of this section:

Theorem 6.20. *$(P, \mathbf{Am}_1)$ is a stratified extension on $J = [(\lambda_0)^+, \kappa)$.*

Proof. We proceed to define a stratification of $\mathbf{Am}_1$. $\mathbf{Am}_1$ is going to be stratified above $(\lambda_0)^+$, but in general not above λ_0, which comes from the fact that possibly B_0 and B_1 conspire to yield antichains of size $(\lambda_0)^+$. [3]

For notational convenience, we define $\bar{q} \mathrel{\bar{\preccurlyeq}^\lambda} \bar{p}$ for arbitrary $\mathbb{Z}$-sequences $\bar{q}, \bar{p} \in {}^{\mathbb{Z}}(P \times B_0 \times B_1)$ and for $\lambda \geq \lambda_0$: $\bar{q} \mathrel{\bar{\preccurlyeq}^\lambda} \bar{p}$ exactly if for every $i \in \mathbb{Z}$, $\bar{q}(i)^P \preccurlyeq^\lambda \bar{p}(i)^P$ and for every $i \in \mathbb{Z} \setminus \{0\}$ we have $\pi(\bar{q}(i)^P) \Vdash_Q \pi_j(\bar{q}(i)) = \pi_j(\bar{p}(i))$—or equivalently,

$$\pi_j(\bar{q}(i)) = \pi(\bar{q}(i)^P) \cdot \pi_j(\bar{p}(i)) \tag{6.21}$$

for both $j \in \{0, 1\}$.

Corollary 6.21. *Using this notation we can state lemma 6.19 in the following way: If for some regular $\lambda \geq \lambda_0$, $\bar{p} \in \mathbf{Am}_1$ and $\bar{q} \colon \mathbb{Z} \to P \times B_0 \times B_1$ satisfy $\bar{q} \mathrel{\bar{\preccurlyeq}^\lambda} \bar{p}$ and moreover $\bar{q}(0) \in P$ and for all $i \in \mathbb{Z}$, $\pi(\bar{q}(i)^P) = \pi(\bar{q}(0)^P)$ holds, then $\bar{q} \in \mathbf{Am}_1$.*

Lemma 6.22. *Observe that if $\bar{q} \colon \mathbb{Z} \to P \times B_0 \times B_1$ and $\bar{p} \in \mathbf{Am}_1$ satisfy $\bar{q}(i)^P \preccurlyeq^\lambda \bar{p}(i)^P$ for all $i \in \mathbb{Z}$ and $\bar{q}(i)^j = \bar{p}(i)^j$ for all $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{0, 1\}$, then $\bar{q} \mathrel{\bar{\preccurlyeq}^\lambda} \bar{p}$.*

Proof. For $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{0, 1\}$, we have

$$\begin{aligned} \pi_j(\bar{q}(i)) &= \bar{p}(i)^j \cdot \pi_j(\bar{q}(i)^P \cdot \bar{p}(i)^{1-j}) \\ &= \bar{p}(i)^j \cdot \pi(\bar{q}(i)^P) \cdot \pi_j(\bar{p}(i)^P \cdot \bar{p}(i)^{1-j}) = \pi(\bar{q}(i)^P) \cdot \pi_j(\bar{p}(i)). \end{aligned}$$

where the second line is equal to the first as $\bar{p}(i)^P \in \mathcal{D}$ and $\bar{q}(i)^P \preccurlyeq^\lambda \bar{p}(i)^P$. Thus, $\bar{q} \mathrel{\bar{\preccurlyeq}^\lambda} \bar{p}$. □

[3] This is the case if we amalgamate e.g. over copies of the Cohen algebra, as was shown by Shelah, which is why he invented sweetness. The same seems to occur for Random algebras, although I have no concrete example.

Now let $\bar{p}, \bar{q} \in \mathbf{Am}_1$ and say $\lambda \in I$ and $\lambda > \lambda_0$. Define

$$\bar{q} \in \bar{\mathbf{D}}(\lambda, x, \bar{p}) \iff \forall i \in \mathbb{Z} \quad \bar{q}(i)^P \in \mathbf{D}(\lambda, x, \bar{p}^P(i)).$$

We say $\bar{q} \mathrel{\bar{\preccurlyeq}^\lambda} \bar{p}$ exactly if

$$\forall i \in \mathbb{Z} \quad \bar{q}(i)^P \preccurlyeq^\lambda \bar{p}(i)^P.$$

Next we define $\bar{\mathbf{C}}^\lambda$. Fix a name $\dot{\mathcal{B}}$ such that

$$\Vdash_P \dot{\mathcal{B}} \colon \dot{B}_0 \cup \dot{B}_1 \to \check{\lambda_0} \text{ is a bijection.}$$

Let $\mathrm{dom}(\bar{\mathbf{C}}^\lambda)$ be the set of all $\bar{p} \in \mathbf{Am}_1$ such that for each $i \in \mathbb{Z}$, we have $\bar{p}(i)^P \in \mathrm{dom}(\mathbf{C}^\lambda)$ and if $i \neq 0$, there is $\lambda' \in \lambda \cap I$ such that for $j \in \{0, 1\}$ we have that $\dot{\mathcal{B}}(\pi_j(\bar{p}(i)))$ is λ'-chromatic below $\pi(\bar{p}(i)^P)$. If $\bar{p} \in \mathrm{dom}(\bar{\mathbf{C}}^\lambda)$, we define $\bar{\mathbf{C}}^\lambda(\bar{p})$ to be the set of all $(c(i), \lambda'(i), H^0(i), H^1(i))_{i \in \mathbb{Z}}$ such that for all $i \in \mathbb{Z}$, $c(i) \in \mathbf{C}^\lambda(\bar{p}(i))$ and for all $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{0, 1\}$, $H^j(i)$ is a $\lambda'(i)$-spectrum of $\dot{\mathcal{B}}(\pi_j(\bar{p}(i)))$ below $\pi(\bar{p}(0)^P)$. Observe that $\lambda'(0)$, $H^0(0)$ and $H^1(0)$ can be chosen arbitrarily—they merely serve as place-holders to facilitate notation. This finishes the definition of the stratification of $\mathcal{D}^{\mathbb{Z}}_f$.

First we check that $\bar{\mathbf{D}}$ and $(\bar{\preccurlyeq}^\lambda)_{\lambda \in I}$ give us a pre-closure system, see 3.1, p. 12. That $\bar{\mathbf{D}}$ is Π^T_1 is immediate (without any further assumptions on the parameter x). For the following, let $\bar{p}$, $\bar{q}$, $\bar{r} \in \mathbf{Am}_1$, $\lambda \in I$ and x be arbitrary.

It is clear that (C 1) holds, for if $\bar{q} \leq \bar{p} \in \bar{\mathbf{D}}(\lambda, x, \bar{r})$, then $\bar{q}(i)^P \leq \bar{p}(i)^P \in \bar{\mathbf{D}}(\lambda, x, \bar{r}(i)^P)$ for each $i \in \mathbb{Z}$. Thus by (C 1) for P, $\bar{q}(i)^P \in \bar{\mathbf{D}}(\lambda, x, \bar{r}(i)^P)$ for each $i \in \mathbb{Z}$ and we are done. For (C 2), we must prove transitivity, so say $\bar{p} \mathrel{\bar{\preccurlyeq}^\lambda} \bar{q} \mathrel{\bar{\preccurlyeq}^\lambda} \bar{r}$ and show $\bar{p} \mathrel{\bar{\preccurlyeq}^\lambda} \bar{r}$. Fix $i \in \mathbb{Z}$ and $j \in \{0, 1\}$. Clearly, $\bar{p}(i)^P \preccurlyeq^\lambda \bar{r}(i)^P$. As $\pi(\bar{p}(i)^P) \Vdash_Q \pi_j(\bar{p}(i)) = \pi_j(\bar{q}(i))$ and $\pi_j(\bar{q}(i)) = \pi_j(\bar{r}(i))$, we get $\pi(\bar{p}(i)^P) \Vdash_Q \pi_j(\bar{p}(i)) = \pi_j(\bar{r}(i))$ and so as i, j were arbitrary, $\bar{p} \mathrel{\bar{\preccurlyeq}^\lambda} \bar{r}$. It remains to show that $\bar{p} \mathrel{\bar{\preccurlyeq}^\lambda} \bar{q} \Rightarrow \bar{p} \leq \bar{q}$. So assume $\bar{p} \mathrel{\bar{\preccurlyeq}^\lambda} \bar{q}$ and fix $i \in \mathbb{Z}$. Firstly, $\bar{p}(i)^P \leq \bar{q}(i)^P$; moreover, (6.21) implies $\pi_j(\bar{p}(i)) \leq \pi_j(\bar{q}(i))$ for $j \in \{0, 1\}$, and so as $i \in \mathbb{Z}$ was arbitrary and by (6.4), we infer $\bar{p} \leq \bar{q}$.

(C 3): Say $\bar{p} \leq \bar{q} \leq \bar{r}$ and $\bar{p} \mathrel{\bar{\preccurlyeq}^\lambda} \bar{r}$. Let $i \in \mathbb{Z}$ be arbitrary; clearly $\bar{p}(i)^P \preccurlyeq^\lambda \bar{q}(i)^P$. Let $j \in \{0, 1\}$ be arbitrary; as

$$\pi_j(\bar{p}(i)) \leq \pi(\bar{p}(i)^P) \cdot \pi_j(\bar{q}(i)) \leq \pi(\bar{p}(i)^P) \cdot \pi_j(\bar{r}(i))$$

and the terms on the sides of the equation are equal, we conclude $\bar{p} \mathrel{\bar{\preccurlyeq}^\lambda} \bar{q}$. Condition (C 4) is trivial.

We continue by checking the remaining conditions of 3.7, i.e. that we have a pre-stratification system on $\mathbf{Am}_1$. The conditions (S 1), (S 2) and (S 3) are immediate by definition. We prove (S 4):

Lemma 6.23. *Density holds; i.e for $\lambda \in J$, $\bar{p} \in \mathbf{Am}_1$ and $\lambda' \in [\lambda_0, \lambda)$ there is $\bar{q} \in \mathbf{Am}_1$ such that $\bar{q} \in \mathrm{dom}(\bar{\mathbf{C}}^\lambda)$ and $\bar{q} \mathrel{\bar{\preccurlyeq}^{\lambda'}} \bar{p}$.*

Proof. First, look through the following definition and find a set of parameters x such that it is Π^T_1 in parameters from x. We define conditions $p^n_i \in P$ for $n \in \mathbb{N}$ and $i \in \mathbb{Z}$ and $q^n \in Q$ for $n \in \mathbb{N}$. We do so by induction on n, in each step using induction on i. First, as Q is stratified we can find $q^0 \in Q$ such that $q^0 \preccurlyeq^{\lambda'} \pi(\bar{p}(0)^P)$ and for all $i \in \mathbb{Z}$ and both $j \in \{0,1\}$, $\pi_j(\bar{p}(i))$ is λ'-chromatic below q^0.

Set $p^0_i = \bar{p}(i)^P$, for $i \in \mathbb{Z}$.

Now say we have already defined a $\mathbb{Z}$-sequence $\bar{p}^n = (p^n_i)_{i\in\mathbb{Z}}$ of conditions in P and $q^n \in Q$. We will define a stronger $\mathbb{Z}$-sequence $\bar{p}^{n+1} = (p^{n+1}_i)_{i\in\mathbb{Z}}$ of conditions in P and a $q^{n+1} \in Q$. We first define $\bar{p}^{n+1}$ on the positive integers by induction, then on the negative ones. At the end we find $q^{n+1} \in Q$.

So find $p^{n+1}_i \in P$ for $i \geq 0$, by induction on i. Find $p^{n+1}_0 \preccurlyeq^{\lambda'} q^n \cdot \bar{p}^n_0$ such that $p^{n+1}_0 \in \mathbf{D}(\lambda', x, q^n \cdot \bar{p}^n_0)$ and $p^{n+1}_0 \in \mathrm{dom}(\mathbf{C}^\lambda)$. Assume by induction that for all $i \in \mathbb{Z}$, $q^n \preccurlyeq^{\lambda'} \pi(p^n_i)$, whence also $\pi(p^{n+1}_0) \preccurlyeq^{\lambda'} q_n \preccurlyeq^{\lambda'} \pi(p^n_i)$. Continue by induction, choosing, for each $i \in \mathbb{N} \setminus \{0\}$, a condition p^{n+1}_i such that

$$\begin{aligned} p^{n+1}_i &\preccurlyeq^{\lambda'} \pi(p^{n+1}_{i-1}) \cdot p^n_i \\ p^{n+1}_i &\in \mathbf{D}(\lambda', x, \pi(p^{n+1}_{i-1}) \cdot p^n_i) \end{aligned} \tag{6.22}$$

and $p^{n+1}_i \in \mathrm{dom}(\mathbf{C}^\lambda)$. By induction hypothesis, $\pi(p^{n+1}_{i-1}) \preccurlyeq^{\lambda'} q_n \preccurlyeq^{\lambda'} \pi(p^n_i)$, so $\pi(p^{n+1}_{i-1}) \cdot p^n_i$ is a well defined condition in P and $\pi(p^{n+1}_{i-1}) \cdot p^n_i \preccurlyeq^{\lambda'} p^n_i$. Thus we have defined p^{n+1}_i for $i \geq 0$. Before we consider the case $i < 0$, observe that for any $i \in \mathbb{N} \setminus \{0\}$, by (6.22), 5.1($\lhd_c$2) and ($\lhd_c$3), we have

$$\begin{aligned} \pi(p^{n+1}_i) &\preccurlyeq^{\lambda'} \pi(p^{n+1}_{i-1}) \\ \pi(p^{n+1}_i) &\in \mathbf{D}(\lambda', x, \pi(p^{n+1}_{i-1})). \end{aligned}$$

We also use that by construction, $\pi(p^{n+1}_{i-1}) \leq q^{n+1} \leq \pi(p^n_i)$. Thus, $(\pi(p^{n+1}_i))_{i\in\mathbb{N}}$ is (λ', x)-strategic and we may assume by choice of x and by lemma 3.14 that it is (λ', x)-adequate. Let q^* be a greatest lower bound for $(\pi(p^{n+1}_i))_{i\in\mathbb{N}}$.

Now we define p^{n+1}_i, by induction on i for $i < 0$: Find $p^{n+1}_{-1} \in P$ such that

$$\begin{aligned} p^{n+1}_{-1} &\preccurlyeq q^* \cdot p^n_{-1} \\ p^{n+1}_{-1} &\in \mathbf{D}(\lambda', x, q^* \cdot p^n_{-1}) \end{aligned}$$

and such that $p^{n+1}_{-1} \in \mathrm{dom}(\mathbf{C}^\lambda)$. Again, continue choosing for each $i \in \mathbb{N}$, $i > 1$ a condition $p^{n+1}_{-i} \in \mathrm{dom}(\mathbf{C}^\lambda)$ such that the sequence $(\pi(p^{n+1}_{-i}))_{i\in\mathbb{N}}$ is (λ', x)-adequate. Finally, let q^{n+1} be a greatest lower bound of $(\pi(p^{n+1}_{-i}))_{i\in\mathbb{N}}$. This finishes the inductive definition of $\bar{p}^n$.

For each $i \in \mathbb{Z}$, $(p^n_i)_{n\in\mathbb{N}}$ is a λ'-adequate sequence and thus has a greatest lower bound which we call $\bar{q}(i)^P$. By lemma 5.7 and by choice of x, $\{\pi(p^n_i)\}_{n\in\mathbb{N}}$

is a λ'-adequate sequence in Q. Since (Q,P) is a quasi-closed extension, by (C II) $\pi(\bar{q}(i)^P)$ is a greatest lower bound of this sequence. As for each $n \in \mathbb{N}$, $q^{n+1} \preccurlyeq^{\lambda'} \pi(p_i^n) \preccurlyeq^{\lambda'} q^n$, $(q_n)_{n\in\mathbb{N}}$ also has greatest lower bound $\pi(\bar{q}(i)^P)$, whence for all $i \in \mathbb{Z}$, $\pi(\bar{q}(i)^P) = \pi(\bar{q}(0)^P)$. Set $\bar{q}(i)^j = \bar{p}(i)^j$ for $j = 0, 1$ and observe that $\bar{q} \bar{\preccurlyeq}^{\lambda'} \bar{p}$. Thus as $\lambda' \geq \lambda_0$, we see $\bar{q}$ satisfies the hypothesis of lemma 6.19 and thus $\bar{q} \in \mathbf{Am}_1$. Lastly, as $\bar{q}(i)^P$ is a greatest lower bound of $\{p_i^n\}_{n\in\mathbb{N}}$, we conclude $\bar{q}(i)^P \in \mathrm{dom}(\mathbf{C}^\lambda)$. For each $i \in \mathbb{Z}$, fix $c(i) \in \mathbf{C}^\lambda(\bar{q}(i)^P)$.

Fix $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{0\}$. At the beginning, we chose q_0 such that $\pi_j(\bar{p}(i))$ is λ'-chromatic below q_0. So we may fix a λ'-spectrum $H^j(i)$ of $\pi_j(\bar{p}(i))$ below q_0—and hence also below $\pi(\bar{q}(0)^P) \leq q_0$. As $\bar{q} \bar{\preccurlyeq}^{\lambda'} \bar{p}$ we have $\pi_j(\bar{q}(i)) = \pi(\bar{q}(i)^P) \cdot \pi_j(\bar{p}(i))$. Thus, as $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{0,1\}$ were arbitrary,

$$(c(i), \lambda', H^0(i), H^1(i))_{i\in\mathbb{Z}} \in \bar{\mathbf{C}}^\lambda(\bar{q}).$$

□

Now we check that the pre-stratification system on $\mathbf{Am}_1$ extends that of P. Conditions $(\lhd_{\mathrm{c}}1)$, $(\lhd_{\mathrm{c}}2)$ and $(\lhd_{\mathrm{c}}3)$ are immediate. For $(\lhd_{\mathrm{s}}1)$, it suffices to check $(\lhd_{\mathrm{s}}\mathrm{A})$ and $(\lhd_{\mathrm{s}}\mathrm{B})$.

$(\lhd_{\mathrm{s}}\mathrm{A})$: Say $q \in P$ and $\bar{p}$ are such that $q \preccurlyeq^\lambda \bar{\pi}(\bar{p})$. Let $i \in \mathbb{Z} \setminus \{0\}$. By $(\lhd_{\mathrm{s}}\mathrm{A})$ for (Q,P) we have $\pi(q) \cdot \bar{p}(i)^P \preccurlyeq^\lambda \bar{p}(i)^P$. Moreover, $\pi_j((\bar{p} \cdot q)(i)) = \pi(q) \cdot \pi_j(\bar{p}(i))$, so as $(\bar{p} \cdot q)(0) = q \preccurlyeq^\lambda \bar{p}(0)^P$, we conclude $\bar{p} \cdot q \bar{\preccurlyeq}^\lambda \bar{p}$.

$(\lhd_{\mathrm{s}}\mathrm{B})$: Say $q \in P$ and $\bar{p}$, $\bar{r} \in \mathbf{Am}_1$ are such that $q \leq \bar{\pi}(\bar{r})$ and $\bar{r} \bar{\preccurlyeq}^\lambda \bar{p}$. Let $i \in \mathbb{Z} \setminus \{0\}$. By $(\lhd_{\mathrm{s}}\mathrm{B})$ for (Q,P) we have $\pi(q) \cdot \bar{r}(i)^P \preccurlyeq^\lambda \pi(q) \cdot \bar{p}(i)^P$. Moreover,

$$\begin{aligned}\pi_j((\bar{r} \cdot q)(i)) &= \pi(q) \cdot \pi_j(\bar{r}(i)) \\ &= \pi(q) \cdot \pi(\bar{p}(0)^P) \cdot \pi_j(\bar{p}(i)) = \pi(q) \cdot \pi_j(\bar{p}(i)) = \pi_j((\bar{p} \cdot q)(i)),\end{aligned}$$

so as $\bar{r} \cdot q(0) = q = \bar{p} \cdot q(0)$, we conclude $\bar{r} \cdot q \bar{\preccurlyeq}^\lambda \bar{p} \cdot q$. Conditions $(\lhd_{\mathrm{s}}2)$, $(\lhd_{\mathrm{s}}3)$ and $(\lhd_{\mathrm{s}}5)$ are left to the reader. Being cautious, we check $(\lhd_{\mathrm{s}}4)$. Say $w \in P$, $\bar{d}$, $\bar{r} \in \mathbf{Am}_1$ and $w \leq \bar{\pi}(\bar{d})$ while $\bar{d} \bar{\preccurlyeq}^\lambda \bar{r}$. By $(\lhd_{\mathrm{s}}4)$ for (Q,P), we have $w \cdot \bar{d}(i)^P \preccurlyeq^\lambda w \cdot \bar{r}(i)^P$. As $\bar{d} \cdot w(0) = w = \bar{r} \cdot w(0)$, we conclude that $w \cdot \bar{d} \preccurlyeq^\lambda w \cdot \bar{r}$.

We check (5.2), i.e. that $(P, \mathbf{Am}_1)$ is a quasi-closed extension. Start with $(\mathrm{E_c I})$, that $\bar{\mathbf{D}}$ is coherently dense. Observe that to find $\bar{q} \in \bar{\mathbf{D}}(\lambda, x, \bar{p})$, with a given $\bar{q}(0)^P$, we need only make a direct extension $\bar{q}(i)^P$ of $\bar{p}(i)^P$ for every $i \in \mathbb{Z} \setminus \{0\}$ such that $\bar{q}(i)^P = \pi(\bar{q}(0)^P)$. This is possible by (5.2) for (Q,P). We obtain $\bar{q}$ wich satisfies all the requirements of 6.19. Thus we can find such $\bar{q} \in \bar{\mathbf{D}}(\lambda, x, \bar{p})$ without falling out of $\mathbf{Am}_1$ and without changing $\bar{q}(0)^P$.

We prove ($\mathrm{E_cII}$): So say $\bar\lambda < \lambda$ and $\bar{\bar{p}} = (\bar p)_{\xi<\rho}$ is a $(\lambda^*, \bar\lambda, x)$-adequate sequence in $\mathbf{Am}_1$ with a $\bar\pi$-bound $p \in P$. Towards finding a greatest lower bound $\bar p$, set $\bar p(0) = p$. Fix $i \in \mathbb{Z} \setminus \{0\}$. By definition of $\bar{\mathbf{D}}$ and $\bar\preccurlyeq^{\lambda}$, the sequence $\{\bar p_\xi(i)^P\}_{\xi<\rho}$ is $(\lambda^*, \bar\lambda, x)$-adequate in P. Since $\{\pi(\bar p_\xi(0)^P)\}_{\xi<\rho}$ is the same as $\{\pi(\bar p_\xi(i)^P)\}_{\xi<\rho}$, the condition $\pi(\bar p(0)) = \pi(p(i)^P) \in Q$ is a π-bound of $\{\pi(\bar p_\xi(i)^P)\}_{\xi<\rho}$. Thus by ($\mathrm{E_cII}$) for (Q, P), the sequence $\{\bar p(i)^P\}_{\xi<\rho}$ has a greatest lower bound $p_i \in P$ such that for all $\xi < \rho$, $p_i \preccurlyeq^{\lambda^*} \bar p_\xi(i)^P$ and $\pi(p_i) = \pi(\bar p(0)^P)$. Moreover, if $\lambda^* < \bar\lambda$, we have $p_i \preccurlyeq^{\bar\lambda} \pi(p_i)$. For each $i \in \mathbb{Z}$, let $\bar p(i)^P = p_i$ and for $j \in \{0, 1\}$ let $\bar p(i)^j = \bar p_0(i)^j$. By corollary 6.21, $\bar p \in \mathbf{Am}_1$ and $\bar p \mathrel{\bar\preccurlyeq^{\lambda^*}} \bar p_0$. We must check that for all $\xi < \rho$, $\bar p \mathrel{\bar\preccurlyeq^{\lambda^*}} \bar p_\xi$. This is clear as for every $i \in \mathbb{Z}$ we have $\bar p(i)^P \preccurlyeq^{\lambda^*} \bar p_\xi(i)^P$ by construction, and for every $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{0, 1\}$ we have

$$\pi_j(\bar p(i)) = \pi(\bar p(i)^P) \cdot \pi_j(\bar p_0(i)) = \pi(\bar p(i)^P) \cdot \pi_j(\bar p_\xi(i)),$$

where the first equation holds since $\bar p \mathrel{\bar\preccurlyeq^{\lambda^*}} \bar p_0$ second equation holds since

$$\pi(\bar p(i)^P) = \pi(p) \leq \pi(\bar p_\xi(i)^P)$$

and $\bar p_\xi \mathrel{\bar\preccurlyeq^{\lambda^*}} \bar p_0$ gives us

$$\pi_j(\bar p_\xi(i)) = \pi(\bar p_\xi(i)^P) \cdot \pi_j(\bar p_0(i)).$$

We check the remaining conditions of 5.19, showing that $(P, \mathbf{Am}_1)$ is a stratified extension on J.

(S 5): Say $\bar{\bar{p}} = (\bar p)_{\xi<\rho}$ and $\bar{\bar{q}} = (\bar q)_{\xi<\rho}$ are both $(\lambda^*, \bar\lambda, x)$-adequate for $\bar\lambda < \lambda$, such that

$$\forall \xi < \rho \quad \bar{\mathbf{C}}^\lambda(\bar p_\xi) \cap \bar{\mathbf{C}}^\lambda(\bar q_\xi) \neq \emptyset. \tag{6.23}$$

Say the sequence $\bar{\bar{p}} = (\bar p)_{\xi<\rho}$ has a greatest lower bound $\bar p$, the sequence $\bar{\bar{q}} = (\bar q)_{\xi<\rho}$ has a greatest lower bound $\bar q$. We show

$$\bar{\mathbf{C}}^\lambda(\bar p) \cap \bar{\mathbf{C}}^\lambda(\bar q) \neq \emptyset. \tag{6.24}$$

First, observe that in each P-component we obtain a common colour for $\bar p$ and $\bar q$: For each $i \in \mathbb{Z}$, as in the previous proof, $\{\bar p_\xi(i)^P\}_{\xi<\rho}$ and $\{\bar q_\xi(i)^P\}_{\xi<\rho}$ are $(\lambda^*, \bar\lambda, x)$-adequate and so by (S 5) for P we can find $c(i) \in \mathbf{C}^\lambda(\bar p(i)^P) \cap \mathbf{C}^\lambda(\bar q(i)^P)$. To obtain the colours fo the boolean values, we can use the spectra of the first condition on either sequence, say $\bar q_0$: fix $(c_0(i), \lambda'_0(i), H^0_0(i), H^1_0(i))_{i\in\mathbb{Z}} \in \bar{\mathbf{C}}^\lambda(\bar p_0) \cap \bar{\mathbf{C}}^\lambda(\bar q_0)$. We shall now check that

$$(c(i), \lambda'_0(i), H^0_0(i), H^1_0(i))_{i\in\mathbb{Z}} \in \bar{\mathbf{C}}^\lambda(\bar p) \cap \bar{\mathbf{C}}^\lambda(\bar q). \tag{6.25}$$

This is clear by definition: fix $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{0,1\}$. Firstly, $\bar{p} \bar{\preccurlyeq}^{\lambda} \bar{p}_0$ and so

$$\pi_j(\bar{p}(i)) = \pi(\bar{p}(i)^P) \cdot \pi_j(\bar{p}_0(i)).$$

Moreover, by choice of $\lambda'_0(i)$ and $H^j_0(i)$ there is $b^j \in B_j$ such that we have

$$\pi_j(\bar{p}_0(i)) = \pi(\bar{p}_0(i)^P) \cdot b^j.$$

and H^j_0 is a $\lambda'_0(i)$-spectrum for b^j below $\pi(\bar{p}(i)^P)$. The last two equations together yield

$$\pi_j(\bar{p}(i)) = \pi(\bar{p}(i)^P) \cdot b^j,$$

and so (6.25) holds. This finishes the proof of (S 5).

($\mathrm{E_s}$I), *coherent expansion*: Assume $\bar{q} \bar{\preccurlyeq}^{\lambda} \bar{p}$ and $\bar{p} \bar{\preccurlyeq}^{\lambda} \bar{a}(\bar{p}(0))$. Moreover, assume $\bar{q}(0) \leq \bar{p}(0)$. We show $\bar{q} \leq \bar{p}$. Let $i \in \mathbb{Z} \setminus \{0\}$ be arbitrary. As $\bar{q}(i)^P \preccurlyeq^{\lambda} \bar{p}(i)^P$ and $\bar{p}(i)^P \preccurlyeq^{\lambda} a(\bar{p}(0)^P)(i)^P = \pi(p(i)^P)$, and as $\pi(\bar{q}(0)^P) \leq \pi(\bar{p}(0))$, we have $\bar{p}(i)^P \leq \bar{q}(i)^P$ by ($\mathrm{E_s}$I) for (Q, P). Say $i \neq 0$ and $j \in \{0,1\}$. Then

$$\pi_j(\bar{p}(i)) \leq \pi(\bar{p}(i)) = \pi(\bar{p}(0)^P) \leq \pi(\bar{q}(0)^P) = \pi(\bar{q}(i)) = \pi_j(\bar{q}(i)),$$

where the last equality holds as $\bar{q}(i)^P \preccurlyeq^{\lambda} \pi(\bar{q}(0)^P) \in Q \subseteq \mathcal{D}$. By (6.4), $\bar{q} \leq \bar{p}$, and we are done.

We show *coherent interpolation* ($\mathrm{E_s}$II): Let $\bar{d}, \bar{r} \in \mathbf{Am}_1$ be such that $\bar{d} \bar{\preccurlyeq}^{\lambda} \bar{r}$, and say $p \in P$ interpolates $\bar{\pi}(\bar{d})$ and $\bar{\pi}(\bar{r})$. We find $\bar{p} \in \mathbf{Am}_1$ such that $\bar{p} \bar{\preccurlyeq}^{\lambda} \bar{r}$ and $\bar{p} \bar{\preccurlyeq}^{\lambda} \bar{d}$ and moreover $\bar{\pi}(\bar{p}) = p$. For $i \in \mathbb{Z} \setminus \{0\}$, use coherent interpolation for (Q, P) to find $p_i \in P$ such that $p_i \preccurlyeq^{\lambda} \bar{r}(i)^P$ and $p_i \preccurlyeq^{\lambda} \bar{d}(i)^P$ and moreover $\pi(p_i) = \pi(p)$. Now we define a sequence $\bar{p} \colon \mathbb{Z} \to P \times B_0 \times B_1$. Set $\bar{p}(0) = (p, 1, 1)$ and set $\bar{p}(i) = (p_i, \bar{r}(i)^0, \bar{r}(i)^1)$ for $i \in \mathbb{Z} \setminus \{0\}$. Clearly, $\bar{p} \bar{\preccurlyeq}^{\lambda} \bar{r}$ and so $\bar{p} \in \mathbf{Am}_1$. By construction, $\bar{p} \bar{\preccurlyeq}^{\lambda} \bar{d}$ and $\bar{\pi}(\bar{p}) = \bar{p}(0)^P = p$.

It remains to demonstrate ($\mathrm{E_s}$III):

Lemma 6.24. *Coherent centering holds: Say $\lambda \in J$, $\bar{p} \bar{\preccurlyeq}^{\lambda} \bar{d}$ and $\bar{\mathbf{C}}^{\lambda}(\bar{p}) \cap \bar{\mathbf{C}}^{\lambda}(\bar{d}) \neq \emptyset$. Say further we have $w_0 \in P$ such that $w_0 \preccurlyeq^{<\lambda} \bar{p}(0)^P$ and $w_0 \preccurlyeq^{<\lambda} \bar{d}(0)^P$. Then there is $\bar{w} \in \mathbf{Am}_1$ such that $\bar{\pi}(\bar{w}) = w_0$ and both $\bar{w} \bar{\preccurlyeq}^{<\lambda} \bar{p}$ and $\bar{w} \bar{\preccurlyeq}^{<\lambda} \bar{d}$.*

Proof. Fix $\bar{p}, \bar{d}$ and w_0 as in the hypothesis. Fix $i \in \mathbb{Z} \setminus \{0\}$ for the moment. Observe we have Since $\mathbf{C}^{\lambda}(\bar{p}(i)^P) \cap \mathbf{C}^{\lambda}(\bar{d}(i)^P) \neq \emptyset$, by coherent centering for (Q, P) we can find $\bar{w}_i \in P$ such that $\pi(w_i) = \pi(w_0)$. If the additional assumption at the end of the lemma holds, we may assume $w_i \preccurlyeq^{<\lambda} \bar{p}(i)^P$ and $w_i \preccurlyeq^{<\lambda} \bar{d}(i)^P$. For $i \in \mathbb{Z}$, set

$$\bar{w}(i) = (w_i, \bar{p}(i)^0, \bar{p}(i)^1).$$

Since $\bar{w} \bar{\preccurlyeq}^{\lambda_0} \bar{p}$ and $\pi(\bar{w}(i)^P) = \pi(w_0)$ for each $i \in \mathbb{Z}$, by lemma 6.19, $\bar{w} \in \mathbf{Am}_1$.

Now say the additional assumption holds. By construction, $\bar{w}(i)^P \preccurlyeq^{<\lambda} \bar{p}$ for each $\lambda' \in [\lambda_0, \lambda)$. Fix $i \in \mathbb{Z}$. Since $\bar{\mathbf{C}}^\lambda(\bar{p}) \cap \bar{\mathbf{C}}^\lambda(\bar{d}) \neq \emptyset$, $\bar{p}(i)^j$ and $\bar{d}(i)^j$ have a common λ-spectrum below $\pi(w_0)$, and so

$$\pi(w_0) \Vdash_Q \bar{p}(i)^j = \bar{d}(i)^j. \tag{6.26}$$

Thus for each $i \in \mathbb{Z}$,

$$\bar{w}(i) = w(i)^P \cdot d(i)^0 \cdot d(i)^1 \leq \bar{d}(i)$$

whence $\bar{w} \leq \bar{d}$. In fact, as $\bar{w}(i)^P \preccurlyeq^{\lambda'} \bar{d}(i)^P$ and (6.26) holds, $\bar{w} \bar{\preccurlyeq}^{\lambda'} \bar{d}$ for each $\lambda' \in [\lambda_0, \lambda)$. □

□

6.4 Stratified type-2 amalgamation

We now consider the simpler case when we want to extend an automorphism already defined on an initial segment of the iteration. Let P be a forcing, Q a complete sub-order, f an automorphism of Q and $\pi \colon P \to Q$ a strong projection. Assume λ_0 is regular and (Q, P) is a stratified extension on $I = [\lambda_0, \kappa)$. We denote the stratification on Q by $\preccurlyeq^\lambda_Q$, $\precsim^\lambda_Q, \ldots$ and write $\preccurlyeq^\lambda$, $\precsim^\lambda, \ldots$ for the stratification of P.

Further we assume that for each regular $\lambda \in I$ and $q, r \in Q$,

1. $q \preccurlyeq^\lambda_Q r \iff f(q) \preccurlyeq^\lambda_Q f(r)$;
2. $q \precsim^\lambda_Q r \iff f(q) \precsim^\lambda_Q f(r)$;
3. $p \in \mathbf{D}_Q(\lambda, x, q)) \iff f(p) \in \mathbf{D}(\lambda, x, f(q))$;
4. $\mathbf{C}^\lambda_Q(q) \cap \mathbf{C}^\lambda_Q(r) \neq \emptyset \iff \mathbf{C}^\lambda_Q(f(q)) \cap \mathbf{C}^\lambda_Q(f(r)) \neq \emptyset$.

We define the type-2 amalgamation $\mathbf{Am}_2(Q, P, f)$ (or just $\mathbf{Am}_2$ where the context allows) as the set of all $\bar{p} \colon \mathbb{Z} \to P$ such that for all but finitely many $i \in \mathbb{Z}$ we have $\bar{p}(i) \preccurlyeq^{\lambda_0} \pi(\bar{p}(i))$ and for all $i \in \mathbb{Z}$,

$$f(\pi(\bar{p}(i))) = \pi(\bar{p}(i+1)). \tag{6.27}$$

The ordering is $\bar{p} \leq \bar{q}$ if and only if for each $i \in \mathbb{Z}$, $\bar{p}(i) \leq \bar{q}(i)$.

Define $\bar{\pi}\colon \mathbf{Am}_2 \to P$ by $\bar{\pi}(\bar{p}) = \bar{p}(0)$. The map $\bar{e}\colon P \to \mathbf{Am}_2$ is defined by $\bar{e}(p)(0) = p$ and $\bar{e}(p)(i) = \pi(p)$ for all $i \in \mathbb{Z}$, $i \neq 0$.

It is straightforward to check that $\bar{e}$ is a complete embedding and $\bar{\pi}$ is the restriction of the canonical projection from r.o.$(\mathbf{Am}_2)$ to r.o.$(\bar{e}[P])$. Moreover, if $q \in P$, $\bar{p} \in \mathbf{Am}_2$ and $q \leq \bar{\pi}(\bar{p})$, then $q \cdot \bar{p} \in \mathbf{Am}_2$.

We now define the stratification of $\mathbf{Am}_2$, consisting of $\bar{\mathbf{C}}^\lambda$, $\bar{\mathbf{F}}^\lambda$, $\bar{\preccurlyeq}^\lambda$, $\bar{\curlyeqprec}^\lambda$ for each regular $\lambda \in I$. We say $\bar{q} \mathrel{\bar{\preccurlyeq}^\lambda} \bar{p}$ exactly if for every $i \in \mathbb{Z}$, $\bar{q}(i) \preccurlyeq^\lambda \bar{p}(i)$, and $\bar{q} \mathrel{\bar{\curlyeqprec}^\lambda} \bar{p}$ exactly if for every $i \in \mathbb{Z}$, $\bar{q}(i) \curlyeqprec^\lambda \bar{p}(i)$. Similarly for $\bar{\mathbf{D}}(\lambda, x, \bar{p}(i))$. For $\bar{p}$ such that for each $i \in \mathbb{Z}$, $\bar{p}(i) \in \operatorname{dom}(\mathbf{C}^\lambda)$, we define $\bar{\mathbf{C}}^\lambda(\bar{p})$ to be the set of all $c\colon \mathbb{Z} \to \lambda$ such that for each $i \in \mathbb{Z}$, $c(i) \in \mathbf{C}^\lambda(\bar{p}(i))$.

Lemma 6.25. $(P, \mathbf{Am}_2)$ *is a stratified extension on* I.

Proof. The proof is a slight modification of the argument for type-1 amalgamation. Therefore, we only touch the main points, and leave the rest to the reader.

Lemma 6.26. $(P, \mathbf{Am}_2)$ *is a quasi-closed extension on* I.

Proof. Let $\lambda \in I$. Let $(\bar{p}_\xi)_{\xi<\theta}$ be λ-adequate. Fix $i \in \mathbb{Z}$ and let $\bar{p}(i)$ be the greatest lower bound of the λ-adequate sequence $(\bar{p}_\xi(i))_{\xi<\theta}$. By coherency, $(\pi(\bar{p}_\xi(i)))_{\xi<\theta}$ is also adequate and its greatest lower bound is $\pi(\bar{p}(i))$. As f is an automorphism, for each $i \in \mathbb{Z}$, $f(\pi(\bar{p}(i)))$ is a greatest lower bound of $(q_\xi(i))_{\xi<\theta}$, where $\bar{q}_\xi(i) = f(\pi(\bar{p}_\xi(i)))$. As the latter is equal to $\pi(\bar{p}_\xi(i-1))$, we obtain (6.27) for $\bar{p}$. So $\bar{p} \in \mathbf{Am}_2$; it is straightforward to check it is a greatest lower bound of $(\bar{p}_\xi)_{\xi<\theta}$. □

Lemma 6.27. *Coherent interpolation holds, i.e whenever* $\bar{r}, \bar{d} \in \mathbf{Am}_2$, $\bar{d} \leq \bar{r}$ *and* $p_0 \in P$ *such that* $p_0 \preccurlyeq^\lambda \bar{\pi}(\bar{r})$ *and* $p_0 \curlyeqprec^\lambda \bar{\pi}(\bar{d})$, *there is* $\bar{p} \in \mathbf{Am}_2$ *such that* $\bar{p} \mathrel{\bar{\preccurlyeq}^\lambda} \bar{r}$, $\bar{p} \mathrel{\bar{\curlyeqprec}^\lambda} \bar{d}$ *and* $\bar{\pi}(\bar{p}) = p_0$.

Proof. Let $\lambda \in I$. Given $\bar{r}, \bar{d}$ and p_0 as above, first set $\bar{p}(0) = p_0$. As $\pi(\bar{r}(i)) = \pi(\bar{r}(0))$ and $\pi(\bar{d}(i)) = \pi(\bar{d}(0))$, $p_0 \curlyeqprec^\lambda \pi(\bar{d}(i))$ and $p_0 \preccurlyeq^\lambda \pi(\bar{r}(i))$, for all $i \in \mathbb{Z}$. Coherent interpolation for (Q, P, π) allows us to find, for each $i \in \mathbb{Z}$, $i \neq 0$ a condition $\bar{p}(i) \in P$ such that $\pi(\bar{p}(i)) = \pi(p_0)$, $p_0 \curlyeqprec^\lambda \bar{d}(i)$ and $p_0 \preccurlyeq^\lambda \bar{r}(i)$. As for each $i \in \mathbb{Z}$, $\pi(\bar{p}(i)) = \pi(p_0)$, $\bar{p} \in \mathbf{Am}_2$. □

Lemma 6.28. *Coherent centering holds. That is: Say* $\bar{p} \mathrel{\bar{\curlyeqprec}^\lambda} \bar{d}$ *and either of the following holds:* $\bar{\mathbf{C}}^\lambda(\bar{p}) \cap \bar{\mathbf{C}}^\lambda(\bar{d}) \neq \emptyset$ *or for some* $q \in Q$, $\bar{p} \mathrel{\bar{\preccurlyeq}^\lambda} q$ *or* $\bar{d} \mathrel{\bar{\preccurlyeq}^\lambda} q$. *Say further* $w_0 \in \mathcal{D}$ *such that for each regular* $\lambda' \in [\lambda_0, \lambda)$, $w_0 \preccurlyeq^{\lambda'} \bar{\pi}(\bar{p})$ *and* $w_0 \preccurlyeq^{\lambda'} \bar{\pi}(\bar{d})$. *Then there is* $\bar{w} \in \mathbf{Am}_2$ *such that for each regular* $\lambda' \in [\lambda_0, \lambda)$, $\bar{w} \mathrel{\bar{\preccurlyeq}^{\lambda'}} \bar{p}$, $\bar{w} \mathrel{\bar{\preccurlyeq}^{\lambda'}} \bar{d}$ *and* $\bar{\pi}(\bar{w}) = w_0$.

Lemma 6.29. *If $\bar{p} \in \mathbf{Am}_2$ and $\lambda', \lambda \in I$ and $\lambda' < \lambda$, there is $\bar{q} \in \mathbf{Am}_2$ such that $\bar{q} \in \mathrm{dom}(\bar{\mathbf{C}}^\lambda)$ and $\bar{q} \mathrel{\bar{\preccurlyeq}^{\lambda'}} \bar{p}$.*

Proof. We define a sequence $(q_i)_{i \in \mathbb{Z}}$ of conditions in P, by induction on i. As usual, read through the following definition and pick x such that it is Π^T_1 with parameters in x. First, find $q_0 \preccurlyeq^{\lambda'} \bar{p}(0)^P$ such that $q_0 \in \mathrm{dom}(\mathbf{C}^\lambda)$. Continue by induction, choosing, for each $n \in \mathbb{N} \setminus \{0\}$, a condition $q_n \preccurlyeq^{\lambda'} \bar{p}(n) \cdot \pi(q_{n-1})$ such that $\pi(q_n) \in \mathbf{D}_Q(\lambda, x, \pi(q_{n-1}))$ $q_n \in \mathrm{dom}(\mathbf{C}^\lambda)$. Let q^*_1 be a greatest lower bound for $(\pi(q_k))_{k \in \mathbb{N}}$; it exists by quasi-closure for Q. Find $q_1 \preccurlyeq^{\lambda'} q^*_1 \cdot \bar{p}(1)$ such that $q_1 \in \mathrm{dom}(\mathbf{C}^\lambda)$. Again, continue by induction, choosing for each $n \in \mathbb{N} \setminus \{0, 1\}$, a condition $q_{-n} \in \mathrm{dom}(\mathbf{C}^\lambda)$ ensuring that $(\pi(q_{-k}))_{k \in \mathbb{N}}$ form an adequate sequence. Finally, let q be a greatest lower bound of $(\pi(q_{-k}))_{k \in \mathbb{N}}$. For each $i \in \mathbb{Z}$, $q \preccurlyeq^{\lambda'} \pi(q_i)$, so $q \cdot q_i \preccurlyeq^{\lambda'} q_i \preccurlyeq^{\lambda'} \bar{p}(i)$. Observe that by coherent stratification, $q \cdot q_i \in \mathrm{dom}(\mathbf{C}^\lambda)$ for each $i \in \mathbb{Z}$. Setting $\bar{q}(i) = q \cdot q_i$, we have $\pi(\bar{q}(i)) = q$, for all $i \in \mathbb{Z}$. Thus $\bar{q} \in \mathbf{Am}_2$, $\bar{q} \mathrel{\bar{\preccurlyeq}^{\lambda'}} \bar{p}$ and $\bar{q} \in \mathrm{dom}(\bar{\mathbf{C}}^\lambda)$. □

We leave the rest of the proof that $(P, \mathbf{Am}_2)$ is a stratified extension on I to the reader. □

6.5 Remoteness and stable meets

The following lemma helps to ensure "coding areas" don't get mixed up by the automorphisms, as we shall see in lemmas 7.4 and 8.2. Also see the discussion at the beginning of section 5.5.

Lemma 6.30. *Say C is remote in P over Q (up to some height κ', where $\kappa' \leq \kappa$). Then $\Phi^k[C]$ is remote in $\mathbf{Am}_1$ over P (up to the same height) for any $k \in \mathbb{Z} \setminus \{0\}$.*

Proof. Let $D^* = \bar{\pi}^{-1}[\mathcal{D}] \subseteq \mathbf{Am}_1 \cap \mathcal{D}^{\mathbb{Z}}_f$ as in lemma 6.18. Let $j \in \{0, 1\}$ arbitrary. If $\hat{p} \in \widehat{\mathcal{D}}$, $c \in C$ and $c \leq \pi_C(\hat{p}^P)$, by the definition of $\mathcal{D}$,

$$\pi_Q(\hat{p} \cdot c) \Vdash \pi_j(\hat{p} \cdot c) = \pi_j(\hat{p}),$$

that is, $\pi_j(\hat{p} \cdot c) = \pi_j(\hat{p}) \cdot \pi_Q(\hat{p}^P \cdot c)$, so as C is independent over Q and thus $\pi_Q(\hat{p}^P \cdot c) = \pi_Q(\hat{p}^P)$, we have $\pi_j(\hat{p} \cdot c) = \pi_j(\hat{p})$. In fact, if we have $\hat{p}^P \in \mathcal{D}$, we have $c \cdot \hat{p} \in \widehat{\mathcal{D}}$. Observe further that for any $c \in C$, $\pi_j(c) = 1$, and moreover, $C \subseteq \mathcal{D}$. Thence, $C \subseteq D^* \subseteq \mathrm{dom}(\Phi^k)$. Moreover, $\Phi^k(c)(0) = e_k(c)(0) = (1, 1, 1)$ and so $\Phi^k(c) \in D^* \subseteq \mathbf{Am}_1$.

We now show $\Phi^k[C] = e_k[C]$ is independent in $\mathbf{Am}_1$ over P: Let $c \in C$, $\bar{p} \in \mathbf{Am}_1$ and say $c \leq (\bar{\pi}_k \circ \pi_C)(\bar{p}) = \pi_C(\bar{p}(k))$.

Since $\pi_j(c \cdot \bar{p}(k)) = \pi_j(\bar{p}(k))$, for every $i \in \mathbb{Z}$,

$$i \neq k \Rightarrow e_k(c) \cdot \bar{p}(i) = \bar{p}(i). \tag{6.28}$$

Thus $e_k(c) \cdot \bar{p} \in \mathbf{Am}_1$. This firstly shows that $\bar{\pi}_k \circ \pi_C$ is a strong projection from $\mathbf{Am}_1$ to C. Moreover $\bar{\pi}(\bar{p} \cdot e_k(c)) = \bar{p}(0) = \bar{\pi}(\bar{p})$, and we are done with the proof that $\Phi^k[C] = e_k[C]$ is independent in $\mathbf{Am}_1$ over P.

It follows that $\Phi^k[C]$ is remote in $\mathbf{Am}_1$ over P, by definition of $\bar{\preccurlyeq}^\lambda$: let $\lambda \in [\lambda_0, \kappa')$. Say $c \leq \bar{\pi}_k(\bar{p})$. Then $e_k(c) \cdot \bar{p}(k) = (\bar{p}(k)^P \cdot c, \bar{p}(k)^0, \bar{p}(k)^1)$ and $\bar{p}(k)^P \cdot c \preccurlyeq^\lambda p(k)^P$, by clause (1) of definition 5.32. So by (6.28), $e_k(c) \cdot \bar{p} \bar{\preccurlyeq}^\lambda \bar{p}$, and we are done. □

The last lemma of this section is the counterpart of lemmas 5.29 and 5.30. Together these lemmas make sure that in the iteration used in our application, we have stable meet operators for every initial segment. We assume P is stratified on J.

Lemma 6.31. *There is a P-stable meet operator $\wedge_P$ on $\mathbf{Am}_1$.*

Proof. Of course we set

$$\operatorname{dom}(\wedge_P) = \{(\bar{p}, \bar{r}) \mid \exists \lambda \in J \quad \bar{r} \bar{\preccurlyeq}^\lambda \bar{\pi}(\bar{r}) \cdot \bar{p}\}.$$

Say we have $\bar{p}$, $\bar{r} \in \mathbf{Am}_1$ such that $(\bar{p}, \bar{r}) \in \operatorname{dom}(\wedge_P)$. This means we can fix a regular $\lambda \in J$ such that for each $i \in \mathbb{Z} \setminus \{0\}$, $\bar{r}(i)^P \preccurlyeq^\lambda \pi(\bar{r}(i)^P) \cdot \bar{p}(i)^P$. Let $w_i = \bar{p}(i)^P \wedge \bar{r}(i)^P$ for $i \in \mathbb{Z} \setminus \{0\}$ and set

$$\bar{p} \wedge_P \bar{r} = \begin{cases} (w_i, \bar{p}(i)^0, \bar{p}(i)^1) & \text{for } i \in \mathbb{Z} \setminus \{0\} \\ \bar{p}(0) & \text{for } i = 0. \end{cases}$$

Let $\bar{p} \wedge_P \bar{r}$ be denoted by $\bar{w}$. By lemma 6.15, for $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{0, 1\}$ we have

$$\begin{aligned} \pi_j(\bar{w}(i)) &= \bar{p}(i)^j \cdot \pi_j(w_i \cdot \bar{p}(i)^{1-j}) \\ &= \bar{p}(i)^j \cdot \pi_j(\bar{p}(i)^P \cdot \bar{p}(i)^{1-j}) = \pi_j(\bar{p}(i)). \end{aligned} \tag{6.29}$$

In particular, as $w_i \preccurlyeq^\lambda \bar{p}(i)^P$ and i was arbitrary, we have

$$\bar{p} \wedge_P \bar{r} \bar{\preccurlyeq}^\lambda \bar{p}. \tag{6.30}$$

Moreover, $\pi(w_i) = \pi(\bar{p}(i)^P) = \pi(\bar{p}(0)) = \pi(\bar{w}(0))$. So $\bar{w}$ satisfies the hypothesis of lemma 6.19 and therefore $\bar{w} \in \mathbf{Am}_1$. Clearly, $\bar{\pi}(\bar{p} \wedge_P \bar{r}) = \bar{p}(0)$. It remains to see that $\bar{\pi}(\bar{r}) \cdot (\bar{p} \wedge_P \bar{r}) \approx \bar{r}$; we have

$$\bar{\pi}(\bar{r}) \cdot \bar{w} = \begin{cases} (\pi(\bar{r}(0)^P) \cdot w_i, \bar{p}(i)^0, \bar{p}(i)^1) & \text{for } i \in \mathbb{Z} \setminus \{0\} \\ \bar{r}(0) & \text{for } i = 0. \end{cases}$$

Write $\bar{u} = \bar{\pi}(\bar{r}) \cdot \bar{w}$ and write $\bar{v} = \bar{\pi}(\bar{r}) \cdot \bar{p}$. For arbitrary $i \in \mathbb{Z} \setminus \{0\}$ and $j \in \{0, 1\}$ we have

$$\begin{aligned} \pi_j(\bar{u}(i)) &= \pi(\bar{r}(0)^P) \cdot \pi_j(\bar{w}(i)) = \pi(\bar{r}(0)^P) \cdot \pi_j(\bar{p}(i)) && \text{by (6.29),} \\ &= \pi(\bar{r}(0)^P) \cdot \pi_j(\bar{v}(i)) = \pi_j(\bar{r}(i)) && \text{as } \bar{r} \preccurlyeq^\lambda \bar{v}. \end{aligned}$$

Thus by (6.4), $\bar{u}(i) \approx \bar{r}(i)$. As $i \in \mathbb{Z} \setminus \{0\}$ was arbitrary and as $\bar{u}(0) = \bar{r}(0)$, we conclude $\bar{u} \approx \bar{r}$, finishing the proof that $\wedge_P$ is a P-stable meet on $\mathbf{Am}_1$.

□

Chapter 7

Projective measure without Baire

We begin with the assumption $V = L$ and fix κ, the least Mahlo. The first step is to force with $\bar{T} = \prod_{\xi<\kappa} T(\xi)$, the (full) product κ-many independent, κ^+-closed κ^{++}-Suslin trees. In fact, $\bar{T}$ itself has the κ^{++}-cc, by construction. This adds a sequence of branches $\bar{B} = (B(\xi))_{\xi<\kappa}$, where $B(\xi)$ denotes the branch through $T(\xi)$. As a notational convenience, we often assume the sequence of trees (resp. branches) is indexed by elements of

$$J = \kappa \times {}^{<\kappa}2 \times \omega \times 2 \times 2$$

rather than by ordinals in κ, that is as $B(\xi, s, n, i, j)$ and $T(\xi, s, n, i, j)$ for $\xi < \kappa$, $s \in {}^{<\kappa}2$, $n \in \mathbb{N}$ and $i, j \in \{0, 1\}$.

Since $\bar{T}$ is κ^{++}-cc, it is also κ^{++}-distributive, whence $W = L[\bar{B}]$ has the same cardinals and the same subsets of κ^+ as L and the GCH still holds in W. In particular, L and W have the same reals.

We now define $(P_\xi)_{\xi\leq\kappa}$, by induction on ξ. We start with $P_0 = \bar{T}$ and identify $\bar{B}$ with G_0. Working in W, setting $P^*_0 = \emptyset$ and interpreting the following definition verbatim in W, we obtain $(P^*_\xi)_{\xi\leq\kappa}$. The only formal difference being limit and amalgamation stages, there is a very canonical equivalence of $\bar{T} * \dot{P}^*_\xi$ with P_ξ. Note though that the proof shows $(P_{\xi+1}, P_\xi)$ to be a stratified extension on $[\lambda_\xi, \kappa]$, but the same *is not directly obvious* for $(P^*_{\xi+1}, P^*_\xi)$ in W. We shall call G_ξ the P_ξ generic over L and G^*_ξ the (P^*_ξ)-generic over W.

We construct this iteration to deal with the following tasks:

Task 1 Add a set of reals Γ^0 such that P_κ forces that the Baire-property fails for Γ^0;

Task 2 For each real r added by P_κ, make sure that P_κ forces $r \in \Gamma^0 \iff \Psi(r, 0) \iff \neg\Psi(r, 1)$, where $\Psi(x, y)$ is Σ^1_3.

Task 3 Make sure every projective set of reals is Lebesgue-measurable in the extension by P_κ.

Task 4 To make the construction more uniform, we force with a Levy-collapse at certain stages.

We force with the Levy-collapse for two reasons: firstly, when we amalgamate, and at limits, whether we collapse the continuum depends on factors beyond our control. So we always make sure we collapse the current continuum at the next stage. Secondly, for the purpose of task 2 (which involves Jensen coding), we want to make sure CH holds all the time.

Task 2 requires the sophisticated technique of Jensen coding, which made its first appearance in [BJW82] and has since undergone a long development culminating in [Fri00]. We will make the real r (along with information about its membership in Γ) definable by coding a subset of our set of branches $\bar{B}$ by a real s, where s is generic for Jensen coding. Say we have iterated for ξ steps and are in $L[G_\xi]$. For now, let's call the set of branches we "code" at the $\xi+1$-th step $\bar{B}^- = \{B(\zeta) \mid \zeta \in I\}$, where $I \subseteq \kappa$ of size κ.

Why do we use a subset of size κ? Since a real carries only a countable amount of information, one would think that a countable set of branches would suffice. The point here is that the automorphisms that arise from amalgamation (task 3) will make any such coding "unreadable" (see section 8). This is not surprising since by [She84], a definable well-ordering of a set of reals of length ω_1 yields a definable non-measurable set. In fact, if the present construction is altered so that each real is coded using a block of trees of length ω, we must fail since this would add such a well-order (since the set of trees is of course well-ordered). It is also easy to see how such a coding is made unreadable: if the trivial condition forces that the real $\dot{r}$ is coded using the branches indexed by the block $[\xi, \xi+\omega)$, for any automorphism Φ, also $\Phi(\dot{r})$ would be coded on the same block. See section 8 for the solution.

We shall pick a set $A_\xi \subseteq \kappa^+$ such that $(\mathbf{H}_\alpha)^{L[A_\xi]} = L_\alpha[A_\xi]$ for every cardinal $\alpha \leq \kappa$ and $\{B(\xi) \mid \xi \in I\}$ is definable in some simple recursive fashion from A_ξ (see below). Note $L[A_\xi]$ will be a proper sub-model of $L[G_\xi]$ (since we don't want to code *all of* $\bar{B}$). We shall then force with $P(A_\xi)$ of section 4 *as defined in the model* $L[A_\xi]$ to obtain a real s such that $A_\xi \in L[s]$ and moreover the following is true in the extension:

$$\begin{array}{l}\text{for all } \alpha,\beta<\kappa \;\; \textit{if} \;\; L_\beta[s] \text{ is a model of ZF}^- \text{ and of "}\alpha \text{ is the} \\ \text{least Mahlo and } \alpha^{++} \text{ exists" } \textit{then:} \\ I\cap\alpha \in L_\beta \text{ and } L_\beta[s] \models \text{"}\forall\xi\in I\cap\alpha\; \bar{T}^\beta(\xi) \text{ has a branch,"}\end{array} \tag{7.1}$$

where $\bar{T}^\beta$ denotes the outcome of the construction of $\bar{T}$ carried out in L_β. This is the vital use Jensen coding with localization (also called "David's trick" or *killing universes*). In order to make lemma 7.6 below go through, we have modified it in several ways (the use of $\diamondsuit$, $\rho(p^*_\delta)$ and most notably, we use Easton support). Note that forcing over $L[A_\xi]$, a proper sub-model of $L[G_\xi]$, means we shall have to argue that this step represents a stratified extension.

There are 4 types of forcing involved, so we fix a simple and convenient partition $E^0, \ldots, E^3$ of κ: let E^n, for $0 \leq n \leq 3$, denote the set of ordinals $\xi < \kappa$ such that for some limit ordinal η and $k \in \omega$, $\xi = \eta + k$ and $k \equiv n \pmod 4$. For an ordinal $\xi < \kappa$, let $E^n(\xi)$ denote the ξ-th element of E^n. Also fix, for each $\rho < \kappa$, an increasing sequence $\bar{\alpha}_\rho = (\alpha^\zeta_\rho)_{\zeta<\kappa}$ of ordinals $> \rho$ cofinal in κ: we let $\alpha^\zeta_\rho = G(\zeta, \rho)$, where G is the Gödel pairing function.

As we have to tackle certain tasks for every real of the extension, our definition will make use of two book-keeping devices, $\bar{s} = (\dot{s}_\xi)_{\xi<\kappa}$ and $\bar{r} = (\bar{\iota}(\xi), \dot{r}^0_\xi, \dot{r}^1_\xi)_{\xi<\kappa}$. We define $\bar{s}$ to list all reals which end up in the complement of Γ^0, in order to handle task 2 for each of these. To make sure all projective sets of reals are measurable (task 3) we ask that for each $\iota < \kappa$, the set of $\dot{r}^0_\xi, \dot{r}^1_\xi$ such that $\bar{\iota}(\xi) = \iota$ list all the pairs of reals in $L[G_\kappa]$ which are random over L^{P_ι}. We also ask that each pair $\dot{r}^0_\xi, \dot{r}^1_\xi$ be λ-reduced over $P_{\bar{\iota}(\xi)}$ for some large enough λ. We shall first proceed with the definition of the iteration, and after that argue that a book-keeping with the requisite properties can be defined at the same time.

We define a sequence $\bar{\lambda} = (\lambda_\xi)_{\xi\leq\kappa}$ by induction, so that for each $\xi \leq \kappa$, P_ξ will be stratified on $[\lambda_\xi, \kappa]$. We let $\lambda_0 = \omega$. For limit ξ, let λ_ξ be the minimum of $\mathbf{Reg} \setminus \bigcup_{\nu<\xi} \lambda_\nu$. For successors, define

$$\lambda_{\xi+1} = \begin{cases} (\lambda_\xi)^+ & \text{if } \xi \in E^0 \text{ or } \exists\rho \text{ s.t. } \xi = E^3(\alpha^0_\rho), \\ \lambda_\xi & \text{otherwise.} \end{cases} \tag{7.2}$$

For the readers orientation, be aware we will always have that P_ξ collapses λ_ξ, except when ξ is limit or P_ξ is an amalgamation of type-1. In those cases P_ξ may or may not preserve λ_ξ. The continuum of $L[G_\xi]$ is always at most $(\lambda_\xi)^+$ so CH always holds in $L[G_\xi]$, except when P_ξ does not collapse λ_ξ.

When we define P_ξ, we will also define some auxiliary sets D_ξ. At stages where we do type-1 amalgamation, they are similar to $\mathcal{D}$ of section 6.3; at other stages $D_\xi = P_\xi$. At all limit stages $\xi \leq \kappa$, we define P_ξ to be the $\bar{\lambda}$-diagonal support limit of the iteration up to that point. We generally write $B_\xi = \text{r.o.}(P_\xi)$ for $\xi \leq \kappa$. In the inductive definition of the iteration, we also define

1. A sequence of sets G^o_ξ, for $\xi < \kappa$; these arise because at coding stages,

the next forcing is not taken from the natural model $L[G_\xi]$ but from a smaller model $L[\bar{B}^-][G^o_\xi]$ (of course, since we do not want to code *all of* $\bar{B}$). The G^o_ξ help to "integrate out the $\bar{T}$-part" and to show that the $(P_\xi, P_{\xi+1})$ is a stratified extension. Note that we will have $\mathcal{P}(\omega)^{L[G\restriction\xi]} = \mathcal{P}(\omega)^{L[G^o_\xi]}$. [1]

2. A sequence $\bar{c} = (\dot{c}_\xi)_{\xi<\kappa}$ of names for reals where each $\dot{c}_\xi$ is Cohen over $L[G \restriction E^0(\xi)]$.

3. A sequence $(C_\xi)_{\xi<\kappa}$ of so-called coding areas, where each $C_\xi \in {}^\kappa 2$ is generic over $L[G \restriction E^1(\xi)]$ but has constructible initial segments.

4. Maps Φ^ζ_ρ, for $\rho, \zeta < \kappa$, where $\Phi^{\bar{\zeta}}_\rho$ extends Φ^ζ_ρ for $\zeta < \bar{\zeta}$. Finally, $\bigcup_{\zeta<\kappa} \Phi^\zeta_\rho$ uniquely determines an automorphism Φ_ρ of r.o.(P_κ) such that $\Phi_\rho(\dot{r}^0_\rho) = \dot{r}^1_\rho$ and $\Phi_\rho \restriction P_{\bar{\iota}(\rho)}$ is the identity. For a more uniform notation, we also write Φ^κ_ρ for Φ_ρ. We call any stage of the iteration $P_{\xi+1}$ such that $\xi = E^3(\alpha^\zeta_\rho)$ for some $\zeta < \kappa$ and thus such that Φ^κ_ρ extends Φ^ζ_ρ, *associated to* Φ_ρ.

7.1 The successor stage of the iteration

For the successor stage, assume by induction that we have already defined P_ξ for $\nu < \xi$ and $\dot{r}^0_\nu, \dot{r}^1_\nu$, $\dot{s}_\nu$ for $\nu \leq \xi$. Fix k and η such that $\xi = E^k(\eta)$. In any case except when $\xi \in E^3(\alpha^0_\rho)$, for some $\rho < \kappa$—that is, ξ is a stage where we do type-1 amalgamation—we let $D_\xi = P_\xi$. Let G_ξ denote a generic for P_ξ. We may assume we have already defined G^o_ξ, letting $G^o_0 = \emptyset$.

$k = 0$ At this stage we collapse the continuum of $L[G_\xi]$ and make sure the GCH holds (task 4). Let $P_{\xi+1} = P_\xi * \dot{Q}_\xi$, where

$$\Vdash_\xi \dot{Q}_\xi = \mathrm{Coll}(\omega, \lambda_{\xi+1}),$$

and let $G(\xi)$ be the $\mathrm{Coll}(\omega, \lambda_{\xi+1})$-generic. Observe that $\lambda_{\xi+1} = (\lambda_\xi)^+ \geq 2^\omega$ in $L[G_\xi]$, so we can pick a $P_{\xi+1}$-name which is fully Cohen over P_ξ, and define $\dot{c}_\eta$ to be this name. In $L[G \restriction \xi + 1]$, simply let $G^o_{\xi+1} = G^o_\xi \times G(\xi)$.

$k = 1$ Let $P_{\xi+1} = P_\xi \times (\mathrm{Add}(\kappa))^L$. We denote by $G(\xi)$ the generic and by C_η the new subset of κ it represents (and let $\dot{C}_\eta$ denote its canonical $P_{\xi+1}$-name). This will be the generic "coding area" used in the next step. Again let $G^o_{\xi+1} = G^o_\xi \times G(\xi)$.

[1] Moreover, $L[G^o_\xi] = L[G^*_\xi] \subsetneq L[\bar{B}][G^*_\xi] = L[G_\xi]$; we only introduce the sets G^o_ξ because $L[G^o_\xi]$ and $L[\bar{B}^-, G^o_\xi]$ are less ambiguous than $L[G^*_\xi]$ and $L[\bar{B}^-, G^*_\xi]$.

$k = 2$ We take care of task 2, making sure $\Psi(c, j)$, holds for some real c given to us by book-keeping ($j = 0, 1$ indicates whether $c \in \Gamma^0$). If η is a limit or $\eta = 0$, let c denote $\dot{c}_\eta^{G_\xi}$ (the Cohen real defined at stage $E^0(\eta)$), and let $j = 0$ (indicating that c will be in Γ^0). If η is a successor, let c denote $\dot{s}_{\eta-1}^{G_\xi}$, and let $j = 1$ (indicating that c will not be in Γ^0). We wish to code a branch through $T(s, n, i, j)$ if and only if s is an initial segment of $C = C_\eta$ (the coding area from the previous step) and $c(n) = i$. That is, we let

$$B(\eta, C, c, j) = \{B(\eta, s, n, i, j) \mid s \lhd C, c(n) = i\}$$

be the set of branches to code, and represent it in a Δ_1 way as a subset of κ^{++}:

$$\bar{B}^- = \{\#(\eta, s, n, i, j, t) \mid s \lhd C, c(n) = i, t \in B(s, n, i, j)\}$$

where $\#x$ denotes the order-type of x in the well-ordering of $L[\bar{B}^-][G_\xi^o]$. Working in $L[\bar{B}^-][G_\xi^o]$, define $A_\xi \subseteq \kappa^{++}$ as in 4.2 so that $L[\bar{B}^-][G_\xi^o] = L[A_\xi]$ and let $P(A_\xi)$ be the forcing discussed in section 4. Finally, define

$$Q_\xi = P(A_\xi)^{L[A_\xi]},$$

and let $\dot{Q}_\xi$ be a P_ξ-name for this forcing. Observe that it takes an argument to show we get a stratified extension; $\dot{Q}_\xi^{G_\xi}$ isn't obviously stratified (on any interval) in $L[G_\xi]$.

Letting $G(\xi)$ denote the generic, let $G_{\xi+1}^o = G_\xi^o \times \{p_{<\kappa^{++}} \mid p \in G(\xi)\}$.

$k = 3$ Say $\eta = \alpha_\rho^\zeta$. We first treat the case where $\zeta = 0$: By induction the book-keeping device $\bar{r}$ gives us $\bar{r}(\rho) = (\bar{\iota}(\rho), \dot{r}_\rho^0, \dot{r}_\rho^1)$, where $\bar{\iota}(\rho) < \xi$ (in fact, $< \rho$) and the pair of names reals $\dot{r}_\rho^0, \dot{r}_\rho^1$ is fully random over $L^{P_{\bar{\iota}(\rho)}}$ and λ_ξ-reduced over $P_{\bar{\iota}(\rho)}$.

Let f be the automorphism of the complete Boolean algebras generated by $\dot{r}_\rho^0$ and $\dot{r}_\rho^1$ in B_ξ and let $P_{\xi+1}$ be the type-1 amalgamation of P_ξ over f and $P_{\bar{\iota}(\rho)}$:

$$P_{\xi+1} = \mathbf{Am}_1(P_{\bar{\iota}(\rho)}, P_\xi, f, \lambda_\xi).$$

Set $D_\xi = \mathbf{D}(P_{\bar{\iota}(\rho)}, P_\xi, f, \lambda_\xi)$. The resulting automorphism of $P_{\xi+1}$ we denote by Φ_ρ^0.

Observe that, in general, this automorphism need not extend to an automorphism of B_κ. Also observe that by induction and theorem 6.20 $(P_\xi, \mathbf{Am}_1(P_{\bar{\iota}(\rho)}, P_\xi, f, \lambda_\xi))$ will be a stratified extension above $\lambda_{\xi+1}$.

In the second case, when $\eta = \alpha^\zeta_\rho$ and $\zeta > 0$, we make sure Φ^0_ρ is extended by an automorphism of $P_{\xi+1}$. So we let

$$P_{\xi+1} = \mathbf{Am}_2(\mathrm{dom}(\Phi), P_\xi, \Phi),$$

where Φ is (an extension of) Φ^0_ρ, constructed at an earlier stage of the iteration:

If ζ is a successor ordinal, at a previous stage $E^3(\alpha^{\zeta-1}_\rho)$, we defined $\Phi^{\zeta-1}_\rho$ extending Φ^0_ρ. Set $\Phi = \Phi^{\zeta-1}_\rho$.

If ζ is a limit, we have a sequence $(\Phi^\nu_\rho)_{\nu<\zeta}$, forming an increasing chain, and all extending Φ^0_ρ. Letting $\delta = \bigcup_{\nu<\zeta} \alpha^\nu_\rho < \xi$, there is a unique automorphism of P_δ, extending each of them. Let Φ be this automorphism.

The resulting automorphism of $P_{\xi+1}$ we denote by Φ^ζ_ρ. In both cases we say $\xi+1$ *or* $P_{\xi+1}$ *is an amalgamation stage associated to* Φ^ζ_ρ.

$G^o_{\xi+1}$ is defined to be the $\mathbb{Z}$-sequence of the sets defined like G^o_ξ in each of the $\mathbb{Z}$-many components of the amalgamation.

We see that in a very concrete sense, we can identify $G^* \restriction \xi$, the $\dot{P}^{\bar{B}}_\xi$-generic over $L[\bar{B}]$ with G^o_ξ.

Lemma 7.1. *1. For all $\xi < \kappa$, $(P_\xi, P_{\xi+1})$ is a stratified extension on $[\lambda_{\xi+1}, \kappa]$.*

2. For all $\xi \leq \kappa$, $\bar{P}_\xi$ is stratified on $[\lambda_\xi, \kappa]$.

3. For all ξ, $P_\xi \Vdash 2^\omega \leq (\lambda_\xi)^+$ over W.

4. If ξ is not of the form $E^3(\alpha^0_\rho)$ for some ρ, i.e. if $P_{\xi+1}$ is not a type-1 amalgamation, we have that $P_{\xi+1} \Vdash |\lambda_{\xi+1}| = \omega$ over W and $P_{\xi+1} \Vdash$ GCH over W.

5. In W, for each $\xi \leq \kappa$, $\dot{P}^{\bar{B}}_\xi$ is κ^+-centered.

Observe it follows that P_ξ preserves all cardinals greater than λ_ξ, for $\xi < \kappa$. We will show later that it also preserves Mahlo-ness of κ and that P_κ preserves κ.

Proof. When $\xi \notin E^2$, the first item holds by induction and lemmas 5.20 (composition), 5.27 (products) and theorem 6.20 (amalgamation); the pre-stratification systems are the ones stemming from the constructions in lemmas 5.20, 5.27 and theorem 6.20 of course.

It remains to show $(P_\xi, P_{\xi+1})$ is a a stratified extension on $[\lambda_{\xi+1}, \kappa]$ when ξ is a coding stage. We let $q \in \mathbf{D}_{\xi+1}(\lambda, p, x)$ if and only if $q \restriction \xi \in \mathbf{D}_\xi(\lambda, p \restriction \xi, x)$,

$q \restriction \xi \Vdash q(\xi) \in \dot{\mathbf{D}}_\xi(\lambda, p(\xi), x)^{L[A_\xi]}$ and both $p(\xi)_{<\kappa^+}$ and $p(\xi)^*_{\kappa^+}$ are κ-chromatic names (with A_ξ defined above in the definition of forcing at coding stages); observe this makes sense since we may view $p(\xi)_{<\kappa^+}$ and $p(\xi)^*_{\kappa^+}$ as functions with domain κ, and since by induction P_ξ is stratified at κ. The proof of the density property of $\mathbf{D}$ goes through verbatim (since we only have to look at $\lambda \leq \kappa$). The rest $\mathbf{C}^\lambda$, $\preccurlyeq^\lambda$, $\preccurlyeq^\lambda$ can be defined in a straightforward manner as in the proof for composition of stratified forcings 3.17.

To see that this is a stratified extension, let $\bar{p} = p^\nu_{\nu<\rho}$ be a $(\lambda, \bar{\lambda}, x)$-adequate sequence of conditions in $P_{\xi+1}$ with $(\bar{\lambda}, x)$-canonical witness $\bar{w}$ and let $q \restriction \xi$ be a greatest lower bound of $(p^\nu \restriction \xi)_{\nu<\rho}$. It suffices to show that q forces that $(p^\nu(\xi))_{\nu<\rho}$ is $(\bar{\lambda}, x)$-adequate in $L[A_\xi] = L[\bar{B}^-, G^o_\xi]$ (where $\bar{B}^-$ is defined as above in the definition of forcing at coding stages). That $\bar{w}$ is a strategic guide is clear by definition of $\mathbf{D}$ and the usual argument for composition; so we check that it is a canonical witness. We check $\bar{w}$ is $\Pi_1^{A_\xi}(\{x\} \cup \bar{\lambda})$; this is clear (as in 3.17), since it is $\Pi_1(\{x\} \cup \bar{\lambda})$ in L and thus also in $L[A_\xi]$. Since each P_ξ-name $q^\nu(\xi)$ is κ-reduced, it's interpretation can be found inside $L[G^0_\xi]$ and hence in $L[A_\xi]$. In fact, the (partial) function assigning to $q^\nu(\xi)$ its interpretation is $\Sigma_1^{A_\xi}(\{x\} \cup \bar{\lambda})$ provided κ^{+++} is among the parameters in x—in fact, the existential quantifier can be bounded by $L_{\kappa^{+++}}[A_\xi]$ since we only have to find a κ-spectrum witnessing the interpretation. Thus $q^\nu(\xi)^{G_\xi}$ can be obtained by application of a $\Sigma_1^{A_\xi}(x)$ (partial) function from the name $q^\nu(\xi)$, which can in turn be obtained by application of recursive function from q^ν, which can be obtained by applying a $\Sigma_1(\{x\} \cup \bar{\lambda})^L$ (partial) function to w^ν. Thus, $q^\nu(\xi)^{G_\xi}$ can be obtained by applying a $\Sigma^{A_\xi}(\{x\} \cup \bar{\lambda})$-function to w^ν in $L[A_\xi]$. This shows that $\bar{w}$ is a canonical witness.

The second item holds by theorem 5.23. The third one is a corollary of the previous ones and lemma 3.15and the next follows since we collapse λ_ξ at the right stage.

Lastly, the centeredness follows since stratification at κ allows us to define $\mathbf{C}^{\kappa^+}$; If $(t, p), (s, q) \in \bar{T} * \dot{P}_\xi$ are such that t and q are compatible and $\mathbf{C}^{\kappa^+}(t, p) \cap \mathbf{C}^{\kappa^+}(s, q) \neq \emptyset$ then clearly $s \cdot t$ forces that $p \cdot q \neq 0$. This implies that cardinals above κ are preserved, which is by the way not essential. □

Remark 7.2. Note that we can also use $\mathbf{C}^\lambda$ of the following explicit form. The abstract definition with guessing systems serves the purpose of proving an abstract iteration theorem. and may help to aid the intuition of the reader. The two forms are equivalent for successor λ, modulo a recursive translation. For $\lambda \in \mathbf{Inacc}$ this equivalence holds on a set which is dense in the same sense as $\mathrm{dom}(\mathbf{C}^\lambda)$.

Let $\lambda \leq \kappa$ be regular and let $\sigma < \lambda$. First, define $D^\sigma_\lambda \subseteq P_\kappa$, by induction on the length of a condition. For the successor step, say $p \in P_{\nu+1}$. We let

$p \in D^\sigma_\lambda$ if and only if $\pi_\nu(p) \in D^\sigma_\lambda$ and the following hold:

1. in case $\nu \in E^0$ (i.e. $\dot{Q}_\nu$ is $\mathrm{Coll}(\omega, \lambda_\nu)$), we require that $\pi_\nu(p) \Vdash \mathrm{ran}(p(\nu)) \subseteq \sigma$,
2. in case $\nu \in E^2$ (i.e. $\dot{Q}_\nu$ is Jensen coding of $L[A_\nu]$), we require that $\pi_\nu(p) \Vdash$ for all $\sup\mathrm{dom}(p(\nu)_{<\lambda}) = \sigma$ and $\rho(p(\nu)^*_\lambda) = \sigma$, if $\lambda \in \mathbf{Inacc}$.
3. in case $\nu \in E^3$ (i.e. $P_{\nu+1}$ is an amalgamation), we require that for all $i \in \mathbb{Z} \setminus \{0\}$ we have $p(i)^P \in D^\sigma_\lambda$ (it would be redundant to require this also for $i = 0$).

For $p \in P_\nu$ where $\nu \leq \kappa$ is a limit ordinal, let $p \in D^\sigma_\lambda$ if and only if for all $\nu' < \nu$, $\pi_{\nu'}(p) \in D^\sigma_\lambda$. Finally, define $p \in D^\Sigma_\lambda$ if and only if there is $\sigma < \lambda$ such that $p \in D^\sigma_\lambda$; if $\lambda \in \mathbf{Inacc}$, we require that $\sigma \in \mathbf{Card}$. Also, for any $p \in D^\Sigma_\alpha$, let $\sigma^2_\lambda(p)$ be the least $\sigma < \alpha$ such that $p \in D^\sigma_\lambda$ and $\mathrm{supp}^\lambda(p) \cap \lambda \subseteq \sigma$.

The sets $\mathbf{C}^\lambda$ can now be defined in a similar fashion: they are binary relations,

$$\mathbf{C}^\lambda \subseteq \{\text{ sequences of length } \leq \kappa\} \times P_\kappa.$$

that is, for any such sequence H, $\mathbf{C}^\lambda(H) \subseteq P_\kappa$.

So let $\lambda \leq \kappa$ be regular. The definition of $\mathbf{C}^\lambda(H)$ is by induction on the length of conditions: for the successor step, assume we have already defined $\mathbf{C}^\lambda$ on

$$\{\text{ sequences of length } \leq \nu\} \times P_\nu.$$

Fix an arbitrary sequence H. Assume $p \in P_{\nu+1}$ and let $p \in \mathbf{C}^\lambda(H)$ if and only if $\pi_\nu(p) \in \mathbf{C}^\lambda(H \restriction \nu)$ and either $p \preccurlyeq^\alpha \pi_\nu(p)$ or the following hold:

1. $H = (H(\xi))_{\xi < \nu+1}$ is a sequence of length $\nu + 1$,
2. $p \in D^\Sigma_\alpha$,
3. in case $\nu \in E^0$, we require that $p(\nu)$ (a collapsing condition) is β-chromatic below $\pi_\nu(p)$, for some $\beta < \alpha$, with spectrum $H(\nu)$,
4. in case $\nu \in E^2$, we require that $H(\nu)((\zeta)_0)$ is defined for each for each $\zeta < \delta = \sigma^2_\lambda(p(\nu))$ and we have $p(\nu)_{<\delta}(\zeta)$ is β-chromatic below $\pi_\nu(p)$, for some $\beta < \lambda$, with spectrum $H(\nu)(\zeta)$. In a similar manner, for inaccessible λ, require $H(\nu)((\zeta)_1)$ is a spectrum for the characteristic function of $B^\rho_p(\nu)$.
5. in case $\nu \in E^3$, we require that $H(\nu) = (\bar{H}^P_i, \bar{H}^0_i, \bar{H}^1_i)_{i \in \mathbb{Z}\setminus\{0\}}$ and for all $i \in \mathbb{Z} \setminus \{0\}$, $p(i)^P \in \mathbf{C}^\lambda(\bar{H}^P_i)$ and for $j \in \{0, 1\}$, $p(i)^j$ is β-chromatic with spectrum $\bar{H}^j_i$ below $\pi_\iota(p(0)^P)$ for some $\beta < \lambda$—where ι is chosen so that P_ι is the base of the amalgamation $P_{\nu+1}$ (see p. 101 for the definition of *base*).

7.2 A word about book-keeping

We give a recipe for cooking up a definition of $\bar{r} = (\bar{\iota}(\rho), \dot{r}^0_\rho, \dot{r}^1_\rho)_{\rho<\kappa}$. The definition is given by induction "on blocks". Assume $\bar{r} \restriction \xi$ has been defined. We shall now define $\bar{r}$ and $\bar{\iota}$ on $[\xi, (\lambda_\xi)^+)$—the "next block".

First, consider some fixed $\iota < \xi$ (or, for the induction start, assume $\xi = \iota = 0$). As $\lambda_{\xi+1} \geq 2^\omega$ in $L[G_\xi]$, we can enumerate all the reals in L^{P_ξ} which are random over L^{P_ι} in order type $\lambda_{\xi+1}$. In other words, find names P_ξ-names $(\dot{x}^\iota_\nu)_{\nu<\beta}$ such that

$$P_\xi \Vdash \mathbb{R} \setminus \dot{N} = \{\dot{x}^\iota_\nu\}_{\nu<\beta},$$

where $\dot{N}$ is a name for the union of the Borel null sets with code in L^{P_ι} and $\beta = \lambda_{\xi+1}$. By assumption, $\xi \in E^0$ is a limit ordinal or 0, so the last forcing of $P_{\xi+1}$ collapses $\lambda_{\xi+1}$ to ω.

For each $\nu, \nu' < \beta$, apply the lemma 7.3 below. You obtain a set $Y = Y(\nu, \nu', \iota)$ of size β consisting of pairs which are $\lambda_{\xi+1}$-reduced over P_ξ. If there are no reals in L^{P_ξ} which are random over L^{P_ι}, let Y be any set of pairs of random reals in $L^{P_{\xi+1}}$ which are $\lambda_{\xi+1}$-reduced over P_ξ (such a set exists—if in doubt, look at the proof of lemma 7.3).

Now define $\bar{r} \restriction \beta$ and $\bar{\iota} \restriction \beta$ (using a bijection of β with $\xi \times \beta^3$) in such a way that all pairs obtained in this way are listed, i.e. for each $\iota < \xi$, each pair and $\nu, \nu' < \beta$ and each $y \in Y(\nu, \nu', \iota)$ there is $\rho \in [\xi, \beta)$ such that $\bar{\iota}(\rho) = \iota$ and $(\dot{r}^0_\rho, \dot{r}^1_\rho) = y$.

Note that by lemma 7.6, we catch our tail and $\bar{r}$ enumerates all the pairs of random reals of the final model $L[G_\kappa]$ (see lemma 7.12).

Lemma 7.3. *Let $\iota < \xi$, where $\xi \in E^0$ and say 1_{P_ξ} forces $\dot{x}^0, \dot{x}^1$ are P_ξ-names random over L^{P_ι}. Then there is a set $Y = \{(\dot{y}^0_\nu, \dot{y}^1_\nu)\}_{\nu<\lambda_{\xi+1}}$ such that*

$$1 \Vdash (\dot{x}^0, \dot{x}^1) \in \{(\dot{y}^0_\nu, \dot{y}^1_\nu)\}_{\nu<\lambda_{\xi+1}}$$

and each pair in Y is $\lambda_{\xi+1}$-reduced over P_ξ.

Proof. Write $\beta = \lambda_{\xi+1}$. Find $\{\dot{q}_\zeta\}_{\zeta<\beta}$ such that $\Vdash_\xi \{\dot{q}_\zeta\}_{\zeta<\beta}$ is a maximal antichain in $\dot{Q}_\xi = Coll(\omega, \beta)$. Note that $\{(1_{P_\xi}, \dot{q}_\zeta)\}_{\zeta<\beta}$ is maximal antichain in $P_{\xi+1}$. Fix a map

$$b \colon \zeta \mapsto (\dot{b}_0(\zeta), \dot{b}_1(\zeta))$$

such that $\Vdash_\xi b \colon \beta \to (\mathbf{Borel}^+)^2$ is onto, where $\mathbf{Borel}^+$ denotes the set of Borel sets with positive measure coded in $L[G_\xi]$. For each $\zeta < \beta$ and $j = 0, 1$ pick R^0_ζ, R^1_ζ such that $(1_{P_\xi}, \dot{q}_\zeta)$ forces R^j_ζ is random over L^{P_ξ} and $R^j_\zeta \in \dot{b}_j(\zeta)$ for both $j = 0, 1$. This is possible since $Coll(\omega, \beta)$ collapses the continuum of

$L[G_\xi]$. Fix $\nu < \theta$ for the moment, in order to define $\dot{y}^0_\nu, \dot{y}^1_\nu$: for both $j = 0, 1$, pick $\dot{y}^j_\nu$ such that $(1_{P_\xi}, \dot{q}_\nu) \Vdash \dot{y}^j_\nu = \dot{x}^j$ and for each $\zeta \in \beta \setminus \{\nu\}$ we have $(1_{P_\xi}, \dot{q}_\zeta) \Vdash \dot{y}^j_\nu = \dot{R}^j_\zeta$.

As $\{(1_{P_\xi}, \dot{q}_\zeta)\}_{\zeta<\beta}$ is maximal, 1_{P_ξ} forces $\dot{r}_j$ is random over L^{P_ι}. For each $\nu < \theta$, the pair $\dot{y}^0_\nu, \dot{y}^1_\nu$ is β-reduced over P_ξ: Let $p \mathrel{\dot{\preccurlyeq}^\beta} q \in P_\xi$, let $\dot{b}_0$, $\dot{b}_1$ be P_ξ-names and fix $w \le \pi_\xi(p)$ such that $w \Vdash_\iota \dot{b}_0$ and $\dot{b}_1$ are codes for positive Borel sets. Find $w' \in P_\xi$ and $\zeta < \theta$, such that $w' \le w$, $\zeta \ne \nu$ and

$$w' \Vdash \dot{b}_j(\zeta) \subseteq \dot{b}_j \text{ for } j = 0, 1. \tag{7.3}$$

We can ask $\zeta \ne \nu$ because we are content with $\subseteq$ instead of $=$ in (7.3). As $w' \Vdash p(\xi) \mathrel{\dot{\preccurlyeq}^\beta_\xi} 1$, $w' \cdot p$ is compatible with $(1_{P_\xi}, \dot{q}_\zeta)$. If $p' \le w' \cdot p$ and $p' \le (1_{P_\xi}, \dot{q}_\zeta)$, we have $p' \Vdash \dot{y}^j_\nu \in \dot{b}_j(\zeta) \subseteq \dot{b}_j$. Clearly, this also shows that $\dot{y}^0_\nu, \dot{y}^1_\nu$ is β-reduced over P_ι: any code for a positive Borel set in $L[G_\iota]$ is remains one in $L[G_\xi]$.

Lastly, as $\{(1_{P_\xi}, \dot{q}_\zeta)\}_{\zeta<\theta}$ is maximal and $(1_{P_\xi}, \dot{q}_\nu) \Vdash \dot{x}^j = \dot{y}^j_\nu$,

$$1 \Vdash (\dot{x}^0, \dot{x}^1) \in \{(\dot{y}^0_\nu, \dot{y}^1_\nu)\}_{\nu<\lambda}.$$

□

We now define Γ^0_ξ, an approximation of Γ^0 at stage $\xi < \kappa$ of the iteration. Let Γ^0_ξ be the smallest superset of $\{\dot{c}_\eta \mid E^0(\eta) < \xi \text{ and } \eta \text{ is limit or } \eta = 0\}$ (for limit η of course $E^0(\eta) = \eta$, but never mind) closed under all of the functions $F = \Phi^\zeta_\rho, (\Phi^\zeta_\rho)^{-1}$ such that $\operatorname{dom} F \subseteq P_\xi$, i.e. closed under functions in

$$\{\Phi^\zeta_\rho, (\Phi^\zeta_\rho)^{-1} \mid E^3(\alpha^\zeta_\rho) \le \xi\}$$

Let

$$\dot{\Gamma}^0_\xi = \{(1_{P_\xi}, \dot{c}) \mid \dot{c} \in \Gamma^0_\xi\},$$

that is, $\dot{\Gamma}^0_\xi$ is the canonical choice for a name whose interpretation consists of the interpretations of the elements of Γ_ξ.

When defining $\bar{s}$ at stage ξ, we need to make sure that all P_ξ-names for reals $\dot{s}$ which have the following property are listed (in the course of the iteration) by $\bar{s}$: for any $\dot{r} \in \Gamma^0_\xi$, $\Vdash_{P_\xi} \dot{r} \ne \dot{s}$. We can easily make sure this is the case using arguments as above. As P_ξ forces $|\mathbb{R}| < \kappa$ (in fact $\le \kappa$ would suffice), we can find $\dot{f}_\xi$ such that

$$\Vdash_\xi \dot{f}_\xi \colon \kappa \to \mathbb{R} \setminus \dot{\Gamma}_\xi \text{ is onto.}$$

We may assume (by induction hypothesis) we have such $\dot{f}_\nu$ for $\nu < \xi$. Pick $\dot{s}_\xi$ such that for $\xi = G(\eta, \zeta)$, $\Vdash_\xi \dot{s}_\xi = \dot{f}_\eta(\zeta)$.

Later (see lemma 7.5), we show that $\bar{s}$ lists exactly the reals of the final model $L[G_\kappa]$ which are not in Γ^0 (which we are about to define). This concludes the definition of $(P_\xi)_{\xi\le\kappa}$, $\bar{c}$, $(C_\xi)_{\xi<\kappa}$, Φ^ζ_ρ for $\zeta \le \kappa$ and $\rho < \kappa$, $\bar{r}$, and $\bar{s}$, as well as that of Γ^0_ξ and $\dot{\Gamma}^0_\xi$.

7.3 Cohen reals, coding areas and the set Γ^0

Let Γ^0 be the least superset of

$$\{\dot{c}_\xi \mid \xi < \kappa,\ \xi \text{ limit ordinal or } \xi = 0\}.$$

closed under all functions $\Phi_\xi, (\Phi_\xi)^{-1}$, $\xi < \kappa$ and let $\dot{\Gamma}^0$ be the P_κ-name $\Gamma^0 \times \{1_{P_\kappa}\}$. Recalling Γ^0_ξ from the previous subsection (defined in the discussion of the coding device $\bar{s}$), note that $\bigcup_{\xi<\kappa} \Gamma^0_\xi \subseteq \Gamma^0$; that the two sets are in fact equal, if not clear, follows from the next lemma. The lemma also helps to see that Γ^0 and Γ^1 (defined below) give rise to disjoint sets in the extension, as intended. Lastly, the lemma also is important to show that the coding areas C_ν behave in the same way as do the reals c_ν, and this will be used in 8.2 to show that the coding does not conflict with the automorphisms coming from amalgamation.

Let $\Gamma^1 = \{\dot{s}_\xi \mid \xi < \kappa\}$ and let $\dot{\Gamma}^1$ be the P_κ-name $\Gamma^1 \times \{1_{P_\kappa}\}$. Let $\dot{x}_\nu$ denote either $\dot{c}_\nu$ or $\dot{C}_\nu$. Say $\dot{y}$ is of the following form:

$$\dot{y} = (\Phi_{\xi_m})^{k_m} \circ \ldots \circ (\Phi_{\xi_1})^{k_1}(\dot{x}_\nu)$$

where $\nu, \xi_1, \ldots, \xi_m < \kappa$ and $k_i \in \mathbb{Z}$ for $1 \leq i \leq m$. For $1 \leq i \leq m$, write

$$\dot{y}_i = (\Phi_{\xi_i})^{k_i} \circ \ldots \circ (\Phi_{\xi_1})^{k_1}(\dot{x}_\nu),$$

and write $\dot{y}_0$ for $\dot{x}_\nu$. Note that we can trivially assume that $\xi_{i+1} \neq \xi_i$, for i such that $1 \leq i < m$. We can also assume $\nVdash_{P_\kappa} \dot{y}_{i+1} = \dot{y}_i$ for such i. We then call $\nu, \xi_1, \ldots, \xi_m, k_1, \ldots, k_m$ an *index sequence of* $\dot{y}$. Observe that every $\dot{y} \in \Gamma^0$ can be written in the form above.

Lemma 7.4. *There are $\rho_0, \ldots, \rho_m < \kappa$ such that*

1. *$\nu < \rho_0 < \ldots < \rho_m$,*
2. *if $1 \leq i \leq m$, P_{ρ_i+1} is an amalgamation stage associated to Φ_{ξ_i},*
3. *if $0 \leq i \leq m$, $\dot{y}_i$ is a P_{ρ_i+1}-name not in $L^{P_{\rho_i}}$. Moreover, $\dot{y}_i$ is either unbounded over $L^{P_{\rho_i}}$ (if $\dot{y}_0 = \dot{c}_\nu$) or remote over P_{ρ_i} up to height κ (if $\dot{y}_0 = \dot{C}_\nu$).*

Moreover, for $\dot{y}, \dot{y}' \in \Gamma^0$, either $\Vdash_{P_\kappa} \dot{y} = \dot{y}'$ or $\Vdash_{P_\kappa} \dot{y} \neq \dot{y}'$. If $\dot{y}$ and $\dot{y}'$ have different index sequences, the latter holds.

Proof. By induction on m. For $m = 0$, since $\dot{y}_0 = \dot{x}_\nu$, we pick ρ_0 so that $\dot{Q}_{\rho_0}$ adds $\dot{x}_\nu$ (over $L[G_{\rho_0}]$). Then all of the above holds.

Now assume by induction the above holds for $i \leq m$. Let ρ_{m+1} be the least $\rho < \kappa$ such that $P_{\rho+1}$ is an amalgamation stage associated to $\Phi_{\xi_{m+1}}$, and $\dot{y}_m$ is a $P_{\rho+1}$-name. Since by induction, $\dot{y}_m$ is forced to be different from any element of $L^{P_{\rho_m}}$, $\rho_m \leq \rho_{m+1}$. Moreover, $\rho_m < \rho_{m+1}$, for otherwise, $\xi_m = \xi_{m+1}$, contrary to assumption.

We have that either $P_{\rho_{m+1}+1} = \mathbf{Am}_1(P_\zeta, D, \dot{r}_0, \dot{r}_1)$ (for some ζ, D a dense subset of $P_{\rho_{m+1}}$, and some $\dot{r}_0$, $\dot{r}_1$), or $P_{\rho_{m+1}+1} = \mathbf{Am}_2(P_\zeta, P_{\rho_{m+1}}, \Phi)$ (for some Φ and ζ). We prove the lemma assuming the first holds; very similar arguments work for the other alternative, which we leave to the reader.

Observe that $\dot{y}_m$ is not a P_ζ-name, as otherwise, contrary to assumption,

$$\Vdash_{P_\kappa} \dot{y}_{m+1} = \Phi_{\xi_{m+1}}(\dot{y}_m) = \dot{y}_m.$$

We have to consider two cases: If $\dot{x}_\nu = \dot{c}_\nu$, we can assume by induction that $\dot{y}_m$ is unbounded over L^{P_ζ} and thus also over $L^{P_\zeta * \dot{B}(\dot{r}_i)}$ for $i = 0, 1$. Thus by lemma 6.8 applied for $P = D$, $\dot{y}_{m+1}$ is unbounded over $L^{P_{\rho_{m+1}}}$ and we are done. If on the other hand, $\dot{x}_\nu = \dot{C}_\nu$, we can assume by induction that $\dot{y}_m$ is remote over P_ζ up to height κ and $\kappa > \lambda_{\xi_{m+1}}$. So by lemma 6.30, $\dot{y}_{m+1}$ is remote over $P_{\rho_{m+1}}$. In either sub-case, we conclude that $\dot{y}_{m+1}$ is not in $L^{P_{\rho_{m+1}}}$ (for the second case, using lemma 5.33).

Lastly, say $\nu, \xi_1, \ldots, \xi_m, k_1, \ldots, k_m$ is an index sequence of $\dot{y}$ and say $\nu', \xi'_1, \ldots, \xi'_m, k'_1, \ldots, k'_m$ is an index sequence of $\dot{y}'$. Assume $\not\Vdash_{P_\kappa} \dot{y} = \dot{y}'$; we show $\Vdash_{P_\kappa} \dot{y} \neq \dot{y}$. Let $\rho_0, \ldots \rho_m$ and $\rho'_0, \ldots \rho'_m$ be obtained as above for $\dot{y}$ and $\dot{y}'$ respectively. Let l be maximal such that $\xi_l \neq \xi'_l$ or $k_l \neq k'_l$, if such l exists. We may assume $\rho_l = \rho'_l$, for otherwise

$$\Vdash_{P_\kappa} \dot{y}_l \neq \dot{y}'_l,$$

and we can apply $(\Phi_{\xi_m})^{k_m} \circ \ldots (\Phi_{\xi_{l+1}})^{k_{l+1}}$ to this to obtain

$$\Vdash_{P_\kappa} \dot{y} \neq \dot{y}',$$

and we are done. So we may also assume $\xi_l = \xi'_l$, as otherwise, also $\rho_l \neq \rho'_l$. Thus $\xi_l = \xi'_l$ and $\rho_l = \rho'_l$, but $k_l \neq k'_l$. From now on may we write ξ and ρ for $\xi_l = \xi'_l$ and $\rho_l = \rho'_l$, respectively.

Observe that $\dot{y}_{l-1} \in V^{P_{\rho_{l-1}+1}}$, and $\dot{y}'_{l-1} \in V^{P_{\rho'_{l-1}+1}}$ where both $\rho_{l-1}, \rho'_{l-1} < \rho$. Also note that $\dot{y}_{l-1}$ is either remote or unbounded over $V^{P_{\rho_{l-1}}}$. Moreover, if we let P_ζ be the base of Φ_ξ, we have $\rho_{l-1} \geq \zeta$, for otherwise again $\Phi^{k-l}(\dot{y}_{l-1}) = \dot{y}_{l-1}$. Thus we have $\dot{y}_{l-1}$ is either remote or unbounded over V^{P_ζ}.

Now apply lemma 6.8 to see that $\Vdash_{P_\kappa} \dot{y}'_{l-1} \neq (\Phi_{\xi_l})^{-k'_l+k_l}(\dot{y}_{l-1})$, noting again $\dot{y}'_{l-1}$ is in V^{P_ρ}.

Apply $(\Phi_{\xi_l})^{k_l}$ to see $\Vdash_{P_\kappa} \dot{y}_l \neq \dot{y}'_l$. As above, apply $(\Phi_{\xi_m})^{k_m} \circ \ldots (\Phi_{\xi_{l+1}})^{k_{l+1}}$ to this to obtain

$$\Vdash_{P_\kappa} \dot{y} \neq \dot{y}'.$$

If no l as above exists, the index sequences for $\dot{y}$ and $\dot{y}'$ are identical except possibly in the first coordinate. Now observe that $\Vdash_{P_\kappa} \dot{y}_0 = \dot{y}'_0$ if $\nu = \nu'$ and $\Vdash_{P_\kappa} \dot{y}_0 \neq \dot{y}'_0$ if $\nu \neq \nu'$. Apply $(\Phi_{\xi_m})^{k_m} \circ \ldots (\Phi_{\xi_0})^{k_0}$ and we're done. □

Lemma 7.5. $\Vdash_{P_\kappa} \dot{\Gamma}^0 = \mathbb{R} \setminus \dot{\Gamma}^1$.

It would be easier to show this in the following way: Show by induction on the number of applications of automorphisms that all names $\dot{c}$ in $\Gamma^0 \setminus \Gamma^0_\xi$ have the following property: there is $\rho \geq \xi$ such that $\dot{c}$ is in $L^{P_{\rho+1}}$ but not in L^{P_ρ}. This would make the slightly more complicated proof of lemma 7.4 unnecessary.

Proof. First we show $\Vdash_{P_\kappa} \dot{\Gamma}^0 \cup \dot{\Gamma}^1 = \mathbb{R}$. Let $r \notin (\dot{\Gamma}^0)^G$. Find $\xi < \kappa$ such that $r \in L[G_\xi]$. As $r \notin (\dot{\Gamma}^0_\xi)^{G_\xi}$, $\bar{s}$ was defined to list a name for $\dot{r}$, so $r \in (\dot{\Gamma}^1)^G$.

Now let $\dot{c} \in \Gamma^0$ and $\dot{s} \in \Gamma^1$, and show $\Vdash_{P_\kappa} \dot{s} \neq \dot{c}$. Fix $\xi < \kappa$ so that $\dot{s}$ is a P_ξ-name and $\Vdash_{P_\xi} \dot{s} \notin \Gamma^0_\xi$. Let $v, \rho_1, \ldots, \rho_n$ be obtained as in the previous lemma from an index sequence for $\dot{c}$ and write $\rho = \rho_n$. By the last lemma $\dot{c}$ is a $P_{\rho+1}$ name not in L^{P_ρ}. If $\rho + 1 \leq \xi$, we are clearly done, for then $\dot{c} \in \Gamma^0_\xi$. Otherwise, if $\rho \geq \xi$, $\dot{c}$ is not in $L^{P_\rho} \supseteq L^{P_\xi}$, so in any case, $\Vdash_{P_\kappa} \dot{c} \neq \dot{s}$. □

7.4 The κ-stage of the iteration

The next lemma shows that κ remains a cardinal in the final model, κ remains Mahlo at each earlier stage, and that all reals appear in some initial segment of the construction. We write conditions of P_θ as (t, p) in the following only to have a convenient way to refer to the $\bar{T}$ part and to aid intuition—we do not need to work in $\bar{T} * P^*_\theta$.

Lemma 7.6. *Let $\theta \leq \kappa$, let $(\dot{\alpha}_\xi)_{\xi<\kappa}$ be a sequence of P_θ-names for ordinals below κ and let $(t, p) \in P_\theta$. Then for any $\beta_0 < \kappa$ there is an inaccessible $\alpha \in (\beta_0, \kappa)$ and a condition $(t', p') \leq (t, p)$ such that for all $\xi < \alpha$, $(t', p') \Vdash \dot{\alpha}_\xi < \alpha$. Moreover if $\theta = \kappa$, there is a sequence of P_α-names $(\dot{\alpha}'_\xi)_{\xi<\alpha}$ such that for each $\xi < \alpha$, $(t', p') \Vdash \dot{\alpha}_\xi = \dot{\alpha}'_\xi$.*

The "moreover" clause is of course meaningless if $\theta < \kappa$. Before we treat the lemma, we draw two corollaries.

Corollary 7.7. *1. If $r \in L[G_\kappa]$ is a real, there is $\alpha < \kappa$ such that $r \in L[G_\alpha]$. In particular, κ remains uncountable in $L[G_\kappa]$ (i.e. $\kappa = \omega_1$ in the final model).*

2. *If $\theta < \kappa$, κ remains Mahlo in $L[G_\theta]$.*

Proof. For the first corollary, fix a real $r \in L[G_\kappa]$ and let $\dot{\alpha}_n$ be a P_κ-name for $r(n)$, for each $n \in \mathbb{N}$. The lemma shows we can find $(t', p') \in \bar{G}_\kappa$, $\alpha < \kappa$ and a sequence of P_α-names $\{\dot{\alpha}'_n \mid n \in \mathbb{N}\}$, such that for each $n \in \mathbb{N}$, $(t', p') \Vdash \dot{\alpha}_n = \dot{\alpha}'_n$. Obviously, $r \in L[G_\alpha]$.

For the second, say $\theta < \kappa$ and fix a P_θ-name $\dot{C}$ for a closed unbounded subset of κ. Let $\dot{\alpha}_\xi$ be a name for the least element of $\dot{C}$ above ξ. By the lemma, we may find an inaccessible $\alpha \in (\lambda_\theta, \kappa)$ and $(t', p') \in G_\theta$ such that for each $\xi < \alpha$, $(t', p') \Vdash \dot{\alpha}_\xi < \alpha$. Thus, $(t', p') \Vdash \check{\alpha} \in \dot{C}$. Now observe that as P_θ is stratified above λ_θ, α is inaccessible in $L[G_\theta]$. □

Before we begin the proof we'd like to remind the reader of the following terminology. Let $\eta < \theta$ a coding stage and work in $L[\bar{B}^-, G^o_\eta] = L[A_\eta]$, where $\dot{B}^-$, G^o, A_η are as defined in section 7.1 (see p. 116; what we call η here is called ξ there unfortunately) and A_η is the set to be coded at this stage, as defined in section 4.2, see p. 31. Let H be any set and let $p, q \in P$, $q \leq p$.

We say that $q \in P(A_\eta)$ is basic generic for (H, p) at κ^+ if and only if

1. $|q_\kappa| > H \cap \kappa^+$;
2. if $\nu \in H \cap [\kappa^+, \kappa^{++})$ then $b^{p_{\kappa^+} \restriction \nu} \in q^*_{\kappa^+}$;
3. if $\nu \in H \cap [\kappa^+, \kappa^{++})$ then there is $\zeta > |p_\kappa|$ such that $q_\kappa((\zeta)_1) = p_{\kappa^+}(\nu)$;
4. if $\xi \in H \cap [\kappa, \kappa^+)$ there is $\nu > |p_\kappa|$ such that $q_\kappa((\langle \xi, \nu \rangle)_0) = 1$ if $\xi \in A_\eta$ and $q_\kappa((\langle \xi, \nu \rangle)_0) = 0$ if $\xi \notin A_\eta$;

We say that $q \in P(A_\eta)$ is basic generic for (H, p) at κ if and only if

1. $|q_{<\kappa}| \geq \sup(H \cap \kappa)$.
2. if $\nu \in H \cap [\kappa, \kappa^+)$ then $b^{p_\kappa \restriction \nu} \setminus \eta' \in q^*_\kappa$ for some $\eta' < \kappa$;
3. if $\nu \in H \cap [\kappa, \kappa^+)$ then there is $\xi \in b^{p_\kappa \restriction \nu} \setminus \eta$ such that $\xi \geq |p_{<\kappa}|$ and $q_{<\kappa}(\{\xi\}_2) = p_\kappa(\nu)$;
4. if $\xi \in H \cap \kappa$ there is $\nu > |p_\delta|$ such that $q_{<\kappa}((\langle \xi, \nu \rangle)_0) = 1$ if $\xi \in A_\eta$ and $q_{<\kappa}((\langle \xi, \nu \rangle)_0) = 0$ if $\xi \notin A_\eta$;

We shall use these notions as a replacement for **D** in the following proof, since this approach simplifies the argument showing that the sequence we build is appropriately definable (or "canonical"). We need to do this because we build the canonical witness (as usual, a sequence of models, this time of two different types) in advance, before we build the associated sequence of conditions.

Proof of lemma 7.6. The proof is by induction on θ, so assume lemma 7.6 holds for all $\theta' < \theta$. Let $\{.\}_i$ for $i \in \{0,1\}$ be a pair of recursive functions such that $\xi \mapsto (\{\xi\}_0, \{\xi\}_1) \in \mathrm{On}^2$ is surjective and each pair in $\alpha \times \alpha$ occurs α-many times as $(\{\xi\}_0, \{\xi\}_1)$ for $\xi < \alpha$, whenever $\alpha \in \mathbf{Card}$. Also, we denote by $h\colon \mathrm{On} \to L$ the map enumerating the constructible universe according to its canonical well-ordering. We assume without loss of generality that $(t,p) \in \mathrm{dom}(\mathbf{C}^\kappa) \cap \mathrm{dom}(\mathbf{C}^{\kappa^+})$.

Start by inductively choosing continuous $\in$-chains N^ξ and M^ξ for $\xi < \kappa$ as follows. Set

$$\vec{x} = \{\kappa^{+++}, (\dot{\alpha})_{\xi<\kappa}, P_\theta, (t,p)\}$$
$$M^{-1} = \vec{x}$$
$$N^{-1} = \kappa \cup \{\vec{x}\}.$$

Assuming we have $M^\xi \in N^\xi$, let $M^{\xi+1} \prec_{\Sigma_1} L$ be least (of size $< \kappa$) such that $N^\xi \in M^{\xi+1}$ and $M^{\xi+1} \cap \kappa \in \kappa$, and choose $N^{\xi+1} \prec_{\Sigma_1} L$ least (of size κ) such that $M^{\xi+1} \in N^{\xi+1}$ and $N^{\xi+1} \cap \kappa \in \kappa^+$. For limit $\rho < \kappa$, let

$$M^\rho = \bigcup_{\xi<\rho} M^\xi$$
$$N^\rho = \bigcup_{\xi<\rho} N^\xi$$

and write $\kappa(\xi) = M^\xi \cap \mathrm{On}$. Observe each M^ξ and N^ξ are closed under h and h^{-1} and each $\kappa(\xi)$ is a strong limit cardinal.

Let $C = \{\kappa(\xi) \mid \xi < \kappa\}$ and find $\alpha \geq \beta_0$, an inaccessible limit point of C such that $\diamondsuit_\alpha = C \cap \alpha$ (remember we have chosen to denote by $(\diamondsuit_\xi)_\xi$ the canonical diamond sequence of L concentrating on inaccessibles below κ). Pick $C^* \subseteq \lim C \cap \mathbf{Sing}$ such that C^* is club in α (where $\lim C$ of course denotes the set of limit point of C). Now we construct a sequence of $(t^\xi, p^\xi) \in \bar{P}_\theta$ by induction on $\xi \leq \alpha$, starting with $(t^0, p^0) = (t,p)$.

The successor step: At successor stages, assume we have constructed $(t^\xi, p^\xi) \in M^{\xi+1}$ and when ξ is a limit, also assume that for each coding stage $\eta < \theta$,

$$(t^\xi, p^\xi \upharpoonright \eta) \Vdash |p^\xi(\eta)_{<\kappa}| = \kappa(\xi).$$

First pick $(t', p') \preccurlyeq^{<\kappa} (t^\xi, p^\xi)$ least such that $(t', p') \in M^{\xi+1}$ meeting the following requirements:

1. If ξ is a limit ordinal, then for each coding stage $\eta < \theta$ we have

$$(t', p' \restriction \eta) \Vdash p'(\eta)_{<\kappa}(\kappa(\xi)) = i,$$

   where $i = 0$ if $\kappa(\xi) \in C^*$ and $i = 1$ otherwise.

2. If there is a condition $(s_0, r_0) \leq (t^\xi, p^\xi)$ with $h(\{\xi\}_0) \in \mathbf{C}^{\kappa(\xi)}(s_0, r_0)$, we demand that for some $(s_1, r_1) \leq (s_0, r_0)$ such that $(s_1, r_1) \in \mathrm{dom}(\mathbf{C}^\kappa) \cap \mathrm{dom}(\mathbf{C}^{\kappa^+})$ which decides $\dot\alpha_{\{\xi\}_1}$, we have that

$$t' \leq s_1,$$

   for every stage $\eta \in E^1$ were we force with κ-Cohen $p'(\eta) \leq r^1(\eta)$ and for every coding stage $\eta < \theta$, $(t', p' \restriction \eta)$ forces both that

$$(p'(\eta)_{<\kappa}, p'(\eta)^*_\kappa) \leq (r_1(\eta)_{<\kappa}), r_1(\eta)^*_\kappa) \text{ in } P^{p'(\eta)_\kappa}$$

   and that

$$(p'(\eta)_\kappa, p'(\eta)^*_{\kappa^+}) \leq (r_1(\eta)_\kappa), r_1(\eta)^*_{\kappa^+}) \text{ in } P^{p'(\eta)_{\kappa^+}}$$

   Moreover we demand $(t', p') \preccurlyeq^{<\kappa} (t^\xi, p^\xi)$. Write $r^\xi_i = r_i$, where $i \in \{0, 1\}$. We also define α^ξ_ν to be the ordinal such that $(s_1, r^\xi_1) \Vdash \dot\alpha_\nu = \check\alpha^\xi_\nu$.

3. For every $\eta < \theta$, $(t', p' \restriction \eta) \Vdash \rho(p'(\eta)_{<\kappa}) \geq \kappa(\xi)$.

4. For every $\eta < \theta$ which is a coding stage, we have that

$$(t', p' \restriction \eta) \Vdash \text{“}p'(\eta)_{<\kappa} \text{ is basic generic for } (M^\xi, p^\xi(\eta)) \text{ at } \kappa \text{ and } p'(\eta)_\kappa \text{ is basic generic for } (N^\xi, p^\xi(\eta)) \text{ at } \kappa^+.\text{”} \quad (7.4)$$

5. Notice that in the previous item we have ensured basic genericity over subsets of L, while we want to capture some information about $L[A_\eta]$; so we ask the following. For every $\eta < \theta$ and every P_η-name for an ordinal $\dot\alpha \in M^\xi$ there is a set a of size $< \kappa$ such that $(t', p' \restriction \eta) \Vdash \dot\alpha \in \check a$. Also, for every $\eta < \theta$ and every P_η-name for an ordinal $\dot\beta \in N^\xi$ there is a set b of size κ such that $(t', p' \restriction \eta) \Vdash \dot\beta \in \check b$.

All of the above requirements can be expressed by a formula which is Σ_1 in parameters from $\vec{x} \cup \{M^\xi, (t^\xi, p^\xi)\}$, so if (t', p') satisfying the above can be found at all then we can demand $(t', p') \in M^{\xi+1}$.

Requirements 2 and 3 are trivial. Requirement 4 can be met by a density argument identical to the one showing the "density" of strategic class $\mathbf{D}$ in the limit of an iteration. The difference is purely notational (we've treated the Mahlo coding as lower part at κ and as upper part for $\lambda < \kappa$, now we are "in-between" these cases):

Lemma 7.8. *The set of all $(t', p' \restriction \eta)$ such that for every $\eta < \theta$, (7.4) holds is dense below (t^ξ, p^ξ)*

Proof of claim. Say we are given $(t', q) \leq (t^\xi, p^\xi)$. Exactly as in 5.12, construct $p'(\eta)$ by induction on η. At successor stages, let $p'(\eta)$ be a name for a condition such that (7.4) is met. This is possible as $(t', q \restriction \eta)$ forces such a condition to exist, by corollary 4.34 (see p. 57). Observe that if $(t', q \restriction \eta) \Vdash q(\eta)_{<\kappa} = \emptyset$, then we can choose $p'(\eta) \preccurlyeq^\kappa q(\eta)$. Thus, the resulting condition (t', p') has legal support.

□

Lastly, we can satisfy 5: for the first line, use induction hypothesis and $\|M^\xi\| < \kappa$, for the second use $\{\kappa\}$-stratification. This shows we can find (t', p') as above.

Find $(t^{\xi+1}, p^{\xi+1}) \preccurlyeq^{<\kappa} (t', p')$ least such that $(t^{\xi+1}, p^{\xi+1}) \in \operatorname{dom}(\mathbf{C}^{\kappa^+})$ and $(t^{\xi+1}, p^{\xi+1}) \in M^{\xi+1}$. Now set $(t^{\xi+1}, p^{\xi+1})$ to be (t', p'). Note that by diagonal support, requirement 5 and since $M^{\xi+1}$ is Σ_1-elementary, for each $\eta < \theta$ we have

$$(t^{\xi+1}, p^{\xi+1}(\eta)) \Vdash |p^{\xi+1}(\eta)_{<\kappa}| < \kappa(\xi + 1). \tag{7.5}$$

(and analogously, a similar equation holds for $|p^{\xi+1}(\eta)_\kappa|$). By induction, we can infer

$$(t^\xi, p^\xi(\eta)) \Vdash \{\zeta \mid p^\xi(\zeta)_{<\kappa} = 1\} \cap \lim C = C^* \cap \kappa(\xi),$$

for all limit ξ.

Taking greatest lower bound at the limit step: At limit $\rho \leq \alpha$, we take (t^ρ, p^ρ) to be the greatest lower bound of $(t^\xi, p^\xi)_{\xi<\rho}$; this is well-defined: we show by induction on $\eta < \theta$ that $(t^\rho, p^\rho \restriction \eta)$ is a condition. Since we are always taking $\preccurlyeq^{<\kappa}$-direct extensions, at amalgamation stages we can simply use induction to take the point-wise limit in each of the $\mathbb{Z}$-many components and the resulting $\mathbb{Z}$-sequence will be a condition in the amalgamation. By κ-closure of $\bar{T}$ and κ-Cohen forcing, we only have to treat coding stages.

So fix a coding stage η and assume $w = (t^\xi, p^\xi \restriction \eta)$ is a condition. We let $p^\xi(\eta)$ by the name for the obvious candidate for a lower bound of the sequence $(p^\nu(\eta))_{\nu<\eta}$ (see (4.2), p. 49).

Claim 7.9. *The condition w is M^ρ-generic, in the sense that*

$$w \Vdash M^\rho[\dot{G} \restriction \eta] \cap \mathrm{On} = M^\rho \cap \mathrm{On}\,.$$

It is N^ρ-generic in the same sense.

Proof. This follows from requirement 5 in the construction of the sequence and by Σ_1-elementarity. □

Observe a consequence of this is that we can treat $M^\rho[\dot{G} \restriction \eta]$ as a generic extension of M^ρ by P_η; likewise for $N^\rho[\dot{G} \restriction \eta]$.

Claim 7.10. *We have that $w \Vdash p^\rho(\eta)_{<\kappa} \in S^*_{<\kappa}$.*

Proof. Let $G \restriction \eta$ denote an arbitrary P_η-generic for the moment. We work in $L[\bar{B}^-, G^o \restriction \eta] = L[A_\eta]$, as usual (see sections 4.2 and 7.1). Let $N = L_\gamma[A_\eta \cap \alpha, p^\xi(\eta)_{<\kappa}]$ be a $<\kappa$-test model such that $N \vDash \alpha$ is the least Mahlo. Let $\bar{M}^\xi$ be the transitive collapse of M^ξ for $\xi \leq \rho$. Letting $\bar{\mu} = \bar{M}^\rho \cap \mathrm{On}$, we show $\gamma < \bar{\mu}$; for otherwise, since $(\bar{M}^\xi)_{\xi<\rho}$ is definable over $\bar{M}^\rho = L_{\bar{\mu}}$ which is in turn definable in N, we find $C \cap \alpha \in N$ and thus $\lim C^* \in N$, contradicting that $N \vDash \alpha$ is Mahlo.

Thus indeed, $\gamma < \bar{\mu}$. Now we can quote the last part of the proof theorem 4.24, which easily shows that by requirement 4 and by claim 7.9 in the construction of the sequence and by elementarity, for $A^* = A^*(p^\rho_{<\kappa}(\eta))$, $L_\gamma[A^*] \vDash r^\eta$ is coded by branches. □

Claim 7.11. *We have that $w \Vdash p^\xi(\eta)_\kappa \in S^*_\kappa$.*

Proof. This is completely analogous to the previous claim: The proof for theorem 4.24 on page 48 carries over to the present situation almost verbatim (in the case $\delta = \kappa$). As in the previous claim, to show that the height of any test-model is less than that of the collapse of N^ρ, use that $(t^\xi, p^\xi)_{\xi<\rho}$ is appropriately definable over the transitive collapse of N^ρ using $\vec{x}$ and C^* as parameters. The additional parameter is unproblematic since $C^* \in \mathbf{H}(\kappa)$ and is therefore an element of each N^ρ. As in the previous claim, we may directly use basic genericity (requirement 4 in the construction) and claim 7.9 instead of $\mathbf{D}$ as we did in the proof of theorem 4.24. □

Observe when $\rho < \alpha$ is a limit ordinal, have $(t^\rho, p^\rho) \in M^{\rho+1}$: this is because $(t^\xi, p^\xi)_{\xi<\rho}$ is definable over $\langle M^\rho, C^* \cap \kappa(\rho)\rangle$ and $C^* \cap \kappa(\rho) \in M^{\rho+1}$ since $\kappa(\bar\rho + 1)$ is a strong limit cardinal. This ends the construction of the sequence (t^ξ, p^ξ), for $\xi \le \alpha$.

Finally, we extend (t^α, p^α) to $(t^\alpha, \bar p^\alpha)$ by making sure α is in the support at each coding stage, i.e.

$$\forall \eta < \theta \cap \alpha \cap E^2 \quad (t^\alpha, \bar p^\alpha \restriction \eta) \Vdash \alpha \in \operatorname{supp}(p^\alpha(\eta)) \tag{7.6}$$

We show $(t^\alpha, \bar p^\alpha) \Vdash \dot\alpha_\nu < \alpha$, for each $\nu < \alpha$. In fact, letting $\dot\alpha'_\nu$ be the name such that whenever $\{\xi\}_1 = \nu$ and r^ξ_1 is defined, $r^\xi_1 \Vdash \dot\alpha'_\nu = \check\alpha^\xi_\nu$, we will show that $(t^\alpha, \bar p^\alpha) \Vdash \dot\alpha_\nu = \dot\alpha'_\nu$. Observe that this is a P_α-name whenever $\alpha \le \theta$, so this proves the theorem.

So let $\nu < \alpha$, $(s, r) \le (t^\alpha, \bar p^\alpha)$ and assume $(s, r) \in \operatorname{dom}(\mathbf{C}^\alpha)$ and (s, r) decides $\dot\alpha_\nu$. Pick ξ such that $h(\{\xi\}_0) \in \mathbf{C}^\alpha(s, r)$, $\{\xi\}_1 = \nu$ and also, $h(\{\xi\}_0) \in \mathbf{H}(\kappa(\xi))$. The latter may be achieved by increasing ξ if needed, without changing $\{\xi\}_i$, $i \in \{0, 1\}$. Observe that it follows that $h(\{\xi\}_0) \in \mathbf{C}^{\kappa(\xi)}(s, r)$, by definition of $\mathbf{C}$.

Since α is inaccessible, we have $\xi < \alpha$, and r witnesses that r^ξ_1 is defined. We now show that $(s, r) \cdot (t^\alpha, r^\xi_1) \ne 0$. This is clear for the $\bar T$-part, since $s \le t^\alpha$. We show that for each $\eta < \theta$, $(s, r \restriction \eta \cdot r^\xi_1 \restriction \eta) \Vdash r(\eta) \cdot r^\xi_1(\eta) \ne 0$. The only non-trivial cases are amalgamation and coding stages. At amalgamation stages, for each $i \in \mathbb{Z}$, we can takes point wise meets by induction (or by running the present argument again). Observe that since the outcome is a $\preccurlyeq^{<\kappa}$-direct extension of $r^\xi_1 \restriction \eta + 1$, the resulting sequence is a condition in the amalgamation, i.e. in $P_{\eta+1}$.

Now assume η is a coding stage ($\eta \in E^2$). Letting $w = (s, r \restriction \eta \cdot r^\xi \restriction \eta)$, we have that w forces that

$$(r(\eta)_{<\kappa}, r(\eta)^*_\kappa) \le (r^\xi_1(\eta)_{<\kappa}, r^\xi_1(\eta)^*_\kappa) \text{ in } P^{r(\eta)_\kappa}$$

and

$$(r(\eta)_\kappa, r(\eta)^*_{\kappa^+}) \le (r^\xi_1(\eta)_\kappa, r^\xi_1(\eta)^*_{\kappa^+}) \text{ in } P^{r(\eta)_{\kappa^+}}.$$

Moreover, w forces that $r^\xi_1(\eta)$ is an extension of a condition $r^\xi_0(\eta)$ which has "lower part $h(\eta)$", i.e. $h(\eta) \in \mathbf{C}^{\kappa(\xi)}(r^\xi_0(\eta))$. Since $w \Vdash h(\eta) \in \mathbf{C}^\kappa(\xi)(r(\eta))$, w forces that $r^\xi_0(\eta) \restriction \kappa(\xi) = r(\eta) \restriction \alpha \ge r^\xi_1(\eta) \restriction \alpha(\xi)$. It follows immediately that that $w \Vdash r(\eta) \restriction [\alpha, \kappa^+] \cup r^\xi_1 \restriction \alpha$ is a condition, call it w'. Moreover, note that w forces that

$$\operatorname{supp}(r^\xi_1(\eta)) \cap \kappa \subseteq \kappa(\xi + 1). \tag{7.7}$$

It remains to show that $w \Vdash w' = r(\eta) \cdot r^\xi_1(\eta)$. It is clear that $w \Vdash w' \le r^\xi_1$; in order to see $w \Vdash w' \le r(\eta)$ we must check that making the extension from

$r(\eta) \upharpoonright \alpha$ to $r_1^\xi(\eta)$ below α did not violate any of the restraints in $r(\eta)$ at and above α. Since $w \Vdash \rho(p^\alpha(\eta)_\kappa) \geq \alpha$, w also forces that making the extension of $r(\eta) \upharpoonright \alpha$ to $r_1^\xi(\eta) \upharpoonright \alpha$ respects all restrains in $r(\eta)^*_\kappa$. Since the restraints in $r(\eta)^*_\alpha$ are spaced by $\diamondsuit_\alpha$ (see the discussion following definition 4.4 on page 32) and by (7.7), making this extension obeys all restraints in $r(\eta)^*_\alpha$. Lastly, since (7.6) makes sure that $w \Vdash \alpha \in \text{supp}(r(\eta))$, no restraints are violated by this extension. In other words, we have shown that $w \Vdash w' = r(\eta) \cdot r_1^\xi \neq 0$. This finishes the proof that $(t^\alpha, r_1^\xi) \cdot (s, r) \neq 0$ and so since (s, r) decides $\dot\alpha_\nu$, we must have $(s, r) \Vdash \dot\alpha_\nu = \check\alpha^\xi_\nu$. This concludes the proof of lemma 7.6 □

It is crucial that by lemma 7.6, the book-keeping devices $\bar r$ and $\bar s$ "catch" all the relevant reals in the final extension by P_κ:

Lemma 7.12. *If $\iota < \kappa$, $\dot r^0, \dot r^1$ are P_κ-names for reals and $p \in P_\kappa$ forces $\dot r^0, \dot r^1$ are random over L^{P_ι}, there is $q \leq p$ and $\nu < \kappa$ such that $\bar\iota(\nu) = \iota$*

$$q \Vdash \dot r^j = \dot r^j_\nu.$$

If $\dot s$ is a P_κ-name for a real and $p \in P_\kappa$, there is $q \leq p$ and $\xi < \kappa$ such that either $q \Vdash \dot s = \dot s_\xi$, or $q \Vdash \dot s \in \Gamma_\xi$.

Proof. By lemma 7.6 there is $q \leq p$, $\xi < \kappa$ and P_ξ-names $\dot x^0, \dot x^1$ such that $q \Vdash \dot r^j = \dot x^j$. As P_κ collapses the continuum of any initial stage of the iteration, we may assume 1_{P_κ} forces $\dot x^j$ is random over L^{P_ν}. Using the notation from lemma 7.3, find ν, ν' and $q' \leq q$ such that $q' \Vdash \dot x^0 = \dot x_\nu$ and $q' \Vdash \dot x^1 = \dot x_{\nu'}$, and find $q'' \leq q'$ and $y \in Y(\nu, \nu')$ such that $q'' \Vdash (\dot x^0, \dot x^1) = y$. Thus, by construction of $\bar r$, we may find ν such that $\bar\iota(\nu) = \iota$ and $y = (\dot r^0_\nu, \dot r^1_\nu)$, and we have $q'' \Vdash \dot r^0 = \dot r^0_\nu$ and $\dot r^1 = \dot r^1_\nu$.

The second claim follows immediately from lemmas 7.6, 7.5 and the definition of $\bar s$. □

7.5 Every projective set of reals is measurable

Write $G = G_\kappa$.

Lemma 7.13. *For any $\nu < \kappa$, $\bigcup N^*_\nu$ is a null set, where*

$$N^*_\nu = \{N \in L[G_\nu] \mid L[G_\nu] \models N \subset \mathbb{R} \textit{ has measure zero}\}$$

Proof. Every null set $N \in L[G_\nu]$ is covered by a null Borel set whose Borel code is also in $L[G_\nu]$. The set C^* of Borel codes for null sets in $L[G_\nu]$ is countable in $L[G]$, so $\bigcup N^*_\nu$, which is equal to the union of all the Borel sets with code in C^*, is a countable union of null sets in $L[G]$. □

The following, together with the last lemma, suffices to show that in the extension by P_κ, every projective set of reals is measurable.

Lemma 7.14. *Let $\nu < \kappa$. There is a name $\dot{r}_*$ which is fully random over L^{P_ν} such that the following hold:*

1. *Let $\dot{B}(\dot{r}_*)$ be a P_ν-name for the complete sub-algebra of $B_\kappa : P_\nu$ generated by $\dot{r}_*$ in $L[G_\nu]$ and let $B^0 = P_\nu * \dot{B}(\dot{r}_*)$. For any $b \in B_\kappa \setminus B^0$, there is an automorphism Φ of B_κ such that $\Phi(b) \neq b$ and $\Phi \restriction B^0 = \mathrm{id}$.*

2. *For any P_κ-name $\dot{r}$ which is random over L^{P_ν} and any $p \in P_\kappa$ there is $q \leq p$ and an automorphism Φ of B_κ such that $q \Vdash \dot{r} = \Phi(r_*)$ and $\pi_\nu \circ \Phi = \Phi \circ \pi_\nu = \pi_\nu$.*

Proof. For $\dot{r}_*$ we may use any $\dot{r}^0_\eta$ (from our list $\bar{r}$) such that $\bar{\iota}(\eta) \geq \nu$ (i.e. its fully random over L^{P_ν}).

Let π be the canonical projection $\pi \colon B_\kappa \to B^0$, where B^0 is as in the hypothesis of item 1 of the lemma. Pick $\xi < \kappa$ such that

1. $\pi_\xi(b) \notin B^0$; this holds for large enough ξ since $b \notin B^0$;

2. r^* is a P_ξ-name, i.e. B^0 is a complete sub-algebra of B_ξ.

3. $P_{\xi+1} = \mathbf{Am}_1(P_\iota, D_\xi, r_*, r_*)$, where $\iota \geq \nu$.

Let b_0 denote $\pi_\xi(b)$. Clearly, there is $p \in D_\xi$, $p \leq b_0$ such that $\pi(p) \not\leq b_0$: for otherwise, the set

$$X = \{d \in B^0 \mid d \leq b_0\}$$

would be predense in P_ξ below b_0, and thus $b_0 = \sum X \in B^0$, contradiction.

So pick p as above and let $q \in D_\xi$, $q \leq \pi(p)$ such that $q \cdot b_0 = 0$. Let $b_1 = \pi(q)$. Look at the condition $\bar{p} \in (D_\xi)^{\mathbb{Z}}_f$ such that $\bar{p}(-1) = q$, $\bar{p}(0) = b_1 \cdot p$ and for $i \in \mathbb{Z} \setminus \{-1, 0\}$, $\bar{p}(i) = b_1$. Then we have $\bar{p} \in (D_\xi)^{\mathbb{Z}}_f$, $\bar{p} \leq p \leq b_0$. Letting Φ denote the automorphism of B_κ resulting from $P_{\xi+1}$, we have $\Phi(\bar{p}) \leq q$ whence $\Phi(\bar{p}) \cdot b_0 = 0$. So as $\bar{p} \leq \pi_\xi(b)$ and $\Phi(\bar{p}) \cdot b = 0$, it follows that $\Phi(b) \neq b$; for otherwise since $\bar{p} \leq \pi_\xi(b)$, we have $\bar{p} \cdot b \neq 0$ but $\Phi(\bar{p} \cdot b) = \Phi(\bar{p}) \cdot b = 0$.

The second claim is clear from the construction, as $\Phi_\rho(\dot{r}^0_\rho) = \dot{r}^1_\rho$ for each $\rho < \kappa$.

□

Finally, we show in $L[G]$:

Lemma 7.15. *Say $s \in [\mathrm{On}]^\omega$, ϕ a formula. If $X = \{r \in \mathbb{R} \mid \phi(r, s)\}$, X is measurable.*

Proof. Let X, s be as above, and say $s = \dot{s}^G$. Without loss of generality, $\dot{s}$ is a P_ν-name, where $\nu < \kappa$ and $\Vdash_\nu \dot{s} \in [\mathrm{On}]^\omega$ (by lemma 7.6). Fix $\dot{r}_*$ as in the previous lemma. Let $\dot{B}(\dot{r}_*)$ be a P_ν-name for the complete sub-algebra of $B_\kappa : P_\nu$ generated by $\dot{r}_*$ in $L[G_\nu]$.

Claim. $\|\phi(\dot{r}_*, \dot{s})\|^{B_\kappa} \in \mathrm{r.o.}(P_\nu) * B(\dot{r}_*)$.

Proof of Claim. Write $B^0 = P_\nu * \dot{B}(\dot{r}_*)$ and $b = \|\phi(\dot{r}_*, \dot{s})\|^{B_\kappa}$. Towards a contradiction, assume $b \notin B^0$. By lemma 7.14, (1), there is an automorphism Φ of B_κ such that $\Phi(b) \neq b$ while $\Phi(\dot{s}) = \dot{s}$ and $\Phi(\dot{r}) = \dot{r}$. This is a contradiction, as we infer

$$b = \|\phi(\dot{r}_*, \dot{s})\|^{B_\kappa} = \|\phi(\Phi(\dot{r}_*), \Phi(\dot{s}))\|^{B_\kappa} = \Phi(b).$$

□

Let N^* denote

$$\bigcup\{N \in L[G_\nu] \mid L[G_\nu] \models N \text{ has measure zero}\},$$

and let $\dot{N}^*$ be a P_ν-name for this set. N^* is null in $L[G]$.

We find a Borel set B such that for arbitrary $r \notin N^*$, we have $r \in X \iff r \in B$. Then $X \setminus N^* = B \setminus N^*$ is measurable, finishing the proof. We may regard $B(\dot{r}_*)$ as identical to the Random algebra in $L[G_\nu]$, so we may write $\|\phi(\dot{r}_*, \dot{s})\|^{B_\kappa : B_\nu} = [B]_{\mathbf{n}}$ for a Borel set B.

To show B is the Borel set we were looking for, let $r \notin N^*$ be arbitrary. Find $\dot{r}$ and $p \in G$ such that $\dot{r}^G = r$ and $p \Vdash \dot{r} \notin \dot{N}^*$, i.e. p forces $\dot{r}$ is random over L^{P_ν}. By 2 of the previous lemma, there is an automorphism Φ of P_κ and $q \in G$ such that $q \Vdash \Phi(\dot{r}_*) = \dot{r}$, and thus $\Phi(\dot{r}_*)^G = \dot{r}_*^{\Phi^{-1}[G]} = \dot{r}^G$. We also have $\pi_\nu \circ \Phi = \Phi \circ \pi_\nu = \pi_\nu$ and so $\Phi(\dot{s})^G = \dot{s}^{\Phi^{-1}[G]} = \dot{s}^G$. We have

$$\begin{aligned} \phi(\dot{r}^G, \dot{s}^G) \iff \|\phi(\dot{r}, \dot{s})\| \in G \iff \\ \iff \bar{\Phi}^{-1}(\|\phi(\dot{r}, \dot{s})\|) \in \bar{\Phi}^{-1}[G] \iff \\ \iff \|\phi(\bar{\Phi}^{-1}(\dot{r}), \dot{s})\| \in \bar{\Phi}^{-1}[G] \iff \\ \iff \|\phi(\dot{r}_*, \dot{s})\| \in \bar{\Phi}^{-1}[G] \cap B(\dot{r}_*) \iff \dot{r}_*^{\bar{\Phi}^{-1}[G]} \in B \end{aligned}$$

As $\dot{r}_*^{\bar{\Phi}^{-1}[G]} = \dot{r}^G$, we are done. □

Chapter 8

The set Γ^0 is Δ^1_3

We now check that Γ^0 is in fact Δ^1_3. By [Bar84], this is optimal, since under the assumption that all Σ^1_2 sets are Lebesgue-measurable, all Σ^1_2 sets do have the property of Baire.

Let $\Theta(r, s, \alpha, \beta)$ be the sentence

> $L_\beta[r, s]$ is a model of ZF^- and of "α is the least Mahlo and α^{++} exists".

Definition 8.1. For an ordinal α and $C \in {}^\alpha 2$, write $\sigma \lhd C$ to express σ is an initial segment of C, i.e. for some $\rho < \alpha$, $\sigma = C \restriction \rho$. Let $\phi(x)$ be a formula. When we write $\forall^* \sigma \lhd C \phi(\sigma)$, we mean there exists $\zeta < \alpha$ such that for all $\rho \in (\zeta, \alpha)$, $\phi(C \restriction \rho)$ holds. In other words, *for almost all initial segments σ of C, $\phi(\sigma)$ holds.*

For $j \in \{0, 1\}$, let $\Psi(r, j)$ denote the formula

> $\exists s \in {}^\omega 2 \quad \forall \alpha, \beta < \kappa$ *if* $\theta(r, s, \alpha, \beta)$ *then:*
> $L_\beta[r, s] \models$"$\exists C \in {}^\alpha 2 \quad \forall^* \sigma \lhd C \quad \forall (n, i) \in \omega \times 2$
> $r(n) = i \Rightarrow T^\alpha(\sigma, n, i, j)$ has a branch."

Lemma 8.2. *For $j \in \{0, 1\}$ and any real r,*

$$r \in (\dot{\Gamma}^j)^G \iff \Psi(r, j).$$

Proof. For $\xi \leq \kappa$, let $\mathcal{F}_\xi$ be the smallest set closed under (relational) composition and containing all functions $F = \Phi^\zeta_\rho, (\Phi^\zeta_\rho)^{-1}$ such that $\operatorname{dom} F \subseteq P_\xi$. In other words, $\mathcal{F}_\xi$ is the closure under relational composition of

$$\{\Phi^\zeta_\rho, (\Phi^\zeta_\rho)^{-1} \mid E^3(\alpha^\zeta_\rho) < \xi\}.$$

First assume $r \in (\dot\Gamma^j)^G$ and show $\Psi(r,j)$ holds. If $j=0$, by definition of $\dot\Gamma^0$ we can find $\eta<\kappa$ and $\Phi \in \mathcal{F}_\kappa$ such that $r=(\Phi(\dot c_\eta))^G$. If $j=1$, we fix $\eta<\kappa$ such that $r=(\dot s_{\eta-1})^G$. Let $\dot r_0$ denote $\dot c_\eta$ if $j=0$ and let $\dot r_0$ denote $\dot s_{\eta-1}$ if $j=1$ and let $r_0=(\dot r_0)^G$.

In either case, at stage $\xi=E^2(\eta)$ we force with Jensen coding, adding a real s_0 such that

> for all $\alpha,\beta<\kappa$, if $\theta(r_0,s_0,\alpha,\beta)$ then $C_\eta \restriction \alpha, r_0 \in L_\beta[s_0]$ and $L_\beta[s_0] \models$ "$\forall\sigma$ such that $\sigma \lhd C_\eta \restriction \alpha$ and for all n,i such that $r_0(n)=i$, $T^\alpha(\sigma,n,i,j)$ has a branch".

So

$$1_{P_\kappa} \Vdash \Psi(\dot r_0, j),$$

which completes the proof in case $j=1$. For $j=0$, apply Φ to get

$$1_{P_\kappa} \Vdash \Psi(\Phi(\dot r_0), j),$$

and we are done as $(\Phi(\dot r_0))^G=r$.

Now assume $\Psi(r,j)$ and show $r\in(\dot\Gamma^j)^G$: Fix s to witness $\Psi(r,j)$. It must be the case that

$$\begin{aligned}&L[r,s]\models \exists C\in {}^\kappa 2 \quad \forall^*\sigma\lhd C \quad \forall(n,i)\in\omega\times 2\\ &(r(n)=i)\Rightarrow T(\sigma,n,i,j) \text{ has a branch .}\end{aligned} \tag{8.1}$$

For let $L_\beta[r,s]$ be isomorphic to an elementary sub-model of $L_{\kappa^{+3}}[r,s]$ which contains r and s, and let α be the least Mahlo in $L_\beta[r,s]$. Then as $\theta(r,s,\alpha,\beta)$ holds, by $\Psi(r,j)$,

$$\begin{aligned}&L_\beta[r,s]\models \exists C\in {}^\alpha 2 \quad \forall^*\sigma\lhd C \quad \forall(n,i)\in\omega\times 2\\ &(r(n)=i)\Rightarrow T^\alpha(\sigma,n,i,j) \text{ has a branch .}\end{aligned}$$

So by elementarity, (8.1) holds.

Fix $\xi<\kappa$ such that $r,s\in W[G_\xi]$ and fix C witnessing (8.1). Pick $\zeta<\kappa$ such that

$$\begin{aligned}&\text{for } \Phi\in\mathcal{F}_\xi \text{ and } \nu,\nu'<\xi \text{ such that } \Phi\neq \mathrm{id} \text{ and } \nu\neq\nu' \text{ we have}\\ &\Phi(C_\nu)\restriction\zeta \neq C_{\nu'}\restriction\zeta.\end{aligned} \tag{8.2}$$

This is possible by lemma 7.4. As (8.1) holds, we can also assume ζ to be large enough so that whenever $r(n)=i$, and $\rho\geq\zeta$, $T(C\restriction\rho,n,i,j)$ has a branch in $L[r,s]$.

Since $r,s\in W[G_\xi]$, lemma 8.3 below gives us: for any n and i, if the tree $T(C\restriction\zeta,n,i,j)$ has a branch in $L[r,s]$ then there is $\Phi\in\mathcal{F}_\xi$ and $\eta<\xi$ such that $C\restriction\zeta\lhd\Phi(C_\eta)$. By (8.2), Φ and η are unique and do not depend on n and i, so let Φ and η be fixed. By the way, it follows that $C = \Phi(C_\eta)$ (which we do not use in the following). More importantly, whenever $r(n) = i$, $T(\Phi(C_\eta) \upharpoonright \zeta, n, i, j)$ has a branch in $L[r, s]$.

Moreover, lemma 8.3 yields that whenever $T(C \upharpoonright \rho, n, i, j)$ has a branch in $L[r, s]$,

1. if $j = 0$ then η is a limit and $(\Phi(\dot{c}_\eta))^G(n) = i$
2. if $j = 1$ then η is a successor and $(\Phi(\dot{s}_{\eta-1}))^G(n) = i$.

Thus, in the first case, $r = \Phi(c_\eta)$ and so $r \in (\dot{\Gamma}^0)^G$. In the second case, $r = (\Phi(\dot{s}_{\eta-1}))^G$. As $(\dot{s}_{\eta-1})^G \in (\dot{\Gamma}^1)^G$ and $(\dot{\Gamma}^1)^G$ is closed under all the automorphisms $\{\Phi^*_\rho \mid \rho < \kappa\}$ (by lemma 7.5), $r \in (\dot{\Gamma}^1)^G$. □

For $\xi \leq \kappa$, let I_ξ be the set of triples (σ, n, i, j) such that for some $\eta < \xi$ and $\Phi \in \mathcal{F}_\xi$, $\sigma \lhd \Phi(C_\eta)$ and

1. if η is limit ordinal, $\Phi(c_\eta)(n) = i$ and $j = 0$
2. if η is a successor ordinal, $\Phi(s_{\eta-1})(n) = i$ and $j = 1$.

Lemma 8.3. *Say $\xi_0 < \kappa$ and let u an arbitrary real in $L[G \upharpoonright \xi\xi_0]$. Then $T(\eta, \sigma, n, i, j)$ has a branch in $L[u]$ only if $(\eta, \sigma, n, i, j) \in I_{\xi_0}$.*

Proof. Fix $\nu \in I$, $\xi_0 < \kappa$ and $p_0 \in P_{\xi_0}$ such that $p_0 \Vdash \check{\nu} \notin \dot{I}_{\xi_0}$ in $L[\bar{B}]$, where $\nu = (\eta, \sigma, n, i, j)$. Let $\bar{B}^-$ denote $\{\bar{B}(\xi)\}_{\xi \in I \setminus \{\nu\}}$.

We will show in a moment that $P_{\xi_0}(\leq p_0)$ is equivalent to a forcing $P^*_{\xi_0} \in L[\bar{B}^-]$, whence $\bar{T} * P_{\xi_0}(\leq p_0)$ is equivalent to

$$\Big[\Big(\prod_{\zeta \in I \setminus \{\nu\}}^{<\kappa} T(\zeta)\Big) * P^*_{\xi_0}(\leq p_0)\Big] \times T(\nu).$$

Assuming this for the moment, we can prove the lemma thus: As $T(\nu)$ doesn't add reals, any real $u \in L[\bar{B}][G_{\xi_0}]$ is actually an element of $L[\bar{B}^-][G_{\xi_0}]$, and as $T(\nu)$ is Suslin in $L[\bar{T}^-]$ and $P^*_{\xi_0}$ is κ^+-centered, $T(\nu)$ remains Suslin in $L[\bar{B}^-][G_{\xi_0}]$ and thus in $L[u]$. It remains to see that $P_{\xi_0}(\leq p_0)$ is equivalent to a forcing which is an element of $L[\bar{B}^-]$. For the purpose of carrying out the inductive proof, we prove a stronger statement, in the claim below. First, note that for $\xi < \bar{\xi} \leq \xi_0$, as $I_\xi \subseteq I_{\bar{\xi}}$,

$$\|\nu \notin \dot{I}_{\bar{\xi}}\| \leq \|\nu \notin \dot{I}_\xi\|,$$

and so

$$\pi_\xi(\|\nu \notin \dot{I}_{\bar{\xi}}\|) \leq \|\nu \notin \dot{I}_\xi\|$$

Let b_ξ denote $\|\nu \notin \dot{I}_\xi\|^{B_\xi}$, for $\xi < \xi_0$.

Claim 8.4. *For each $\xi \leq \xi_0$, there is an isomorphism*[1]

$$j_\xi \colon P_\xi(\leq b_\xi) \to P^*_\xi,$$

such that

$$\text{for } \xi < \bar{\xi} \leq \xi_0 \text{ and } p \in P_{\bar{\xi}}(\leq b_\xi),\ j_\xi(\pi_\xi(p)) = \pi_\xi(j_{\bar{\xi}}(p)). \tag{8.3}$$

Moreover, $P^*_\xi \in L[\bar{B}^-]$ and $P^*_\xi = P_\xi(\leq b_\xi) \cap L_{\kappa^{++}}[B^-]$.

There is no need to distinguish between G_ξ and $j_\xi[G_\xi]$, we write G_ξ for either one. Observe $\bar{T}$ is κ^{++}-distributive, so we have

$$\mathbf{H}(\kappa^{++})^{L[\bar{B}]} = \mathbf{H}(\kappa^{++})^L \tag{8.4}$$

At heart, the claim is a consequence of this simple observation:

Fact 8.5. If P has the κ^{++}-chain condition and $p \Vdash \dot{x} \in H(\kappa^{++})$, there is $\dot{x}' \in H(\kappa^{++})$ such that $p \Vdash \dot{x} = \dot{x}'$.

Proof. Use nice names. □

The induction splits into cases. For the successor case, assume j_ξ is already defined and define $j_{\xi+1}$. Observe that by induction, $\bar{T} * P_\xi$ is equivalent to

$$\Big[\big(\prod^{<\kappa}_{\zeta \in I \setminus \{\nu\}} T(\zeta)\big) * P^*_\xi\Big] \times T(\nu),$$

and since $T(\nu)$ is κ^{++}-distributive in $L[\bar{B}^-][G_\xi]$ (because it is still Suslin in that model),

$$H(\kappa^{++})^{L[\bar{B}][G_\xi]} \subseteq L[\bar{B}^-][G_\xi], \tag{8.5}$$

Easiest Case: As a warm up, assume $\xi \in E^1$. Thus $P_{\xi+1} = P_\xi \times Q$ for $Q \in L$. Of course, $P^*_\xi \times Q \in L[\bar{B}^-]$. We can set $j_{\xi+1}(p, q) = (j_\xi(p), q)$.

Observe that the claim asks for an isomorphism of $P^*_{\xi+1}$ with $P_\xi * \dot{Q}_\xi(\leq b_\xi)$, not $P_\xi(\leq b_\xi) * \dot{Q}_\xi$. So to formally satisfy the claim – and to make the induction work in the next step – restrict j_ξ to $P_{\xi+1}(\leq b_{\xi+1})$. We should check $P^*_\xi \times Q(\leq b_{\xi+1}) \in L[\bar{B}^-]$, though:

[1] By isomorphism, we mean of course j_ξ is injective on the separative quotient of its domain.

Lightheartedly identify $P_{\xi+1}$ names and $P^*_\xi \times Q$-names and assume (by fact 8.5 and (8.4)) that $\dot{I}_\xi \in L[\bar{B}^-]$. Then for $p \in P^*_\xi \times Q$,

$$p \Vdash \nu \in \dot{I}_{\xi+1}$$

is absolute between $L[\bar{B}]$ and $L[\bar{B}^-]$, so $P^*_\xi \times Q(\leq b_{\xi+1}) \in L[\bar{B}^-]$. So set $P^*_{\xi+1} = P^*_\xi \times Q(\leq b_{\xi+1})$.

Jensen Coding (and another easy case): Now, assume $\xi \in E^2$, i.e. $P_{\xi+1} = P_\xi * \dot{Q}_\xi$ where $\dot{Q}_\xi$ is a name for Jensen coding. Now it is crucial that we work below $b_\xi = \|\nu \notin \dot{I}_\xi\|^{B_\xi}$: Work in $L[\bar{B}][G_\xi]$ for now, where G_ξ is P^*_ξ-generic over $L[\bar{B}]$. Then $\nu \notin (\dot{I}_\xi)^{G_\xi}$, so the set of branches we code at this stage does not contain $B(\nu)$. Thus $\dot{Q}_\xi^{G_\xi}$ is a subset of $H(\kappa^{++})$ (of the extension), which is definable over $\langle H(\kappa^{++}), \bar{B}^-\rangle$. By (8.5), $\dot{Q}_\xi^{G_\xi} \in L[\bar{B}^-][G_\xi]$. This immediately implies that $P_{\xi+1}$ is equivalent to a forcing which is an element of $L[\bar{B}^-]$, but in order to carry out the inductive proof at limits, we need (8.3). For this, let $\phi(x)$ be a formula defining membership in $\dot{Q}_\xi^{G_\xi}$ over $\langle H(\kappa^{++}), \bar{B}^-\rangle$ in $L[\bar{B}][G_\xi]$. Set

$$P^*_{\xi+1} = \{(p, \dot{q}) \mid p \in P^*_\xi, \dot{q} \in H(\kappa^{++}) \text{ is a } P^*_\xi\text{-name, } P^*_\xi \Vdash \dot{\phi}(q)^{H(\kappa^{++})}\} \quad (8.6)$$

As P^*_ξ has the κ^{++}-chain condition, any $x \in H(\kappa^{++})^{L[\bar{B}][G_\xi]}$ has a P^*_ξ-name in $H(\kappa^{++})^{L[\bar{B}]}$. Therefore, by (8.4),

$$P^*_\xi \Vdash \dot{\phi}(q)^{H(\kappa^{++})}$$

is absolute between $L[\bar{B}]$ and $L[\bar{B}^-]$ and thus (8.6) witnesses that $P^*_{\xi+1} \in L[\bar{B}^-]$. For $(p, \dot{q}) \in P_{\xi+1}$, we can now define $j_{\xi+1}(p, \dot{q})$. Since

$$P^*_\xi \Vdash j_\xi(\dot{q}) \in j_\xi(\dot{Q}_\xi) \subseteq H(\kappa^{++})$$

using fact 8.5, we can find a P^*_ξ-name $\dot{q}' \in H(\kappa^{++})$ such that $P^*_\xi \Vdash j_\xi(\dot{q}) = \dot{q}'$. Let $j_{\xi+1}(p, \dot{q}) = (j_\xi(p), \dot{q}')$.

Clearly, $j_\xi(p, \dot{q}) \in P^*_{\xi+1}$. It is straightforward to check that $j_{\xi+1}$ preserves the ordering and is onto. It is injective on the separative quotient of $P_{\xi+1}$. Again, as in the previous case, restrict j_ξ to $P_{\xi+1}(\leq b_{\xi+1})$ to formally satisfy the claim. The case $\xi \in E^0$ can be treated in an analogous – but simpler – way.

Remark 8.6. Observe, by the way that for any P^*_ξ name $\dot{Q}$ such that $P^*_\xi \Vdash \dot{Q} = j_\xi(\dot{Q}_\xi)$,

$$P^*_{\xi+1} = (P^*_\xi * \dot{Q}) \cap H(\kappa^{++}).$$

In particular, it follows by induction that $P^*_{\xi+1} = P_{\xi+1} \cap H(\kappa^{++})$. Moreover, if $\dot{Q} \in L[\bar{B}^-]$ then

$$P^*_{\xi+1} = ((P^*_\xi * \dot{Q}) \cap H(\kappa^{++}))^{L[\bar{B}^-]}.$$

If $\dot{Q}_\xi$ is chosen reasonably (e.g. in the most obvious way), in fact $j(\dot{Q}_\xi) \in L[\bar{B}^-]$. [2] This means that we could find P^*_ξ by interpreting the definition of P_ξ *in* $L[B^-]$ if we commit to using only *names which have size at most* κ^+.

Amalgamation: Now let $\xi \in E^3$ and say $\xi = E^3(\alpha^0_\rho)$, i.e. $P_{\xi+1} = \mathbf{Am}_1(P_\iota, P_\xi, f, \lambda_\xi)$, where f is the isomorphism of the algebras generated by some P_ξ-names $\dot{r}_0$ and $\dot{r}_1$, and let π_i denote the canonical projection from P_ξ to the domain and range of f. Let Φ be the resulting automorphism. Let R denote the set of $\bar{p} \in P_{\xi+1}$ such that for all $i \neq 0$, $\bar{p}(i) \in \widehat{D_\xi}(\leq b_\xi)$ and $\bar{p}(0) \in P_\xi(\leq b_\xi)$. We show $P_{\xi+1}(\leq b_{\xi+1}) \subseteq R$ and that R is in $L[\bar{B}^-]$.

For the first, it is crucial that $\dot{I}_{\xi+1}$ is closed under Φ. Say $\bar{p} \in P_{\xi+1}(\leq b_{\xi+1})$, that is, $\bar{p} \Vdash \nu \notin \dot{I}_{\xi+1}$. By the definition of $\dot{I}_{\xi+1}$, for each $i \in \mathbb{Z}$,

$$\Phi^i(\bar{p}) \Vdash_{P_{\xi+1}} \nu \notin \dot{I}_{\xi+1},$$

and so

$$\bar{p}(i) = \bar{\pi}(\Phi^i(\bar{p})) \Vdash_{\widehat{P_\xi}} \nu \notin \dot{I}_\xi,$$

where $\bar{\pi}$ denotes the canonical projection from $P_{\xi+1}$ to $\widehat{P_\xi}$. Thus, $\bar{p} \in R$.

By induction, $P_\xi(\leq b_\xi)$ is isomorphic to P^*_ξ. A little care is needed to see $\widehat{D_\xi}(\leq b_\xi)$ is in $L[\bar{B}^-]$: $\widehat{D_\xi}(\leq b_\xi)$ is not the same as $\widehat{D_\xi(\leq b_\xi)}$ in general. The two orderings are equivalent, but once more, this doesn't mean that we can use them interchangeably in the definition of R. At the same time, $(p, \dot{b}_0, \dot{b}_1) \in \widehat{D_\xi}(\leq b_\xi)$ *does not imply* that $p \leq b_\xi$, and so p needn't be in the domain of j_ξ.

So we have to check that in fact, $B^*_\xi = \text{r.o.}(P^*_\xi) \in L[\bar{B}^-]$. This is because we may regard B^*_ξ the collection of regular open cuts which are given by antichains in P^*_ξ. As P^*_ξ has the κ^{++}-chain condition all such antichains and hence all regular cuts are in $L[\bar{B}^-]$ (once more by (8.4)). So $B^*_\xi \in L[\bar{B}^-]$ and j_ξ can be viewed as an isomorphism of $B_\xi(\leq b_\xi)$ with B^*_ξ. Thus, as we have assumed $\dot{r}_0$, $\dot{r}_1$ are in $L[\bar{B}^-]$, we can define $B(\dot{r}_i)^{B^*_\xi}$ and canonical projections from B^*_ξ to $P^*_\iota * (\langle \dot{r}_i \rangle^{B^*_\xi : P^*_\iota})$ in $L[\bar{B}^-]$. In fact,

$$P^*_\iota * (\langle \dot{r}_i \rangle^{B^*_\xi : P^*_\iota}) = j[C],$$

[2] It would be tempting to define $j_{\xi+1}(p, \dot{q}) = (j_\xi(p), j_\xi(\dot{q}))$, but we do not know if $j_\xi(\dot{q}) \in L[\bar{B}^-]$.

where C is the algebra obtained from $P_\iota * (\langle \dot{r}_i \rangle^{B_\xi : P_\iota})$ by factoring through the ideal of elements below $-b_\xi$. Thus also $D^*_\xi = j_\xi[\widehat{D_\xi}(\leq b_\xi)] \in L[\bar{B}^-]$ (it is a subset of $B^*_{\xi+1}$ with a sufficiently absolute definition). We leave it to the reader to check that this suffices to find an isomorphic copy R^* of R in $L[\bar{B}^-]$. Finally, let $P_{\xi+1} = R^*(\leq b_{\xi+1})$ and let $j_{\xi+1}$ be defined by $j_{\xi+1}(\bar{p})(i) = j_\xi(\bar{p}(i))$. A very similar but simpler argument works if $\xi = E^3(\alpha^\zeta_\rho)$ for $\zeta > 0$ and $P_{\xi+1} = \mathbf{Am}_2(\mathrm{dom}(\Phi), P_\xi, \Phi)$ for some Φ. This completes the successor cases.

For ξ limit, check that the λ_ξ-diagonal limit is absolute between $L[\bar{B}^-]$ and $L[\bar{B}]$. So let P^*_ξ be the λ_ξ-diagonal limit of the sequence constructed so far, inside $L[\bar{B}^-]$, restricted to conditions below b_ξ. By (8.3), the isomorphisms constructed at earlier stages can be glued together to form j_ξ. This finishes the proof of the claim. □

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