COLLAPSING THE CARDINALS OF *HOD*

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Abstract. Suppose GCH holds and κ is a κ^{+4}-supercompact cardinal. Then there is a generic extension V^* of V in which κ remains inaccessible and for all infinite cardinals $\alpha < \kappa$, $(\alpha^+)^{HOD} < \alpha^+$. In particular $W = V_\kappa^{V^*}$ is a model of ZFC in which for all infinite cardinals α, $(\alpha^+)^{HOD} < \alpha^+$.

1. Introduction

An important development in large cardinal theory is the construction of inner models M all of whose sets are definable from ordinals and which serve as good approximations to the entire universe V. The former means that M is contained in HOD, the universe of hereditarily ordinal definable sets, and the latter can be interpreted in a number of ways. One such interpretation is that the cardinal structure of M is "close" to that of V in the sense that α^+ of M equals α^+ of V for many cardinals α. This is for example the case if V does not contain $0^\sharp$ and M equals L [1], or if V does not contain an inner model with a Woodin cardinal and M is the core model K for a Woodin cardinal [4].

In this paper we show that we can't hope to approximate the cardinals of V by those of (inner models of) HOD in general:

Theorem 1.1. *Suppose GCH holds and κ is a κ^{+4}-supercompact cardinal. Then there is a generic extension V^* of V in which κ remains inaccessible and for all infinite cardinals $\alpha < \kappa$, $(\alpha^+)^{HOD} < \alpha^+$. In particular $W = V_\kappa^{V^*}$ is a model of ZFC in which for all infinite cardinals α, $(\alpha^+)^{HOD} < \alpha^+$.*

2. Measure sequences

Definition 2.1. *Let $P_\kappa(\lambda) = \{x \subseteq \lambda : \mathrm{ot}(x) < \kappa, x \cap \kappa \in \kappa\}$, and for $x \in P_\kappa(\lambda)$ set $\kappa_x = x \cap \kappa$ and $\lambda_x = \mathrm{ot}(x)$.*

Given $x, y \in P_\kappa(\lambda)$ we define the relation $x \prec y$ (due to Magidor [6]) by

$$x \prec y \Leftrightarrow x \subseteq y \text{ and } \mathrm{ot}(x) < y \cap \kappa.$$

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The authors thanks Hugh Woodin for a very helpful discussion of this problem, including the suggestion to use supercompact Radin forcing. In an email exchange with the first author [9] Woodin suggested an alternative to using our projected forcing $\mathbb{R}^\pi$, but we have opted to retain our original formulation.

Definition 2.2. *For infinite cardinals $\kappa < \lambda$, let $S(\kappa, \lambda)$ be the set of sequences w such that $\mathrm{lh}(w) < \kappa$, $w(0) \in P_\kappa(\lambda)$ and $w(\alpha) \in V_\kappa$ for $0 < \alpha < \mathrm{lh}(w)$.*

Definition 2.3. *Given $\kappa < \lambda$ and $j : V \to M$ witnessing that κ is λ-supercompact, we generate a sequence by the rules:*

- *$u_j(0) = j[\lambda]$.*
- *For $\alpha > 0$, $u_j(\alpha) = \{X \subseteq S(\kappa, \lambda) : u_j \restriction \alpha \in j(X)\}$.*

$u_j(\alpha)$ is defined as long as $u_j \restriction \alpha \in M$. We denote the least α such that $u_j \restriction \alpha \notin M$ by $\mathrm{lh}(u_j)$.

Definition 2.4. *w is a (κ, λ)-*measure sequence *if $w(0)$ is a set of ordinals of order type λ and $w(\alpha)$ is a measure on $S(\kappa, \lambda)$ for each $0 < \alpha < \mathrm{lh}(w)$,*

Definition 2.5. *Given a (κ, λ)-measure sequence w, say that j* is a constructing embedding for *w if j witnesses κ is λ-supercompact and for all α with $0 < \alpha < \mathrm{lh}(w)$ we have that $u_j(\alpha)$ is defined with $u_j(\alpha) = w(\alpha)$. Note that possibly $w(0) \neq u_j(0)$.*

Definition 2.6. *We define a class $\mathcal{U}_\infty$ of* measure sequences *as follows:*

- *$\mathcal{U}_0$ is the class of w such that for some κ and λ, w is a (κ, λ)-measure sequence which has a constructing embedding.*
- *$\mathcal{U}_{n+1} = \{w \in \mathcal{U}_n :$ for all nonzero $\alpha < \mathrm{lh}(w)$, $w(\alpha)$ concentrates on $\mathcal{U}_n\}$.*
- *$\mathcal{U}_\infty = \bigcap_{n<\omega} \mathcal{U}_n$.*

Note that if $u \in \mathcal{U}_\infty$, then it follows from the countable completeness of the measures in u that every measure in u concentrates on $\mathcal{U}_\infty$.

Notation 2.7. *Given a measure sequence $u \in \mathcal{U}_\infty$, we denote the cardinals $\kappa_{u(0)} = u(0) \cap \kappa$ and $\lambda_{u(0)} = \mathrm{ot}(u(0))$ by κ_u and λ_u respectively.*

Given a $u \in \mathcal{U}_\infty$, α is called a *weak repeat point* for u if for all $X \in u(\alpha)$ there exists $\beta < \alpha$ such that $X \in u(\beta)$.

Remark 2.8. *A (κ, λ)-measure sequence of length $(2^{\lambda^{<\kappa}})^+$ contains a weak repeat point.*

Lemma 2.9. *Let GCH hold and let $j : V \to M$ witness that κ is κ^{+4}-supercompact. Then j constructs a (κ, κ^+)-measure sequence u such that $u \in \mathcal{U}_\infty$ and u has a weak repeat point.*

Proof. Evidently M contains every (κ, κ^+)-measure sequence of length less than κ^{+3}, so that the construction of u_j runs for at least κ^{+3} steps. Since there are only κ^{++} subsets of $S(\kappa, \kappa^+)$, a weak repeat point α appears before stage κ^{+3}, so it suffices to check that $u \in \mathcal{U}_\infty$, where $u = u_j \restriction \alpha + 1$.

The main point is to verify that $u \in \mathcal{U}_1$, which amounts to showing that each measure $u(\beta)$ for $0 < \beta \leq \alpha$ concentrates on $\mathcal{U}_0$. By the recursive definition of u_j, we need to show that $u \restriction \beta \in \mathcal{U}_0^M$ for all $0 < \beta \leq \alpha$, which we will do in a uniform way by exhibiting a constructing embedding for $u \restriction \alpha$ in M.

Let W be the supercompactness measure on $P_\kappa(\kappa^{+3})$ induced by the embedding j and let $j_0 : V \to M_0 = Ult(V, W)$ be the usual ultrapower map. The following claims are all standard and easy to verify:

1. $W \in M$.
2. If we define a map k from M_0 to M by setting $k : [F]_W \mapsto j(F)(j[\kappa^{+3}])$ then k is elementary, $k \circ j_0 = j$, $crit(k) > \kappa^{+3}$ and $k(j_0[\kappa^+]) = j[\kappa^+]$.

(3) Since $\kappa^{+3} \subseteq \operatorname{ran}(k)$, it follows easily from GCH that $H_{\kappa^{+3}} \subseteq \operatorname{ran}(k)$, and so that $k \restriction H_{\kappa^{+3}} = id$.
(4) Let $j_0^* : M \to M^* = Ult(M, W)$. For every $a \in M$, ${}^{P_\kappa(\kappa^{+3})}TC(a) \subseteq M$ and so $j_0(M) = j_0^*(M)$, that is $j_0^* = j_0 \restriction M$.

Now let $u^* = u^M_{j_0^*}$, the measure sequence constructed by the embedding j_0^* in the model M. We claim that the construction of u^* proceeds for at least $\alpha + 1$ steps and that $u(\beta) = u^*(\beta)$ for $0 < \beta \leq \alpha$. At the start, $u^*(0) = j_0^*[\kappa^+] = j_0[\kappa^+]$, and we now proceed by induction on β with $0 < \beta \leq \alpha$. Note that all the models V, M, M_0 and M^* agree on the computation of $P(S(\kappa, \kappa^+))$.

Suppose that $u^*(\eta) = u(\eta)$ when $0 < \eta < \beta$. Since M_0 is closed under κ^{+3}-sequences, $u^* \restriction \beta \in M_0$ and the properties of k listed above imply that $k(u^* \restriction \beta) = u \restriction \beta$. Now for every $X \subseteq S(\kappa, \kappa^+)$ we have that

$$\begin{aligned} X \in u^*(\beta) &\iff u^* \restriction \beta \in j_0^*(X) \\ &\iff u^* \restriction \beta \in j_0(X) \\ &\iff u \restriction \beta \in j(X) \\ &\iff X \in u(\beta), \end{aligned}$$

where the equivalences follow respectively from the definition of u^*, the agreement between j_0 and j, the properties of k, and the definition of u.

The rest of the argument is straightforward. We start by observing that since $\kappa^{+3} = (\kappa^{+3})^M$, every measure in u concentrates on the set of measure sequences x such that $x(0) \in P_\kappa(\kappa^+)$ and $\operatorname{lh}(x) < \kappa_x^{+3}$. By the agreement between V and M, a routine induction shows that for all n and for all such measure sequences x we have $x \in \mathcal{U}_n \iff x \in \mathcal{U}_n^M$.

We now establish by induction on $n \geq 1$ that $u \in \mathcal{U}_n$. We just did the base case $n = 1$, so suppose that we established $u \in \mathcal{U}_n$. By definition, $u \in \mathcal{U}_{n+1}$ if and only if $u(\beta)$ concentrates on $\mathcal{U}_n$ for $0 < \beta \leq \alpha$, that is if $u \restriction \beta \in \mathcal{U}_n^M$ for all such β. By definition $u \restriction \beta \in \mathcal{U}_n^M$ if and only if $u(\gamma)$ concentrates on $\mathcal{U}_{n-1}^M$ for $0 < \gamma < \beta$, and by the remarks in the previous paragraph this amounts to verifying that $u(\gamma)$ concentrates on $\mathcal{U}_{n-1}$ which is true since $u \in \mathcal{U}_n$. □

3. Supercompact Radin forcing

In this section we define the *supercompact Radin forcing* and present some of its properties. Our presentation is essentially based on [3] where supercompact Radin forcing was first introduced.

Definition 3.1. *A* good pair *is a pair (u, A) where $u \in \mathcal{U}_\infty$, $A \subseteq \mathcal{U}_\infty$, $A \subseteq S(\kappa_u, \lambda_u)$ and A is u-*large*, that is $A \in u(\alpha)$ for $0 < \alpha < \operatorname{lh}(u)$.*

We define, for each $u \in \mathcal{U}_\infty$, the corresponding *supercompact Radin forcing* $\mathbb{R}_u$.

Definition 3.2. *A condition in $\mathbb{R}_u$ is a finite sequence*

$$p = \langle (u^0, A^0), \ldots, (u^i, A^i), \ldots, (u^n, A^n) \rangle$$

where:

(1) $u^n = u$.
(2) *Each (u^i, A^i) is a good pair.*
(3) $u^i(0) \in P_{\kappa_{u^n}}(\lambda_{u^n})$, *for $i < n$.*

(4) $u^i(0) \prec u^{i+1}(0)$, *for* $i < n-1$.

Definition 3.3. *Given* $p \in \mathbb{R}_u$, $p = \langle (u^0, A^0), \ldots, (u^i, A^i), \ldots, (u^n = u, A^n)\rangle$ *and* $w \in \mathcal{U}_\infty$, *we say* w appears *in* p, *if* $w = u^i$ *for some* $i < n$.

Given $w \in \mathcal{U}_\infty$, let $\pi_w : w(0) \to \lambda_w$ be the collapse map. Given $v \in \mathcal{U}_\infty$ with $v(0) \prec w(0)$, let $\pi_{v,w} : \lambda_v \to \lambda_w$ be defined by $\pi_{v,w} = \pi_w \circ \pi_v^{-1}$ and let $\pi_w(v)$ be obtained from v by replacing $v(0)$ by $\pi_w[v(0)]$.

We now define the notion of extension.

Definition 3.4. *Let*

$$p = \langle (u^0, A^0), \ldots, (u^i, A^i), \ldots, (u^n = u, A^n)\rangle$$

and

$$q = \langle (v^0, B^0), \ldots, (v^i, B^i), \ldots, (v^m = u, B^m)\rangle$$

be in $\mathbb{R}_u$. *Then* $q \leq p$ *(*q *is an extension of* p*) iff:*

(1) *There exist natural numbers* $i_0 < \ldots < i_n = m$ *such that* $v^{i_k} = u^k$ *and* $B^{i_k} \subseteq A^k$.
(2) *If* j *is such that* $0 \leq j \leq m$ *and* $j \notin \{i_0, \ldots, i_n\}$, *and* i *is least such that* $\kappa_{v^j} < \kappa_{u^i}$, *then:*
   - *If* $i = n$, *then* $v^j \in A^n$ *and for all* $x \in B^j$, $\pi_{v^j}^{-1}(x) \in A^n$, *where* $\pi_{v^j}^{-1}(x) = \langle \pi_{v^j}^{-1}[x(0)]\rangle^\frown \langle x(\alpha) : 0 < \alpha < \mathrm{lh}(x)\rangle$.
   - *If* $i < n$, *then* $v^j(0) \prec u^i(0)$, $\pi_{u^i}(v^j) \in A^i$ *and for all* $x \in B^j$, $\pi_{v^j,u^i}(x) \in A^i$, *where* $\pi_{v^j,u^i}(x) = \langle \pi_{v^j,u^i}[x(0)]\rangle^\frown \langle x(\alpha) : 0 < \alpha < \mathrm{lh}(x)\rangle$.

We also define $q \leq^* p$ *(*q *is a direct or a Prikry extension of* p*) iff* $q \leq p$ *and* $m = n$.

Let $u \in \mathcal{U}_\infty$ be a (κ_u, κ_u^+)-measure sequence. Let us mention the main properties of the corresponding forcing $\mathbb{R}_u$ (see [3] and [5] for proofs).

Theorem 3.5. *Let* G *be* $\mathbb{R}_u$*-generic over* V. *The following hold in* $V[G]$*:*

(1) *Let* $C = \{w(0) : w$ *appears in some* $p \in G\}$. *Then* C *is a* $\prec$*-increasing and continuous sequence in* $P_{\kappa_u}(\kappa_u^+)$ *of order type* $\leq \kappa_u$. *Furthermore if* $\mathrm{lh}(u) \geq \kappa_u$, *then* $\mathrm{ot}(C) = \kappa_u$.
(2) $\kappa_u^+ = \bigcup C$, *in particular* κ_u^+ *is collapsed.*
(3) $\kappa_C = \{\kappa_{w(0)} : w(0) \in C\}$ *is a club in* κ_u.
(4) $(\mathbb{R}_u, \leq)$ *satisfies the* $\kappa_u^{++} -$ *c.c..*
(5) $(\mathbb{R}_u, \leq, \leq^*)$ *satisfies the Prikry property: Given any* $b \in \mathrm{RO}(\mathbb{R}_u)$ *and any condition* $p \in \mathbb{R}_u$ *there exists* $q \leq^* p$ *which decides* b.

We now give a factorization property of $\mathbb{R}_u$.

Theorem 3.6. *Suppose that*

$$p = \langle (u^0, A^0), \ldots, (u^i, A^i), \ldots, (u^n = u, A^n)\rangle \in \mathbb{R}_u$$

and $m < n$. *Let*

$$p^{>m} = \langle (u^{m+1}, A^{m+1}), \ldots, (u^n = u, A^n)\rangle$$

and

$$p^{\leq m} = \langle (w^0, B^0), \ldots, (w^{m-1}, B^{m-1}), (u^m, \pi_{u^m}[A^m])\rangle$$

where for $i < m, w^i = \pi_{u^m}(u^i)$ *and* $B^i = \pi_{u^i,u^m}[A^i]$.

Then $p^{\leq m} \in \mathbb{R}_{u^m}, p^{>m} \in \mathbb{R}_u$ and there exists

$$i : \mathbb{R}_u/p \to \mathbb{R}_{u^m}/p^{\leq m} \times \mathbb{R}_u/p^{>m}$$

which is an isomorphism with respect to both $\leq$ and $\leq^*$.

Theorem 3.7. *Let G be $\mathbb{R}_u$-generic. Let $\vec{w} = \langle w_i : i < \mathrm{ot}(C)\rangle$ enumerate $\{w : w$ appears in some $p \in G\}$ such that for $i < j < \mathrm{ot}(C), \kappa_{w_i} < \kappa_{w_j}$. Then:*

1. *$V[G] = V[\vec{w}]$.*
2. *For every limit ordinal $j < \mathrm{ot}(C), \langle w_i^* : i < j\rangle$ is $\mathbb{R}_{w_j}$-generic over V, where $w_i^* = \pi_{w_j}(w_i)$, and $\vec{w} \restriction [j, \mathrm{ot}(C))$ is $\mathbb{R}_u$-generic over $V[\langle w_i^* : i < j\rangle]$.*
3. *For every $\gamma < \kappa$ and $A \subseteq \gamma$ with $A \in V[\vec{w}]$, $A \in V[\langle w_i^* : i < j\rangle]$. where $i < \mathrm{ot}(C)$ is the least ordinal such that $\gamma < \kappa_{w_i}$.*

Theorem 3.8. *Let G be $\mathbb{R}_u$-generic, and let λ be a cardinal of V with $\lambda < \kappa_u$. Then λ is collapsed in $V[G]$ if and only if $\lambda = (\kappa^+_{w(0)})^V$ for some w which is a $\prec$-limit element of C.*

Proof. Let $\delta = \kappa_{w(0)}$, where w is a $\prec$-limit element of C. Then $w(0) = \bigcup\{v(0) : v \in C, v(0) \prec w(0)\}$, $w(0)$ has order type δ^+ and $\{v(0) : v \in C, v(0) \prec w(0)\}$ has order type at most δ, so δ^+ is collapsed. The converse is immediate from Theorem 3.7.

$\square$

The next theorem is implicit in [3] and we have just given a proof here to make our paper more self-contained. We follow Radin's idea [7].

Theorem 3.9. *Let $j : V \to M$ witness κ is κ^{+4}-supercompact and let $v \in \mathcal{U}_\infty$ be a (κ, κ^+)-measure sequence constructed from j which has a weak repeat point α. Let $u = v \restriction \alpha$ and let G be $\mathbb{R}_u$-generic over V. Then in $V[G]$, κ remains λ-supercompact, where $\lambda = (\kappa^{+4})^V = (\kappa^{+3})^{V[G]}$.*

Proof. We prove the theorem in a sequence of claims.

Claim 3.10. *If $A \in u(\beta)$ for all $0 < \beta < \alpha$, then $u \in j(A)$.*

Proof. It suffices to show that A is $v(\alpha)$-large. Suppose not. Then $A^c = S(\kappa, \kappa^+) \setminus A \in v(\alpha)$, so for some $0 < \beta < \alpha$, $A^c \in v(\beta) = u(\beta)$ which is in contradiction with $A \in u(\beta)$. $\square$

Note that any condition $p \in \mathbb{R}_u$ is of the form $p_d{}^\frown\langle(u, A)\rangle$, for some unique p_d and A. By the above Claim $u \in j(A)$, so we can form the condition

$$p^* = q_d{}^\frown\langle(u, A)\rangle^\frown\langle(j(u), j(A))\rangle \in \mathbb{R}^M_{j(u)},$$

where q_d is obtained from p_d by type changing to make the above condition well-defined. The following can be proved easily.

Claim 3.11. *p^* and $j(p)$ are compatible in $\mathbb{R}^M_{j(u)}$.*

Given any condition $p = p_d{}^\frown\langle(u, A)\rangle \in \mathbb{R}_u$ and any $j(u)$-large set E, set

$$p^* \restriction E = q_d{}^\frown\langle(u, A)\rangle^\frown\langle(j(u), E)\rangle.$$

We now define U on $P_\kappa(\lambda)$ as follows: $X \in U$ if and only if there exist $p \in G$ and E which is $j(u)$-large such that $p^* \restriction E \Vdash j[\lambda] \in j(\dot{X})$, where $\dot{X}$ is a name for X.

Claim 3.12. *The above definition of U does not depend on the choice of the name $\dot{X}$.*

Proof. Suppose not. Then we can find $p \in G$, $\mathbb{R}_u$-names $\dot{X}_1$, $\dot{X}_2$ and a $j(u)$-large set E such that $p \Vdash \dot{X}_1 = \dot{X}_2$, but $p^* \restriction E \Vdash j(\dot{X}_1) \neq j(\dot{X}_2)$. But then $j(p) \Vdash j(\dot{X}_1) = j(\dot{X}_2)$ and the conditions $p^* \restriction E$ and $j(p)$ are compatible, which is a contradiction. □

We show that U is a normal measure on $P_\kappa(\lambda)$. U is easily seen to be a filter.

Claim 3.13. *U is an ultrafilter.*

Proof. Let $X \subseteq P_\kappa(\lambda)$ and let $p \in \mathbb{R}_u$. By the Prikry property we can find $q \leq^* p$ and a $j(u)$-large set E such that $q^* \restriction E$ decides $j[\lambda] \in j(\dot{X})$, and hence q forces $\dot{X} \in \dot{U}$ or $P_\kappa(\lambda) \setminus \dot{X} \in \dot{U}$. □

Claim 3.14. *U is fine.*

Proof. Suppose that $\alpha < \lambda$, $X = \{x \in P_\kappa(\lambda) : \alpha \in x\}$ and that $p \in \mathbb{R}_u$. It is clear that $p^* \Vdash j[\lambda] \in j(\dot{X})$, so p forces $\dot{X} \in \dot{U}$. □

Claim 3.15. *U is normal.*

Proof. Let $F : P_\kappa(\lambda) \to \lambda$ be regressive, that is $F(x) \in x$ for all non-empty $x \in P_\kappa(\lambda)$. Let $p \in \mathbb{R}_u$ and $p \Vdash \dot{F}$ is regressive, where $\dot{F}$ is a name for F. Suppose also for contradiction that

$$p \Vdash \forall \delta < \lambda \, \dot{F}^{-1}[\{\delta\}] \notin \dot{U}.$$

So for $\delta < \lambda$ we can find a $j(u)$-large set E_δ such that

$$p^* \restriction E_\delta \Vdash j(\dot{F})(j[\lambda]) \neq j(\delta).$$

Let $E = \bigcap_{\delta < \lambda} E_\delta$. Then E is $j(u)$-large and

$$p^* \restriction E \Vdash \forall \delta < \lambda \, j(\dot{F})(j[\lambda]) \neq j(\delta).$$

On the other hand $j(p)$ forces that $j(\dot{F})$ is regressive, so in particular $j(p)$ forces that $j(\dot{F})(j[\lambda]) \in j[\lambda]$. But this is a contradiction, since $p^* \restriction E$ and $j(p)$ are compatible. Hence F is constant on a set in U, and the claim follows. □

Claim 3.16. *U is κ-complete.*

Proof. This follows by a similar argument to the one we gave for normality. □

It follows that U is a normal measure on $P_\kappa(\lambda)$, and the theorem follows. □

4. Projected forcing

In this section we define the projected forcing. We note that our projected forcing is different from that of [3]. The reason is that we need our projected forcing to be as close to the supercompact Radin forcing as far as possible, so that the quotient forcing is sufficiently homogeneous.

Given any $u \in \mathcal{U}_\infty$ we first define $\pi(u)$, and then we will define the projected forcing $\mathbb{R}^\pi_{\pi(u)}$.

Definition 4.1. *Suppose (u, A) is a good pair. Let $\pi(u, A) = (\pi(u), \pi(A))$ where*

- $\pi(u) = \kappa_u{}^\frown u \restriction [1, \mathrm{lh}(u))$.
- $\pi(A) = \{\pi(v) : v \in A\}$.

Also let $\mathcal{U}^\pi_\infty = \{\pi(u) : u \in \mathcal{U}_\infty\}$. *For* $u \in \mathcal{U}_\infty$ *set* $\pi(u(\alpha)) =$ *the Rudin-Keisler projection of* $u(\alpha)$, *for* $\alpha > 0$. *Note that* $\pi(u(\alpha)) \neq \pi(u)(\alpha)$.

Definition 4.2. *A* good pair for projected forcing *is a pair* (u, A) *where* $u \in \mathcal{U}^\pi_\infty$, $A \subseteq \mathcal{U}^\pi_\infty$, $A \subseteq V_{\kappa_u}$ *and* A *is of measure one for all* $\pi(u(\alpha)), 0 < \alpha < \mathrm{lh}(u)$.

Remark 4.3. *If* (u, A) *is a good pair, then* $(\pi(u), \pi(A))$ *is a good pair for projected forcing.*

Given $u \in \mathcal{U}^\pi_\infty$, we define the projected forcing $\mathbb{R}^\pi_u$.

Definition 4.4. *A condition in* $\mathbb{R}^\pi_u$ *is a finite sequence*

$$p = \langle (u^0, A^0), \ldots, (u^i, A^i), \ldots, (u^n, A^n)\rangle$$

where:

1. $u^n = u$.
2. *Each* (u^i, A^i) *is a good pair for projected forcing.*
3. $\kappa_{u^i} < \kappa_{u^{i+1}}$, *for all* $i < n$.

We now define the extension relation.

Definition 4.5. *Let*

$$p = \langle (u^0, A^0), \ldots, (u^i, A^i), \ldots, (u^n = u, A^n)\rangle$$

and

$$q = \langle (v^0, B^0), \ldots, (v^i, B^i), \ldots, (v^m = u, B^m)\rangle$$

be in $\mathbb{R}^\pi_u$. *Then* $q \leq p$ *(q is an extension of p) iff:*

1. *There exists an increasing sequence of natural numbers* $i_0 < \ldots < i_n = m$ *such that* $v^{i_k} = u^k$ *and* $B^{i_k} \subseteq A^k$.
2. *If* j *is such that* $0 \leq j \leq m$ *and* $j \notin \{i_0, \ldots, i_n\}$, *and if* i *is least such that* $\kappa_{v^j} < \kappa_{u^i}$, *then* $v^j \in A^i$ *and* $B^j \subseteq A^i$.

We also define $q \leq^* p$ *(q is a direct or a Prikry extension of p) iff* $q \leq p$ *and* $m = n$.

It is easy to see that $(\mathbb{R}^\pi_u, \leq)$ satisfies the $\kappa_u^+ - c.c.$. The next theorem can be proved easily.

Theorem 4.6. *Let* G *be* $\mathbb{R}^\pi_u$*-generic over* V, *and let*

$$C = \{\kappa_w : w \text{ appears in some } p \in G\}.$$

Then C *is a club of* κ_u. *Furthermore if* $\mathrm{lh}(u) \geq \kappa_u$, *then* $\mathrm{ot}(C) = \kappa_u$,

As before we have a factorization property for $\mathbb{R}^\pi_u$.

Theorem 4.7. *Suppose that*

$$p = \langle (u^0, A^0), \ldots, (u^i, A^i), \ldots, (u^n = u, A^n)\rangle \in \mathbb{R}^\pi_u$$

and $m < n$. *Let*

$$p^{>m} = \langle (u^{m+1}, A^{m+1}), \ldots, (u^n = u, A^n)\rangle$$

and

$$p^{\leq m} = \langle (u^0, A^0), \ldots, (u^m, A^m)\rangle.$$

Then $p^{\leq m} \in \mathbb{R}^\pi_{u^m}, p^{>m} \in \mathbb{R}^\pi_u$ *and there exists*

$$i : \mathbb{R}^\pi_u / p \to \mathbb{R}^\pi_{u^m}/p^{\leq m} \times \mathbb{R}^\pi_u / p^{>m}$$

which is an isomorphism with respect to both $\leq$ *and* $\leq^*$.

5. Weak projection

Suppose that $u \in \mathcal{U}_\infty$ and consider the forcing notions $\mathbb{R}_u$ and $\mathbb{R}^\pi_{\pi(u)}$. We define a map $\pi : \mathbb{R}_u \to \mathbb{R}^\pi_{\pi(u)}$ in the natural way by

$$\pi(\langle (u^0, A^0), \ldots, (u^i, A^i), \ldots, (u^n, A^n) \rangle) = \langle \pi(u^0, A^0), \ldots, \pi(u^i, A^i), \ldots, \pi(u^n, A^n) \rangle$$

In general π is not a projection map, but we show that it has a weaker property, introduced by Foreman-Woodin [3].

Definition 5.1. *$\pi : \mathbb{Q} \to \mathbb{P}$ is called a* weak projection *if*

1. $\pi(1_\mathbb{Q}) = 1_\mathbb{P}$.
2. *π is order preserving.*
3. *For all $p \in \mathbb{Q}$ there is $p^* \leq p$ such that for all $q \leq \pi(p^*)$ there exists $r \leq p$ such that $\pi(r) \leq q$.*

The next lemma shows that $\pi : \mathbb{Q} \to \mathbb{P}$ being a weak projection is sufficient to imply that $\mathbb{Q}$-generics yield $\mathbb{P}$-generics.

Lemma 5.2. *Suppose $\pi : \mathbb{Q} \to \mathbb{P}$ is a weak projection and H is $\mathbb{Q}$-generic over V. Let G be the filter generated by $\pi[H]$. Then G is $\mathbb{P}$-generic over V.*

Proof. Let D be a dense open subset of $\mathbb{P}$. Let $E = \pi^{-1}[D]$.

Claim 5.3. *E is dense in $\mathbb{Q}$.*

Proof. Let $p \in \mathbb{Q}$ and let $p^* \leq p$ be as in 5.1(3). Let $q \leq \pi(p^*)$ be such that $q \in D$ and let $r \leq p$ be such that $\pi(r) \leq q$. Then $\pi(r) \in D$, hence $r \in E$. □

Let $p \in H \cap E$. Then $\pi(p) \in G \cap D$, and hence $G \cap D \neq \emptyset$. □

Let us now consider $\pi : \mathbb{R}_u \to \mathbb{R}^\pi_{\pi(u)}$.

Theorem 5.4. *$\pi : \mathbb{R}_u \to \mathbb{R}^\pi_{\pi(u)}$ is a weak projection, in fact for all $p \in \mathbb{R}_u$ there is $p^* \leq^* p$ such that for all $q \leq \pi(p^*)$ we can find $r \leq p$ such that $\pi(r) \leq^* q$.*

Proof. It is easily seen that $\pi(1_{\mathbb{R}_u}) = 1_{\mathbb{R}^\pi_{\pi(u)}}$ and that π is order preserving.

For a measure sequence u, recall that a set X is *u-large* iff $X \in u(\alpha)$ for all $0 < \alpha < \mathrm{lh}(u)$.

Definition 5.5. *Suppose (u, A) is a good pair and $v \in \mathcal{U}_\infty$. v is* addable *to (u, A) iff:*

1. *$v \in A$.*
2. *$\pi_v[\{x \in A : x(0) \prec v(0)\}]$ is v-large.*

Claim 5.6. *The set $\{v \in A : v$ is addable to $(u, A)\}$ is u-large.*

Proof. Suppose that $j : V \to M$ constructs u and let $0 < \alpha < \mathrm{lh}(u)$. We need to show that $\{v \in A : v \text{ is addable to } (u, A)\} \in u(\alpha)$, or equivalently $u \restriction \alpha \in j(\{v \in A : v \text{ is addable to } (u, A)\})$. Thus it suffices to show that (in M) $u \restriction \alpha$ is addable to $(j(u), j(A))$.

1. $u \restriction \alpha \in j(A)$: This is trivial, since $A \in u(\alpha)$.
2. The set $X = \pi_{u \restriction \alpha}[\{x \in j(A) : x(0) \prec (u \restriction \alpha)(0)\}]$ is $u \restriction \alpha$-large: We have $X = \pi_u[\{x \in j(A) : x(0) \prec u(0)\}]$, and since $\{x \in j(A) : x(0) \prec u(0)\} = j[A]$, we have $X = \pi_u[j[A]]$. so it suffices to show that $\pi_u[j[A]]$ is $u \restriction \alpha$-large. This follows easily from the fact that $\pi_u^{-1} = j \restriction \lambda_u$.

The claim follows. □

Iterating this process ω times, we can find $B \subseteq A$ such that B is u-large and any $v \in B$ is addable to (u, B).

Claim 5.7. *Suppose that (u, A) is a good pair. Let A' be the set of $w \in A$ such that for all $x \in A$ with $\kappa_x < \kappa_w$, there is $\bar{x} \in A$ such that $\pi(\bar{x}) = \pi(x)$, $\bar{x}(0) \prec w(0)$ and w is addable to (u, A). Then A' is u-large.*

Proof. Suppose that $j : V \to M$ constructs u and let $0 < \alpha < \mathrm{lh}(u)$. We need to show that $A' \in u(\alpha)$ or equivalently $u \restriction \alpha \in j(A')$. Clearly $u \restriction \alpha \in j(A)$ and by Claim 5.6 $u \restriction \alpha$ is addable to $(j(u), j(A))$. Thus we need to prove the following:
$(*)$ If $x \in j(A)$ and $\kappa_x < \kappa_u$, then there is $\bar{x} \in j(A)$ such that $\pi(\bar{x}) = \pi(x)$, $\bar{x}(0) \prec u(0)$.

Choose $\bar{x} \in \mathrm{ran}(j)$ such that $\bar{x} \in j(A)$, $\kappa_{\bar{x}} = \kappa_x$ and the measures on $\bar{x}$ are the same as the measures on x, so that $\pi(\bar{x}) = \pi(x)$. Since $\bar{x} \in \mathrm{ran}(j)$ we have $\bar{x} = j(z)$ for some $z \in A$. Then $z(0) \in P_{\kappa_u}(\lambda_u)$ and hence we have $\bar{x}(0) = j[z(0)]$. It is now clear that $\bar{x}(0) = j[z(0)] \prec j[\lambda_u] = u(0)$. □

We are now ready to complete the proof of Theorem 5.4 by showing that π satisfies conditions 1–3 of Definition 5.1. Conditions 1 and 2 are trivial. For condition 3 it suffices, using the factorization property from Theorem 3.6, to prove it for the special case of $p = \langle (u, A) \rangle$. Let $A^{(\omega)} = \bigcap_{n<\omega} A^{(n)}$ where the sets $A^{(n)}, n < \omega$, are defined by:

- $A^{(0)} = A$.
- $A^{(n+1)} = (A^{(n)})'$.

Now apply Claim 5.6 to shrink A^{ω} to a u-large set B such that any $v \in B$ is addable to (u, B). We show that $p^* = \langle (u, B) \rangle$ is as required. Thus let

$$q = \langle (v^0, B^0), \ldots, (v^i, B^i), \ldots, (v^n, B^n) \rangle \in \mathbb{R}^{\pi}_{\pi(u)},$$

with $q \leq \pi(p^*)$. We find a sequence $(u^i, A^i), i \leq n$, such that:

(1) $u^n = u$.
(2) $u^0(0) \prec u^1(0) \prec \ldots \prec u^{n-1}(0)$.
(3) $u^i(0) \in P_{\kappa_{u^n}}(\lambda_{u^n}), i < n$.
(4) $\pi(u^i) = v^i, i \leq n$.
(5) $\pi(A^i) \subseteq B^i, i \leq n$.

By induction on $i \leq n$ we choose (u^{n-i}, A^{n-i}). For $i = 0$ let $u^n = u$ and let $A^n \subseteq B$ be such that $\pi(A^n) \subseteq B^n$. Suppose now that we have chosen (u^{n-i}, A^{n-i}). Let $w \in A^{n-i}$ be such that $\pi(w) = v^{n-i-1}$. Since $u^{n-i} \in A^{(n-i+1)}$ and $A^{(n-i+1)} = (A^{(n-i)})'$ we can find $u^{n-i-1} \in A^{n-i}$ such that $\pi(u^{n-i-1}) = \pi(w) = v^{n-i-1}$, $u^{n-i-1}(0) \prec u^{n-i}(0)$ and u^{n-i-1} is addable to (u^{n-i}, A^{n-i}). Let $A^{n-i-1} = \pi_{u^{n-i-1}}[\{x \in A^{n-i} : x(0) \prec u^{n-i-1}\}]$. Then A^{n-i-1} is u^{n-i-1}-large and by shrinking A^{n-i-1}, if necessary, we may suppose that $\pi(A^{n-i-1}) \subseteq B^{n-i-1}$. This completes the inductive choice of (u^i, A^i), $i \leq n$.

Let $r = \langle (u^0, A^0), \ldots, (u^i, A^i), \ldots, (u^n, A^n) \rangle$. Then $r \in \mathbb{R}_u, r \leq p$ and $\pi(r) \leq^* q$. □

By weak projection we can find $1^* \leq^* 1_{\mathbb{R}_u}$ such that for all $q^* \leq \pi(1^*)$ there exists $r \in \mathbb{R}_u$ such that $\pi(r) \leq q^*$.

Lemma 5.8. *We may choose 1^* to be $1_{\mathbb{R}_u}$.*

Proof. Let $A = S(\kappa_u, \lambda_u) \cap \mathcal{U}_\infty$ and consider $1_{\mathbb{R}_u} = \langle (u, A) \rangle$. It suffices to show that $A' = A$. So suppose that $w \in A$. Then $\pi_w[\{x \in A : x(0) \prec w(0)\}]$ is clearly w-large, so w is addable to (u, A). Suppose that $x \in A$ and $\kappa_x < \kappa_w$. Let

$$\bar{x} = \langle (x \cap \kappa_u) \cup \text{the first } \lambda_x\text{-many elements of } w(0) \setminus \kappa_u \rangle ^\frown \langle x(\alpha) : 0 < \alpha < \mathrm{lh}(x) \rangle.$$

Then $\bar{x} \in A$, $\pi(\bar{x}) = \pi(x)$ and $\bar{x}(0) \prec w(0)$. Hence $w \in A'$ and the lemma follows. □

It follows that $\pi[\mathbb{R}_u]$ is dense in $\mathbb{R}^\pi_{\pi(u)}$. Also by weak projection and the Prikry property for $\mathbb{R}_u$ we have the following:

Theorem 5.9. *$\pi[\mathbb{R}_u]$ satisfies the Prikry property.*

Proof. Let $b \in \mathrm{RO}(\pi[\mathbb{R}_u)]$ and $q \in \pi[\mathbb{R}_u]$. Let $p \in \mathbb{R}_u$ be such that $q = \pi(p)$. By part 5 of Theorem 3.5 there is $p^* \leq^* p$ such that p^* decides $\|b \in \pi[\dot{G}]\|_{\mathrm{RO}(\pi[\mathbb{R}_u)]}$ where $\dot{G}$ is the canonical name for a generic filter over $\mathbb{R}_u$. Let $q^* = \pi(p^*)$. Then $q^* \leq^* q$ and it decides b. □

By Theorem 5.4, $\pi[\mathbb{R}_u]$ is in fact $\leq^*$-dense in $\mathbb{R}^\pi_{\pi(u)}$, so by the above theorem:

Corollary 5.10. *$\mathbb{R}^\pi_{\pi(u)}$ satisfies the Prikry property.*

Theorem 5.11. *Let G be $\mathbb{R}^\pi_u$-generic over V, and let C be as in Theorem 4.6. Also let $\vec{w} = \langle w_i : i < \mathrm{ot}(C) \rangle$ enumerate $\{w : w$ appears in some $p \in G\}$ such that for $i < j < \mathrm{ot}(C)$, $\kappa_{w_i} < \kappa_{w_j}$. Then:*

1. *$V[G] = V[\vec{w}]$.*
2. *For every limit ordinal $j < \mathrm{ot}(C)$, $\vec{w} \restriction j$ is $\mathbb{R}_{w_j}$-generic over V, and $\vec{w} \restriction [j, \mathrm{ot}(C))$ is $\mathbb{R}_u$-generic over $V[\vec{w} \restriction j]$.*

Theorem 5.12. *Suppose $\gamma < \kappa$, $A \subseteq \gamma$, $A \in V[\vec{w}]$. Let $i < \mathrm{ot}(C)$ be the least ordinal such that $\gamma < \kappa_{w_i}$. Then $A \in V[\vec{w} \restriction j]$.*

From Theorem 4.6 and the above results we have the following:

Theorem 5.13. *Suppose that $u \in \mathcal{U}^\pi_\infty$ and let G be $\mathbb{R}^\pi_u$-generic over V. Then V and $V[G]$ have the same cardinals.*

The same argument as in Theorem 3.9 yields the following:

Theorem 5.14. *Let $j : V \to M$ witness κ is κ^{+4}-supercompact and let $v \in \mathcal{U}_\infty$ be a (κ, κ^+)-measure sequence constructed from j which has a weak repeat point α. Let $u = v \restriction \alpha$ and let G^π be $\mathbb{R}^\pi_{\pi(u)}$-generic over V. Then in $V[G^\pi]$, κ remains λ-supercompact, where $\lambda = (\kappa^{+4})^V = (\kappa^{+4})^{V[G^\pi]}$.*

6. Homogeneity property

Suppose that $u \in \mathcal{U}_\infty$ is a (κ, κ^+)-measure sequence.

Theorem 6.1. *For all $p, q \in \mathbb{R}_u$, if $\pi(p) = \pi(q)$, then there exists q^* compatible with q such that $\mathbb{R}_u/p \simeq \mathbb{R}_u/q^*$.*

Proof. We give the proof in a sequence of claims. First note that if σ is a permutation of λ, then σ naturally extends to a permutation of $S(\kappa, \lambda)$ defined by $\sigma(x) = \langle \sigma[x(0)] \rangle ^\frown \langle x(\alpha) : 0 < \alpha < \mathrm{lh}(x) \rangle$. We also denote this extension by σ.

Claim 6.2. *Suppose that u is a measure on $S(\kappa,\lambda)$ which has a generating embedding j, σ is a permutation of λ, $0<\alpha<\mathrm{lh}(u)$ and $A \in u(\alpha)$. Then $\sigma[A]\in u(\alpha)$.*

Proof. We have

$$j(\sigma)(u\restriction\alpha) = j(\sigma)[j[\lambda]]^\frown\langle u(\beta):0<\beta<\alpha\rangle = j[\lambda]^\frown\langle u(\beta):0<\beta<\alpha\rangle = u\restriction\alpha.$$

On the other hand by our assumption $u\restriction\alpha \in j(A)$, which implies $u\restriction\alpha\in j(\sigma)[j(A)]$. Since

$$\sigma[A]\in u(\alpha) \Leftrightarrow u\restriction\alpha\in j(\sigma)[j(A)],$$

we get $\sigma[A]\in u(\alpha)$. □

Corollary 6.3. *If (u,A) is a good pair, then $(u,\sigma[A])$ is also a good pair.*

Claim 6.4. *Let κ be inaccessible. Let A and A' be sets of ordinals with $\mathrm{ot}(A)=\mathrm{ot}(A')=\kappa^+$ and let $B\subseteq A$, $B'\subseteq A'$ be such that $\mathrm{ot}(B)=\mathrm{ot}(B')<\kappa$. Let ρ be the order preserving map from A to A'. Then there is a bijection $\sigma:A\to A'$ such that $\sigma\restriction B$ is the order preserving map from B to B' and $\{i:\sigma(i)\neq\rho(i)\}$ has size less than κ.*

Proof. Let $C\subseteq A$ be such that $B\subseteq C$, $B'\subseteq\rho[C]$ and $\mathrm{ot}(C)<\kappa$. Let $\tau:C\leftrightarrow\rho[C]$ be such that $\tau\restriction B:B\to B'$ is order preserving. Then $\sigma=\tau\cup(\rho\restriction(A\setminus C))$ is as required. □

We now complete the proof of Theorem 6.1. Thus let $p,q\in\mathbb{R}_u$ be such that $\pi(p)=\pi(q)$. Then p,q have the same length. Let

$$p=\langle(u^0,A^0),\dots,(u^i,A^i),\dots,(u^n=u,A^n)\rangle$$

and

$$q=\langle(v^0,B^0),\dots,(v^i,B^i),\dots,(v^n=u,B^n)\rangle.$$

By our assumption $\kappa_{u^i}=\kappa_{v^i}$ and we call this κ_i. Also we have $\pi(A^i)=\pi(B^i)$. Using the above claim, by starting at the top and working down, we can find a permutation σ of κ^+ such that:

1. (1) $\sigma\restriction\kappa=\mathrm{id}\restriction\kappa$.
2. (2) The support of σ has size less than κ.
3. (3) For each $i<n$, $\sigma[u^i(0)]=v^i(0)$.
4. (4) For each $i<n$, σ and the order preserving map from $u^i(0)$ to $v^i(0)$ differ at fewer than κ_i many points.

Note that for each $i<n$, σ induces a permutation σ_i of κ_i^+ defined by $\sigma_i=\pi_{v^i}\circ\sigma\circ\pi_{u^i}^{-1}$. By convention set $\sigma_n=\sigma$. Let $q^*=\langle(v^0,\sigma_0[A^0]),(v^1,\sigma_1[A^1]),\dots,(v^n=u,\sigma_n[A^n])\rangle$. By Corollary 6.3 $q^*\in\mathbb{R}_u$, and clearly q^* is compatible with q. Define

$$\varphi:\mathbb{R}_u/p\to\mathbb{R}_u/q^*$$

as follows:

Suppose that

$$\langle(x^0,C^0),(x^1,C^1),\dots,(x^m,C^m)\rangle\in\mathbb{R}_u/p.$$

For each $k\le m$ let $i_k\le n$ be the least such that $x^k=u^{i_k}$ or $x^k\prec u^{i_k}$. We define

$$\varphi(\langle(x^0,C^0),(x^1,C^1),\dots,(x^m,C^m)\rangle)=\langle(y^0,D^0),(y^1,D^1),\dots,(y^m,D^m)\rangle$$

where for $k\le m$:

- If $x^k=u^{i_k}$, then $y^k=v^{i_k}$ and $D^k=\sigma_{i_k}[C^{i_k}]$.

- If $i_k = n$ and $x^k \prec u^n = u$, then $y^k = \sigma_n(x^k)$ and $D^k = \pi_{\sigma_n(x^k)} \circ \sigma_n \circ \pi^{-1}_{x^k}[C^k]$.
- If $i_k < n$ and $x^k \prec u^{i_k}$, then $y^k = \sigma_{i_k}(x^k)$ and $D^k = \pi^{-1}_{\sigma_{i_k}(x^k), v^{i_k}} \circ \sigma_{i_k} \circ \pi_{x^k, u^{i_k}}[C^k]$.

The following can be proved easily.

Claim 6.5. (1) *Each (y^k, D^k) is a good pair.*
(2) *Suppose that $i_k = n$ and $x^k \prec u^n = u$. Then $y^k \in \sigma_n[A^n]$ and $\pi^{-1}_{y^k}[D^k] \subseteq \sigma_n[A^n]$.*
(3) *Suppose that $i_k < n$ and $x^k \prec u^{i_k}$. Then $y^k \in \sigma_{i_k}[A^{i_k}]$ and $\pi_{y^k, v^{i_k}}[D^k] \subseteq \sigma_{i_k}[A^{i_k}]$.*

It follows from Claim 6.5 that φ is well-defined.

Claim 6.6. *$\varphi : \mathbb{R}_u/p \to \mathbb{R}_u/q^*$ is an isomorphism.*

Proof. Define
$$\psi : \mathbb{R}_u/q^* \to \mathbb{R}_u/p$$
as follows:

Suppose that
$$\langle (y^0, D^0), (y^1, D^1), \dots, (y^m, D^m) \rangle \in \mathbb{R}_u/q^*.$$
For each $k \le m$, let i_k be least such that $y^k = v^{i_k}$ or $y^k \prec v^{i_k}$. Set
$$\psi(\langle (y^0, D^0), (y^1, D^1), \dots, (y^m, D^m) \rangle) = \langle (x^0, C^0), (x^1, C^1), \dots, (x^m, C^m) \rangle$$
where for $k \le m$:

- If $y^k = v^{i_k}$, then $x^k = u^{i_k}$ and $C^k = \sigma^{-1}_{i_k}[D^{i_k}]$.
- If $i_k = n$ and $y^k \prec v^n = u$, then $x^k = \sigma^{-1}_n(y^k)$ and $C^k = \pi_{\sigma^{-1}_n(y^k)} \circ \sigma^{-1}_n \circ \pi^{-1}_{y^k}[D^k]$.
- If $i_k < n$ and $y^k \prec v^{i_k}$, then $x^k = \sigma^{-1}_{i_k}(y^k)$ and $C^k = \pi^{-1}_{\sigma^{-1}_{i_k}(y^k), u^{i_k}} \circ \sigma^{-1}_{i_k} \circ \pi_{y^k, v^{i_k}}[D^k]$.

As above we can show that ψ is well-defined and it is easily seen that $\varphi \circ \psi = \mathrm{id}_{\mathbb{R}_u/q^*}$ and $\psi \circ \varphi = \mathrm{id}_{\mathbb{R}_u/p}$, which implies both of φ and ψ are bijections. It is also easy to show that both of them preserve the forcing relations $\le$ and $\le^*$. □

This completes the proof of Theorem 6.1. □

Corollary 6.7. *(Weak homogeneity). Suppose $p, q \in \mathbb{R}_u$ and $\pi(p) = \pi(q)$. If $p \Vdash \phi(\alpha, \vec{\gamma})$, where $\alpha, \vec{\gamma}$ are ordinals, then it is not the case that $q \Vdash \neg\phi(\alpha, \vec{\gamma})$.*

It follows that:

Corollary 6.8. *Suppose that G is $\mathbb{R}_u$-generic and let G^π be the filter generated by $\pi[G]$ Then:*
(1) *G^π is $\mathbb{R}^\pi_{\pi(u)}$-generic over V.*
(2) *$HOD^{V[G]} \subseteq V[G^\pi]$.*

Proof. Part 1 follows from Lemma 5.2, Theorem 5.4 and the fact that $\pi[\mathbb{R}_u]$ is dense in $\mathbb{R}^\pi_{\pi(u)}$.

For part 2 suppose that a is a set of ordinals in $V[G]$ which is definable in $V[G]$ with ordinal parameters. We show that a belongs to $V[G^\pi]$. Write $a = \{\alpha : V[G] \models \phi(\alpha, \vec{\gamma})\}$. Then

$$\alpha \in a \Leftrightarrow p \Vdash \phi(\alpha, \vec{\gamma}) \text{ for some } p \in G.$$

By corollary 6.7, if $p \Vdash \phi(\alpha, \vec{\gamma})$ and if $\pi(p) = \pi(q)$, then it is not the case that $q \Vdash \neg\phi(\alpha, \vec{\gamma})$. It follows that

$$a = \{\alpha : p \Vdash \phi(\alpha, \vec{\gamma}) \text{ for some } p \text{ with } \pi(p) \in G^\pi \},$$

and hence $a \in V[G^\pi]$. □

7. Proof of main theorem

In this section we will present the proof of Theorem 1.1. Suppose κ is a κ^{+4}-supercompact cardinal and let $j : V \to M$ witness this. Let $v \in \mathcal{U}_\infty$ be a (κ, κ^+)-measure sequence constructed from j which has a weak repeat point α and let $u = v \restriction \alpha$. Consider the forcing notions $\mathbb{R}_u$ and $\mathbb{R}^\pi_{\pi(u)}$. Let G be $\mathbb{R}_u$-generic over V and let G^π be the filter generated by $\pi[G]$. In summary we showed:

- κ remains λ-supercompact in $V[G]$, where $\lambda = (\kappa^{+4})^V = (\kappa^{+3})^{V[G]}$ (by Theorem 3.9).
- κ remains λ-supercompact in $V[G^\pi]$, where $\lambda = (\kappa^{+4})^V = (\kappa^{+4})^{V[G^\pi]}$ (by Theorem 5.14).
- There exists a club $C \in V[G^\pi]$ of κ such that for every limit point α of C we have $(\alpha^+)^{V[G^\pi]} = (\alpha^+)^V < (\alpha^+)^{V[G]}$ (by Theorems 3.8 and 5.13).

By part 2 of Corollary 6.8 we have $HOD^{V[G]} \subseteq V[G^\pi]$, in particular for every limit point α of C we have

$$(\alpha^+)^{HOD^{V[G]}} \leq (\alpha^+)^{V[G^\pi]} = (\alpha^+)^V < (\alpha^+)^{V[G]}.$$

Claim 7.1. *Let $\langle \kappa_i : i < \kappa \rangle$ be an increasing enumeration of C. Working in $V[G]$, let $\mathbb{Q}$ be the reverse Easton iteration for collapsing each κ_{i+1} to κ_i^+ for each each $i < \kappa$, and let H be $\mathbb{Q}$-generic over $V[G]$. Clearly*

1. *$CARD^{V[G*H]} \cap (\kappa_0, \kappa) = \{\kappa_i^+ : i < \kappa\} \cup \{\kappa_i : i < \kappa,\ i \text{ is a limit ordinal}\}$.*
2. *κ remains inaccessible in $V[G * H]$.*

It also follows from the results of [2] that $\mathbb{Q}$ is *cone homogeneous*, that is for all $p, q \in \mathbb{Q}$ there are $p^* \leq p, q^* \leq q$ and an isomorphism $\phi : \mathbb{Q}/p^* \to \mathbb{Q}/q^*$. Hence by [2] we have

$$HOD^{V[G*H]} \subseteq HOD^{V[G]}.$$

Finally force with $\mathbb{P} = Col(\aleph_0, \kappa_0)^{V[G*H]}$ over $V[G*H]$ and let K be $\mathbb{P}$-generic over $V[G*H]$. It is now easily seen that κ remains inaccessible in $V[G*H*K]$ and by homogeneity of $\mathbb{P}$

$$HOD^{V[G*H*K]} \subseteq HOD^{V[G*H]}.$$

Hence

$$HOD^{V[G*H*K]} \subseteq HOD^{V[G*H]} \subseteq HOD^{V[G]} \subseteq V[G^\pi].$$

Thus for all infinite cardinals $\alpha < \kappa$ of $V[G*H*K]$ we have

$$(\alpha^+)^{HOD^{V[G*H*K]}} \leq (\alpha^+)^{V[G^\pi]} = (\alpha^+)^V < (\alpha^+)^{V[G*H*K]}.$$

Let $V^* = V[G*H*K]$. Then V^* is the required model and the theorem follows.

Remark 7.2.
(1) *In fact we can show that κ remains measurable in Theorem 1.1.*
(2) *If we start with a supercompact cardinal κ, then our proof shows that we can preserve the supercompactness of κ and have $(\alpha^+)^{HOD} < \alpha^+$ for a closed unbounded set of $\alpha < \kappa$:*

Open Question 7.3. *What is $HOD^{V[G]}$?*

Open Question 7.4. *Is it consistent that all uncountable cardinals are inaccessible cardinals of HOD?*

Open Question 7.5. *Is it consistent that κ is supercompact and $(\alpha^+)^{HOD} < \alpha^+$ for all cardinals $\alpha < \kappa$?*

As a consequence of his "HOD Conjecture" (see [8]), Woodin has conjectured a negative answer to the previous question.

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