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DEFINABLE NORMAL MEASURES

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Abstract. A normal measure U on a measurable cardinal κ cannot be definable over $H(\kappa^+)$, as otherwise it would belong to its own ultrapower. In this article we show that it may however be Δ_1 definable over $H(\kappa^{++})$ when the GCH fails at κ. In [8] it is shown that there can be a unique normal measure on κ when the GCH fails at κ; we show here that this unique normal measure can in addition be Δ_1 definable over $H(\kappa^{++})$.

1. Introduction and Preliminaries

Definability is a central theme in set theory and in this paper we examine it in the large cardinal setting. Important work in inner model theory (see [10]) reveals that the smaller of the large cardinals can exist in L-like models which share many of the definability features of Gödel's L. This can be extended with forcing (via the *outer model programme*) to the strongest of large cardinals (see [3, 4, 5], for example). When GCH fails, obtaining large cardinal definability results is more of a challenge; an example is [6], which provides a definable wellorder of $H(\kappa^+)$ when the GCH fails at the measurable cardinal κ.

In this paper we continue the study of large cardinal definability when GCH fails, by showing that there can be a normal measure on κ which is Δ_1 definable over $H(\kappa^{++})$ with $2^\kappa = \kappa^{++}$; we can even add the requirement that there is only one normal measure on κ.

Core model theory

We begin with some facts from core model theory (see [10]). Let $L[E]$ be the core model for a κ^{++}-hypermeasurable cardinal κ (i.e. a cardinal κ which is the critical point of an elementary embedding $j : V \to M$ with $H(\kappa^{++}) \subseteq M$, or equivalently, a cardinal κ which is $\kappa + 2$-strong). Then in $L[E]$ we have local definability in the following sense: For every successor cardinal $\eta \geq \omega_2$, $E \upharpoonright \eta$ is Σ_1 definable over $H(\eta)$ with parameters, and this remains true in all cardinal-preserving set-generic extensions of $L[E]$. Using the fact that $\alpha \mapsto L_\alpha[E]$ is Σ_1 in $\langle L_\eta[E], E \upharpoonright \eta\rangle$ and an appropriate form of condensation for substructures of $L_\eta[E]$ we may carry out the canonical construction of a $\diamondsuit$ sequence:

Fact 1.1. *In $L[E]$, for each regular cardinal α there is a $\diamondsuit_{\alpha^+}(\alpha^+ \cap Cof(\alpha))$-sequence $\vec{D}_\alpha$ which is Σ_1 definable over $\langle L_{\alpha^+}[E], E \upharpoonright \alpha^+\rangle$ in parameter $\{\alpha\}$, uniformly in α.*

Let $\langle A_i \mid i < \alpha^{++}\rangle$ be the enumeration of $P(\alpha^+)^{L[E]}$ along $<_{L[E]}$. Then $\vec{S} = \langle S_i \mid i < \alpha^{++}\rangle$ is an almost disjoint family of stationary sets where $S_i = \mathrm{Cof}(\alpha) \cap \{j < \alpha^+ \mid A_i \cap j = D_j\}$. It follows that $\vec{S}$ is Σ_1 definable with parameters over $\langle L_{\alpha^{++}}[E], E \upharpoonright \alpha^{++}\rangle$ and therefore also over the $H(\alpha^{++})$ of any cardinal-preserving set-generic extension of $L[E]$. For technical reasons, we assume that $\vec{S}$ starts with S_{-1} instead of S_0.

For inaccessible α, let $Sacks(\alpha)$ denote the following forcing. A condition is a subset T of $2^{<\alpha}$ such that:

(1) $s \in T$, $t \subseteq s \to t \in T$.

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(2) Each $s \in T$ has a proper extension in T.
(3) If $s_0 \subseteq s_1 \subseteq \cdots$ is a sequence in T of length less than α then the union of the s_i's belongs to T.
(4) Let Split(T) denote the set of s in T such that both $s * 0$ and $s * 1$ belong to T. Then for some closed unbounded $C(T) \subseteq \alpha$,

$$Split(T) \supset \{s \in T \mid \text{length}(s) \in C(T)\}.$$

Extension is defined by $S \leq T$ iff S is a subset of T. $Sacks(\alpha)$ is an α-closed forcing of size α^+. For $i < \alpha$, we define the i-th splitting level of T, $Split_i(T)$ to consist of all $s \in Split(T)$ such that $\{j < |s| \mid s \restriction j \in Split(T)\}$ has ordertype i. Say $S \leq_\beta T$ iff $S \leq T$ and $Split_i(T) = Split_i(S)$ whenever $i < \beta$.

For any condition S and node $s \in S$, $S|s$ denotes the subtree $\{t \in S \mid t \subseteq s \wedge s \subseteq t\}$ of S. Suppose $\langle P_\beta, \dot{Q}_\beta \mid \beta \in I\rangle$ is an iterated forcing where some of the $\dot{Q}_\beta$'s are of the form $Sacks(\alpha)^{P_\beta}$ and f is a function from such β's to $2^{<\alpha}$. Then for any condition p, we define $p|f$ by setting $(p|f)(\alpha) = p(\alpha)$ if $\alpha \notin \text{dom}(f)$ and $(p|f)(\alpha) = p(\alpha)|f(\alpha)$ otherwise. We are only interested in the case when $p|f$ is also a condition in the iteration.

2. A definable normal measure

Theorem 2.1. *$Con(ZFC + \kappa$ is a κ^{++}-hypermeasurable cardinal$) \rightarrow Con(ZFC + 2^\kappa = \kappa^{++} +$ there is a normal measure U on κ such that U is Δ_1 definable over $H(\kappa^{++})$ with parameters$)$.*

We begin with an outline of the proof. We use the method of [1] to obtain a model where κ is measurable and $2^\kappa = \kappa^{++}$ using iterated κ-Sacks forcing, after a preparation below κ using iterated α-Sacks forcings for inaccessible $\alpha < \kappa$. However we mix into the κ-Sacks iteration forcings that kill the stationarity of canonical stationary subsets of κ^+ of the ground model, so as to code certain subsets of κ. The subsets of κ that get coded are the ones that appear in the final measure on κ once the iteration is complete. To see that κ is indeed measurable in the final model, we project our iteration onto the standard κ-Sacks iteration and argue that this projected iteration, which by [1] produces a measure U' on κ, captures all subsets of κ. We make a canonical choice of the measure U' (as the interpretation of a name relative to the iterated κ-Sacks generic). Thus during the construction we code subsets of κ which are forced by the projection of a condition in the generic into the standard κ-Sacks iteration to belong to U'.

Throughout this section, we assume $V = L[E]$, where $L[E]$ is the core model witnessing that κ is κ^{++}-hypermeasurable. For each regular $\lambda \leq \kappa$, let $\langle S^\lambda_\alpha \mid \alpha < \lambda^{++}$ or $\alpha = -1\rangle$ be the almost disjoint family of stationary subsets of $\lambda^+ \cap \text{Cof}(\lambda)$ as defined in the last section.

Proof. Let $j : V \to M$ witness the κ^{++}-strength of κ and let U be the normal measure derived from j. Let F be a suitable bookkeeping function from κ^{++} onto $\kappa^{++} \times \kappa^+$. We define a length $\kappa + 1$ Easton support iterated forcing poset $P = \langle P_\alpha, \dot{Q}_\alpha \mid \alpha \leq \kappa\rangle$ as follows:

Case 1: ($\alpha < \kappa$ and α is inaccessible) $\dot{Q}_\alpha$ is an iteration of $Sacks(\alpha)$ of length α^{++} with supports of size at most α.

Case 2: ($\alpha = \kappa$) $\dot{Q}_\kappa$ is defined as follows: $\dot{Q}_\kappa = \langle P^\kappa_\gamma, \dot{Q}^\kappa_\gamma\rangle_{\gamma<\kappa^{++}}$ is an iteration of length κ^{++} with supports of size at most κ. Assume that P^κ_γ is defined and write $P(\gamma) = P_\kappa * P^\kappa_\gamma$. Then $\dot{Q}^\kappa_\gamma$ is a two-step iteration $Q^\kappa_{\gamma 0} * Q^\kappa_{\gamma 1}$. Let $Q^\kappa_{\gamma 0}$ be $Sacks(\kappa)$ defined in $V^{P(\gamma)}$.

Inductively, we will show that if $P'(\gamma)$ is P_κ followed by the length γ iteration of $Sacks(\kappa)$ with supports of size at most κ then there is a canonical projection $p \mapsto p'$ of a dense subset $D(\gamma)$ of $P(\gamma)$ onto $P'(\gamma)$. Thus if $G(\gamma)$ denotes the $P(\gamma)$-generic we obtain via projection a $P'(\gamma)$-generic $G'(\gamma)$. Now let A'_γ be the $F(\gamma)_1$-th subset of κ in $V[G'(F(\gamma)_0)]$.

Let $Q^\kappa_{\gamma 1} = \langle P^\kappa_{\gamma 1\beta}, \dot{Q}^\kappa_{\gamma 1\beta} \mid \kappa * \gamma \leq \beta < \kappa * \gamma + \kappa\rangle$ be the following length κ iteration with full support (where $*$ denotes ordinal multiplication). We say that A'_γ is *valid* if there is a $P(\gamma)$-condition p in $G(\gamma) \cap D(\gamma)$ such that the image p' of p under projection satisfies

$$p' \Vdash_{P'} A'_\gamma \in \dot{U}_{P'},$$

where $P' = P'(\kappa^{++})$ and $\dot{U}_{P'}$ denotes the canonical measure on κ in $V^{P'}$ (it suffices to take $\dot{U}_{P'}$ to be the $L[E]$-least P'-name for a measure on κ).

(1) If $\beta = \kappa * \gamma + 2\delta$, $\delta \in \dot{A}_\gamma$ and A'_γ is valid, then let $Q^\kappa_{\gamma 1\beta}$ be Shoot$((\kappa^+ \setminus S_\gamma) \cup S^{\kappa^+}_{<\kappa})$ (where Shoot denotes the forcing for adding a club subset and $S^{\kappa^+}_{<\kappa}$ denotes the set of ordinals less than κ^+ of cofinality less than κ).
(2) If $\beta = \kappa * \gamma + 2\delta + 1$, $\delta \notin \dot{A}_\gamma$ and A'_γ is valid, then again let $Q^\kappa_{\gamma 1\beta}$ be Shoot$((\kappa^+ \setminus S_\gamma) \cup S^{\kappa^+}_{<\kappa})$.
(3) Otherwise let $Q^\kappa_{\gamma 1\beta}$ be the trivial forcing.

Case 3: (Otherwise) Let $\dot{Q}_\alpha$ be the trivial forcing.

We will prove a pair of lemmas to guarantee that P is well-defined. Note that P_κ preserves cardinals as well as the stationarity of S_γ for all $\gamma \in \kappa^{++} \cup \{-1\}$.

Fix $\gamma < \kappa^{++}$, assume that $P(\gamma)$ has been defined. Consider $\dot{Q}^\kappa_{\gamma 0}$. Since each iterant of P^κ_γ is $<\kappa$-closed. κ remains inaccessible in $V^{P(\gamma)}$. Thus $\dot{Q}^\kappa_{\gamma 0} = Sacks(\kappa)$ can be defined. Note that $Sacks(\kappa)$ preserves cardinals and the stationarity of S_γ for all $\gamma \in \kappa^{++} \cup \{-1\}$.

The next lemma shows that for all $\gamma \leq \kappa^{++}$, $P(\gamma)$ preserves cardinals and the stationarity of all S_β whose stationarity is not explicitly killed. The proof is based on the fusion argument first developed in [6] and falls into the context of good κ-fusion introduced in [7]. Since we need this argument for other purposes, we present a complete proof here instead of just showing that the forcing satisfies good κ-fusion. We fix some notation before proceeding. For each $p \in P^\kappa_\gamma$ and $\xi < \gamma$, $p(\xi)$ denotes the P^κ_ξ name for a $\dot{Q}^\kappa_\xi$ condition specified by p. We write $p(\xi)_0$ for the Sacks component of $p(\xi)$ and $p(\xi)_1$ for the club-shooting component of $p(\xi)$. We write $p \restriction \xi$ for the restriction of p to the forcing P^κ_ξ and $p \restriction \xi 0$ for the restriction of p to the forcing $P^\kappa_\xi * \dot{Q}^\kappa_{\xi 0}$.

Lemma 2.1. *For each $\gamma \leq \kappa^{++}$, assuming that $P(\gamma)$ is well-defined, $P(\gamma)$ preserves the stationarity of S_β for all $\beta \in \kappa^{++} \cup \{-1\}$ where the $P(\gamma)$-generic does not specify that the stationarity of S_β is to be killed. In particular, $P(\gamma)$ preserves cardinals up to κ^+.*

Proof. We shall work in V^{P_κ} and focus on P^κ_γ. Fix $p \in P^\kappa_\gamma$ forcing β to be as in the hypothesis of the lemma. In order to show that the stationarity of S_β is preserved, fix a P^κ_γ-name $\dot{C}$ for a club subset of κ^+; it suffices to show that there is $q \leq_{P^\kappa_\gamma} p$ such that $q \Vdash_{P^\kappa_\gamma} \dot{C} \cap S_\beta \neq \emptyset$. Pick $M \prec H(\eta)$ such that $|M| = \kappa$, $M \cap \kappa^+$ is an ordinal, $M \cap \kappa^+ \in S_\beta$, $M^{<\kappa} \subset M$ and M contains all relevant parameters. We can moreover require $M \cap \kappa^+ \notin S_\xi$ for all $\xi \in (M \cap \kappa^{++}) \setminus \{\beta\}$; this is possible using a Fodor argument, as $\vec{S}$ is an almost disjoint family. Note that $cof(M \cap \kappa^+) = \kappa$, so we can also fix an increasing sequence $\langle \epsilon_\eta \mid \eta < \kappa\rangle$ with supremum $\epsilon = M \cap \kappa^+$, all of whose proper initial segments belong to M. We construct a descending fusion sequence of conditions $\langle p_\eta \mid \eta < \kappa\rangle$ all of whose proper initial segments belong to

M. Let $p_0 = p$. The desired q will be the fusion limit of the p_η-sequence and will satisfy the following property:

For every $\eta < \kappa$, there is a size $< \kappa$ subset F_η of $supp(q)$ such that if $r <_{P^\kappa_\gamma} q$ is such that for all $\xi \in F_\eta$, $r \restriction \xi \Vdash_{P^\kappa_\xi} r(\xi)_0 < q(\xi)_0|\sigma$, where $r \restriction \xi \Vdash_{P^\kappa_\xi} \sigma \in Split_\eta(q(\xi)_0)$, then $r \Vdash_{P^\kappa_\gamma} \tau \in \dot{C}$ for some $\tau \in (\epsilon_\eta, \epsilon)$.

Then the resulting q forces that $\epsilon \in \dot{C} \cap S_\beta$, which completes the proof. We will construct our fusion sequence, along with the auxiliary sequence $\langle F_\eta \mid \eta < \kappa \rangle$, to satisfy the following properties.

(1) $p_0 = p$.
(2) $F_\eta \subset supp(p_\eta)$, $|F_\eta| < \kappa$ and $F_{\eta_1} \subset F_{\eta_2}$ whenever $\eta_1 < \eta_2$.
(3) $p_{\eta+1} <_{F_\eta,\eta} p_\eta$ and $p_{\eta+1} \in M$. Here we say $p <_{F,\eta} q$ if $p < q$ and for all $\gamma \in F$, $p \restriction \gamma 0 \Vdash p(\gamma)_0 <_\eta q(\gamma)_0$.
(4) $\forall \xi \in supp(p_{\eta_1}) \exists \eta_2 (\xi \in F_{\eta_2})$.
(5) $p_{\eta+1}$ is η-determined over F_η, i.e. there is a tree T_η such that
  (a) $T_\eta \cong T'_\eta$, where $T'_\eta = \langle |2^{\eta+1}|^{<\mathrm{ot}(F_\eta)}, \subset \rangle$.
  (b) Every node of T_η is a sequence of elements of $2^{<\kappa}$ and $s <_{T_\eta} t$ iff t extends s.
  (c) For every node s of T_η, define $f_s : F_\eta \restriction |s| \to 2^{<\kappa}$ by mapping the δ-th member of F_η to the δ-th member of s (where $F_\eta \restriction |s|$ denotes the first $|s|$ members of F_η). Then for all $\xi \in F_\eta \restriction |s|$ such that ξ is the δ-th member of F_η,
$$(p_{\eta+1}|f_s) \restriction \xi \Vdash Split_\eta(p_{\eta+1}(\xi)_0) = \{t(\delta) \mid s \restriction \delta <_{T_\eta} t\}$$
  and
$$(p_{\eta+1}|f_s) \restriction \xi 0 \text{ decides } p_{\eta+1}(\xi)_1 \restriction \epsilon_\eta,$$
  (d) For every branch r of T_η, let $f_r = \bigcup_{\delta < \mathrm{ot}(F_\eta)} f_{r \restriction \delta}$; then $\exists \tau_r \in (\epsilon_\eta, \epsilon)$ such that $p_{\eta+1}|f_r \Vdash_{P^\kappa_\gamma} \tau_r \in \dot{C}$.

We now give the construction of the p_η's and ensure that it takes place in M.

For $\eta < \kappa$ limit, let p_η be the greatest lower bound of $\langle p_\zeta \mid \zeta < \eta \rangle$, which exists since P^κ_γ is $< \kappa$-closed.

For $\eta = \zeta + 1$, assume p_ζ has been defined. Let F_ζ be the support of p_ζ. We construct the tree T_ζ and a system of conditions $\{p_s \mid s \in T_\zeta \cup [T_\zeta]\}$ associated to T_ζ. Let $\lhd^{1\zeta}$ be a wellordering of $[T'_\zeta]$. For any node $s \in T'_\zeta$, let $r_s \in [T'_\zeta]$ be the $\lhd^{1\zeta}$-least branch extending s. Let $\lhd^{2\zeta}$ be the induced wellordering of nodes in T'_ζ as following: $s \lhd^{2\zeta} t$ iff $r_s \lhd^{1\zeta} r_t$ or $r_s = r_t \wedge s \subset t$. Let $\lhd^\zeta$ be a wellordering of $T'_\zeta \cup [T'_\zeta]$ by letting $s \lhd^\zeta t$ iff $s \lhd^{1\zeta} t$ or $s \lhd^{2\zeta} t$ or $s \lhd^{1\zeta} r_t$ or $r_s \lhd^{1\zeta} t$.

Note that the order type of $\lhd^\zeta$ is less than κ. The construction is along $\lhd^\zeta$. At each stage $s \in T'_\zeta \cup [T'_\zeta]$, we define a pair $\langle t_s, p_s \rangle$, where t_s will be a corresponding node or branch of T_ζ and p_s will be the associated condition to t_s. We will ensure that $p_s <_{F_\zeta,\zeta} p_t$ iff $t \lhd^\zeta s$.

During the construction, the p_s's for $s \in T'_\zeta \cup [T'_\zeta]$ will satisfy the following assumptions $(*)$: Let $s' \in T_\zeta \cup [T_\zeta]$ be the node or branch corresponding to s, then

(1) If $s' \in T_\zeta$,then there exists $A \subset 2^{<\kappa}$ such that for all $\xi \in F_\zeta \restriction \mathrm{ot}(s))$,
$$(p_s|f_{s'}) \restriction \xi \Vdash Split_\zeta(p_s(\xi)_0) = \check{A}$$
and
$$(p_s|f_{s'}) \restriction \xi 0 \text{ decides } p_s(\xi)_1 \restriction \epsilon_\zeta.$$
For such A and s, fix an enumeration $\vec{A}_s = \langle A(\delta) \mid \delta < |2^{\zeta+1}| \rangle$ of A.
(2) If $s' \in [T_\zeta]$, then $\exists \tau_s \in (\epsilon_\zeta, \epsilon)$ s.t $p_s|f_{s'} \Vdash_{P^\kappa_\gamma} \tau_s \in \dot{C}$.

Assume now we are in the situation to define s and p_s. Let p'_s be the greatest lower bound of $\langle p_t \mid t \lhd^\zeta s\rangle$. Note that by the induction hypothesis, for all $s \lhd^\zeta t$, $p'_s \leq_{F_\zeta,\zeta} p_t$. The definition of p_s splits into three cases:

Case 1: $s \in T'_\zeta$ and has no $<_{T'_\zeta}$ immediate predecessor. In this case we let $p_s = p'_s$.

Case 2: $s \in T'_\zeta$ and has an $<_{T'_\zeta}$ immediate predecessor $t \in T'_\zeta$. Choose $\delta < |2^{\zeta+1}|$ such that $s = t^\frown\langle\delta\rangle$. By the definition of $\lhd^\zeta$, p_t has been defined and satisfies $(*)(1)$. Let $t' \in T_\zeta$ be the node corresponding to t. Let A_t be the corresponding witness of $(*)(1)$. Since $p'_s \leq_{F_\zeta,\zeta} p_t$, p'_s also satisfies $(*)(1)$. Let $s' = t'^\frown A_t(\delta)$ and define $f_{s'}$ respectively. Now denote $p'_s|f_{s'}$ by p''_s, the existence of which is justified by $(*)(1)$. Using the Maximality Principle, we can pick a $P^\kappa_{F_\zeta(\mathrm{ot}(s))}$ condition p'''_s, a $P^\kappa_{F_\zeta(\mathrm{ot}(s))}$-name $\dot T$ for a subtree of 2^κ, a $P^\kappa_{F_\zeta(\mathrm{ot}(s))} * \dot Q^\kappa_{F_\zeta(\mathrm{ot}(s))}$ name $\dot S$ for a subset of κ^+ and a set $A_s \subset 2^{<\kappa}$ such that

$$p''_s \restriction_{F_\zeta(\mathrm{ot}(s))} > p'''_s \Vdash_{P^\kappa_{F_\zeta(\mathrm{ot}(s))}} \dot T \leq_\zeta p''_s(F_\zeta(\mathrm{ot}(s)))_0 \wedge Split_\zeta(\dot T) = A_t$$

and

$$p'''^\frown_s \dot T \Vdash S < p''_s(F_\zeta(ot(s)))_1 \text{ and } \sup \dot S > \epsilon_\zeta.$$

By elementarity, we can choose all these objects in M. We complete Case 2 by inductively defining $p_s(\zeta)$ to be a name as follows:

- $p_s|f_{s'} \restriction_\xi \Vdash p_s(\xi) = p'_s(\xi)$ if $\xi > F_\zeta \restriction (\mathrm{ot}(s)+1)$.
- $p_s|f_{s'} \restriction_\xi \Vdash p_s(\xi) = p'''_s(\xi)$ if $\xi \leq F_\zeta \restriction (\mathrm{ot}(s)+1)$ and $\xi \notin F_\zeta$.
- $(p_s|f_{s'}) \restriction \xi \Vdash p_s(\xi)_0$ be the tree amalgamating $p'''_s(\xi)_0$ to $p'_s(\xi)_0$ and $p_s(\xi)_1 = p'''_s(\xi)_1$ if $\xi \in F_\zeta \cap F_\zeta(\mathrm{ot}(s))$.
- otherwise $(p_s|f_{s'}) \restriction \xi \Vdash p_s(\xi)_0 = \dot T$ and $(p_s|f_{s'}) \restriction \xi^\frown \dot T \Vdash p_s(\xi)_1 = \dot S$, if $\xi = F_\zeta(\mathrm{ot}(s))$.
- Whenever $p < p_s \restriction \xi$ and $p \parallel (p_s|f_{s'}) \restriction \xi$, $p \Vdash p_s(\xi) = p'_s(\xi)$.

Case 3: $s \in [T'_\zeta]$. By the definition of $\lhd^\zeta$, for all $t \subset s$ and $t \in T'_\zeta$, p_t has been defined. As $f_{s'} = \bigcup_{t' \subset s' \wedge t' \in T_\zeta} f_{t'}$, by applying $(*)(1)$ to all such t's, $p'_s|f_{s'}$ is welldefined. Choose $p''_s < p'_s|f_{s'}$ be a condition such that there is $\tau_s > \epsilon_\zeta$ such that $p''_s|f_{s'} \Vdash \tau_s \in \dot C$. By elementarity, we can choose p''_s and τ_s in M. Finally we define p_s by amalgamating $p''_s|f_s$ to p'_s via the same procedure as in Case 2.

Let p_η be the infimum of $\langle p_s \mid s \in T'_\zeta \cup [T'_\zeta]\rangle$. p_η is a condition as $\mathrm{ot}(\lhd^\zeta) < \kappa$. Moreover $p_\eta <_{F_\zeta,\zeta} p_s$ for all $s \in T'_\zeta \cup [T'_\zeta]$. It can be verified that p_η is the desired condition.

Finally, let q be the fusion limit of $\langle p_\eta \mid \eta < \kappa\rangle$. To be precise, $q \restriction \xi \Vdash q(\xi)_0 = \bigcap_{\eta<\kappa} p(\xi)_0$ and $q \restriction \xi 0 \Vdash q(\xi)_1 = \bigcup_{\eta<\kappa} p(\xi)_1 \cup \{\epsilon\}$ if $\xi \in \bigcup_{\eta<\kappa} supp_1(p_\eta)$ and $q \restriction \xi 0 \Vdash q(\xi)_1$ be trivial condition otherwise. Since $\epsilon \in S_\beta \setminus \bigcup_{\xi\in\alpha\cap M} S_\xi$, it is routine to verify that q is a condition and $q <_{F_\eta,\eta} p_\delta$ for all $\eta \leq \delta < \kappa$.

For any $\eta < \kappa$, $p_{\eta+1}$ is η-determined over F_η. Now if $r <_{P^\kappa_\gamma} p_{\eta+1}$ is such that for all $\xi \in F_\eta$, $r \restriction \xi \Vdash_{P^\kappa_\xi} r(\xi)_0 < p_{\eta+1}(\xi)_0|\sigma$, where $r \restriction \xi \Vdash_{P^\kappa_\xi} \sigma \in Split_\eta(q(\xi)_0)$, then $\exists s \in [T_\eta]$ such that $\sigma = f_s$. It follows that $r < p_{\eta+1}|f_s < p_s|f_s \Vdash_{P^\kappa_\gamma} \tau_s \in \dot C$. Since $q <_{F_\eta,\eta} p_{\eta+1}$, the same statement when replacing $p_{\eta+1}$ by q. $\square$

It follows that cardinals up to κ^+ and the stationarity of $\langle S_\xi \mid \xi \in [\kappa\cdot\gamma, \kappa^{++})\rangle$ are preserved by $P(\gamma)$. Thus assuming $P(\xi)$ is well-defined for all $\xi < \gamma$, in order to prove that $P(\gamma)$ is well-defined, it suffices to define the projection of a dense subset $D(\gamma)$ of $P(\gamma)$ onto $P'(\gamma)$ as promised in the definition of $P(\gamma+1)$.

Fix $\gamma < \kappa^{++}$. Suppose that for all $\xi < \gamma$, $P(\gamma)$ has been defined. Moreover, assume we keep the following induction hypothesis:

(1) For any $\xi_0 < \xi_1$, $D(\xi_0) = D(\xi_1) \cap P_(\xi_0)$.

(2) For any ξ, Lemma 2.2 holds for $D(\xi)$.

Now for a condition $q \in P(\gamma)$ define the "derived condition" q' in $P'(\gamma)$ ($= P_\kappa$ followed by the length γ iteration of κ-Sacks) as follows: $q'(\xi)$ is the least $P'(\xi)$-name $\dot{A}$ such that $q \upharpoonright \xi \Vdash_{P(\xi)} \dot{A}' = q(\xi)$, where the $\dot{A}'$ is the canonical $P(\xi)$-name derived from $\dot{A}$ via the complete embedding h_ξ defined in Lemma 2.2 for $D(\xi)$. Clearly, $D(\gamma)$ satisfies (1). Let $D(\gamma)$ denote the set of conditions q such that q' is defined. It remains to verify Lemma 2.2 for $D(\gamma)$, which ends the inductive definition.

Lemma 2.2. *$D(\gamma)$ is dense. Moreover:*
(a) The map $h_\gamma : q \mapsto q'$ is a projection from $D(\gamma)$ onto $P'(\gamma)$.
(b) Let $G(\gamma)$ be $P(\gamma)$-generic and $G'(\gamma) = \{q' \mid q \in D(\gamma) \cap G(\gamma)\}$ the derived $P'(\gamma)$-generic. Then

$$H(\kappa^+)^{V[G(\gamma)]} = H(\kappa^+)^{V[G'(\gamma)]}.$$

Proof. Suppose that p belongs to $P(\gamma)$; we want to construct $q < p$ such that q belongs to $D(\gamma)$. As the construction of q is very similar to the construction in the proof of Lemma 2.1, we will only present the requirements that have to be met and make some comments about some new features. Fix $M \prec H(\eta)$ for some sufficiently large η such that M contains all relevant parameters and satisfies the same requirements stated in the proof of Lemma 2.1. The desired condition q is constructed as a fusion limit of a sequence $\langle q_\eta \mid \eta < \kappa \rangle$ obeying the following requirements:

(1) $q_0 = p$.
(2) $F_\eta \subset supp(q_\eta)$, $|F_\eta| < \kappa$ and $F_{\eta_1} \subset F_{\eta_2}$ whenever $\eta_1 < \eta_2$.
(3) $q_{\eta+1} <_{F_\eta,\eta} q_\eta$ and $q_{\eta+1} \in M$.
(4) $\forall \eta \in supp(p_{\eta_1}) \exists \eta_2 \geq \eta_1 (\eta \in F_{\eta_2})$.
(5) $q_{\eta+1}$ is η-determined over F_η, i.e. there is a tree T_η such that
  (a) $T_\eta \cong T'_\eta$, where $T'_\eta = \langle |2^{\eta+1}|^{<\mathrm{ot}(F_\eta)}, \subset \rangle$.
  (b) Every node of T_η is a sequence of elements of $2^{<\kappa}$ and $s <_{T_\eta} t$ iff t extends s.
  (c) For every node s of T_η, define $f_s : F_\eta \upharpoonright |s| \to 2^{<\kappa}$ by mapping the δ-th member of F_η to the δ-th member of s (where $F_\eta \upharpoonright |s|$ denotes the first $|s|$ members of F_η). Then for all $\xi \in F_\eta \upharpoonright |s|$ such that ξ is the δ-th member of F_η,
$$(q_{\eta+1}|f_s) \upharpoonright \xi \Vdash Split_\eta(q_{\eta+1}(\xi)_0) = \{t(\delta) \mid s \upharpoonright \delta <_{T_\eta} t\}$$
  and
$$(q_{\eta+1}|f_s) \upharpoonright \xi 0 \text{ decides } q_{\eta+1}(\xi)_1 \upharpoonright \epsilon_\eta,$$
  (d) For every branch r of T_η, let $f_r = \bigcup_{\delta < \mathrm{ot}(F_\eta)} f_{r \upharpoonright \delta}$, then for every $\xi \in F_{\eta+1}$, $q_{\eta+1}|f_r \upharpoonright \xi \Vdash q_{\eta+1}(\xi) \in V[G'(\xi)]$.

Here to achieve (5)(d), we need to inductively use Lemma 2.2(b). Let q be the fusion limit of $\langle q_\eta \mid \eta < \kappa \rangle$. It follows from (5)(d) that for all $\xi < \gamma$, $q \upharpoonright \xi \Vdash q(\xi) \in V[G'(\xi)]$ as $q(\eta)$ is the fusion limit of $q_\eta(\xi)$ and each of $q_\eta(\xi)$ is in $V[G'(\xi)]$. Hence, there is a condition q' derived from q and q belongs to $D(\gamma)$.

(a) Suppose that q belongs to $D(\gamma)$ and choose $r' < q'$. Define a condition r by inductively replacing $q(\xi 0)$ by a $P(\xi)$-name for $r'(\xi)$. Then r' is the condition derived from r and $r < q$.

(b) The proof of the density of $D(\gamma)$ above can be adapted to show that $P(\gamma)/P'(\gamma)$ is $< \kappa^+$ distributive. The point is that for any $P(\gamma)$-name $\dot{B}$ for a subset of κ, we can add one more requirement

$$q_{\eta+1}|f_r \upharpoonright \xi \text{ decides "} \eta \in \dot{B} \text{"}$$

to (5)(d). Then the fusion limit q reduces $\dot{B}$ to a $P'(\gamma)$ name $\dot{B}'$. It follows that all subsets of κ in $V[G(\gamma)]$ in fact belong to $V[G'(\gamma)]$ and therefore these models have the same $H(\kappa^+)$. □

This completes the definition of $P(\gamma)$. We have that $V[G'(\gamma)] = V[G_\kappa * G'^{\kappa}_\gamma]$ is an inner model of $V[G(\gamma)] = V[G_\kappa * G^\kappa_\gamma]$ such that $H(\kappa^+)^{V[G_\kappa * G'^{\kappa}_\gamma]} = H(\kappa^+)^{[G_\kappa * G^\kappa_\gamma]}$ and P^κ_γ preserves cardinals up to κ^+. We need to show that P^κ_γ preserves cardinals greater than κ^+. To do so we apply a higher analog of proper forcing.

Definition 2.1. *Let N be an elementary submodel of $H(\theta)$. We say that N is relevant if*

- $|N| = \kappa$.
- $N^{<\kappa} \subset N$.
- $N = \bigcup_{\beta<\kappa} N_\beta$, *where $\langle N_\beta : \beta < \kappa\rangle$ is a continuous $\in$-increasing sequence of elementary submodels of $H(\theta)$ such that $\langle N_\gamma : \gamma \leq \beta\rangle \in N_{\beta+1}$ and $|N_\beta| < \kappa$.*

Definition 2.2. *Suppose S is a stationary subset of $S^{\kappa^+}_\kappa$, the set of ordinals less than κ^+ of cofinality κ. A notion of forcing P is said to be S-proper if for all sufficiently large regular cardinals θ, there is some $x \in H(\theta)$ such that whenever M is a relevant elementary submodel of $H(\theta)$ with $P, x \in M$, $M \cap \kappa^+ \in S$ and p is an element of $M \cap P$, there is a condition $q \leq p$ such that*

$$q \Vdash \text{“}M[\dot{G}_P] \cap \mathit{Ord} = M \cap \mathit{Ord}\text{”}.$$

Such a condition q is said to be (M, P)-generic.

The following Lemma generalizes Proposition 3.1 of [2]. The proof is almost identical to the proof in [2] with the one difference that the chosen submodel M must satisfy $M \cap \kappa^+ \in S$.

Lemma 2.3. *Suppose S is a stationary subset of $S^{\kappa^+}_\kappa$. Assume $\kappa^{<\kappa} = \kappa$, $2^\kappa = \kappa^+$, and let $P = \langle P_i, Q_i : i < \kappa^{++}\rangle$ be an α-support iteration such that*

(1) *P_i is S-proper for $i \leq \kappa^{++}$*
(2) *$P_i \Vdash$ “$|Q_i| \leq \kappa^+$”.*

Then $P_{\kappa^{++}}$ satisfies the κ^{++}-chain condition.

Lemma 2.4. *P^κ_γ is S_{-1}-proper for all $\gamma < \kappa^{++}$.*

Proof. By the definition of P, S_{-1} was not intended to be killed.. Let $M \prec H(\theta)$ be relevant and such that $p, P \in M$, $M \cap \kappa^+ \in S_{-1}$. It is clear that $M \cap \kappa^+ \in S_{-1} \setminus \bigcup_{\xi \in M\cap\gamma} S_\xi$. Thus an (M, P^κ_γ) condition $q < p$ can be constructed as in proof of Lemma 2.1. □

By Lemma 2.2, for all $\gamma < \kappa^{++}$, $V[G_\kappa][G^\kappa_\gamma] \models 2^\kappa = \kappa^+$, and therefore $|Q'^{\kappa}_\gamma| = \kappa^+$. Inductively applying Lemma 2.3, for all $\gamma \leq \kappa^{++}$, P^κ_γ is κ^{++}-c.c and hence preserves cardinals greater than κ^+.

As mentioned in the definition of P, κ is a measurable cardinal in $V[G_\kappa * G'^\kappa]$ and $U_{P'}$ denotes the canonical normal measure in $V[G_\kappa * G'^\kappa]$ (derived from a lifting of j). $U_{P'}$ is also a normal measure in $V[G_\kappa * G^\kappa]$ as this model has the same subsets of κ as $V[G_\kappa * G'^\kappa]$.

Lemma 2.5. *$U_{P'}$ is Δ_1-definable with parameters over $H(\kappa^{++})$ in $V[G_\kappa * G^\kappa]$.*

Proof. $U = U_{P'}$ is coded by the following sentence:

$$x \in U \leftrightarrow \exists\beta\forall\alpha < \kappa((\alpha \in x \to \kappa^+ \setminus S_{\beta+2*\alpha+1} \cup S^{\kappa^+}_{<\kappa} \text{ contains a club })$$
$$\wedge(\alpha \notin x \to \kappa^+ \setminus S_{\beta+2*\alpha} \cup S^{\kappa^+}_{<\kappa} \text{ contains a club }))$$

Note that $\langle S_\alpha \mid \alpha < \kappa^{++}\rangle$ is Σ_1 definable with parameters over $H(\kappa^{++})$ and $S_\kappa^{\kappa^+}$ is an element of $H(\kappa^{++})$ in $V[G_\kappa * G^\kappa]$. It follows that U is Σ_1 definable (and hence Δ_1 definable) with parameters over $H(\kappa^{++})$ in $V[G_\kappa * G^\kappa]$. □

□

3. A Unique and Definable Normal Measure

In [8], the following theorem is proved:

Theorem 3.1. *Suppose $V = L[E]$, the canonical inner model for one κ^{++}-hypermeasurable cardinal κ. Then in a forcing extension, $2^\kappa = \kappa^{++}$ and κ carries a unique normal measure.*

In this section we prove:

Theorem 3.2. *$Con(ZFC + \kappa$ is a κ^{++}-hypermeasurable cardinal$)\rightarrow Con(ZFC + 2^\kappa = \kappa^{++}+$ there is a unique normal measure U on κ, and U is Δ_1 definable over $H(\kappa^{++})$ with parameters$)$.*

We begin by describing the general framework of the proof of Theorem 3.1. Let $L[E]$ be the canonical inner model for one κ^{++}-hypermeasurable cardinal κ and j the κ^{++}-hypermeasurable embedding. We define a forcing P' to force that $2^\kappa = \kappa^{++}$ in such a way that the generic filter G can code itself in the sense that G is the unique generic filter in $L[E][G]$. Via a lifting argument, we show that j can be uniquely lifted and hence κ is measurable. Moreover, we take advantage of the properties of $L[E]$ to prove that all embeddings in $L[E][G]$ with critical point κ lift j.

For our present purposes we use a different version of P', in which we code using clubs in κ^+ instead of κ^{++}.

As in [8], P' is a length $\kappa + 1$ iteration which is nontrivial only at inaccessible $\alpha \leq \kappa$.

For an inaccessible cardinal α, let $Sacks^*(\alpha)$ be the following variant of $Sacks(\alpha)$, where S^α_ω denotes the set of ordinals less than α of cofinality ω: A condition is a subset T of $2^{<\alpha}$ such that:

(1) $s \in T$, $t \subseteq s \rightarrow t \in T$.
(2) Each $s \in T$ has a proper extension in T.
(3) If $s_0 \subseteq s_1 \subseteq \cdots$ is a sequence in T of length less than α then the union of the s_i's belongs to T.
(4) Let Split(T) denote the set of s in T such that both $s*0$ and $s*1$ belong to T. Then $s \in$Split$(T) \rightarrow$ length$(s) \in S^\kappa_\omega$ and for some closed unbounded $C(T) \subseteq \alpha$,
$$\text{Split}(T) \supset \{s \in T \mid \ \text{length}(s) \in C(T) \cap S^\alpha_\omega\}.$$

$Sacks^*(\alpha)$ is $< \alpha$ closed and satisfies α-fusion. We can define $Split_i(T)$ and $<_i$ for $Sacks^*(\alpha)$ as for $Sacks(\alpha)$. All the occurences of $Sacks(\alpha)$ in the last section can be replaced by $Sacks^*(\alpha)$.

Suppose $r \subset \alpha$ and $\vec{X} = \langle X_i \mid i < \alpha^+\rangle$ is a sequences of almost disjoint stationary subsets of $\alpha^+ \cap \text{Cof}(\alpha)$. Define $Code_\alpha(r, \vec{X})$ as follows. A condition in $Code_\alpha(r, \vec{X})$ is a closed, bounded subset C of α^+. For conditions c, d in $Code_\alpha(r, \vec{X})$, $d \leq c$ iff

(1) d end-extends c.
(2) For $i < \alpha$, if $i \in r$ then $d \setminus c$ is disjoint from X_{4i}, otherwise $d \setminus c$ is disjoint from X_{4i+1}.
(3) For $i < \max(c)$, if $i \in c$, then $d \setminus c$ is disjoint from X_{4i+2}, otherwise $d \setminus c$ is disjoint from X_{4i+3}.

For all inaccessible cardinals $\alpha \leq \kappa$ let $\vec{S}_\alpha$ be the sequence of almost disjoint stationary subsets of $\alpha^+ \cap \mathrm{Cof}(\alpha)$ derived from the canonical $\diamondsuit_{\alpha^+}$ sequence as in section 1. Via some coding we regroup $\vec{S}^\alpha$ as $\langle S^\alpha_{i,j} \mid i = j = -1$ or $i < \alpha^+, j < \alpha^{++}\rangle$.

Now we define the iteration P' of length $\kappa + 1$:

Q'_α is trivial if α is not inaccessible. Otherwise, Q'_α is an iteration $\langle Q'^\alpha_i \mid i < \alpha^{++}\rangle$ with supports of size at most α. For each i, Q'^α_i is defined as the forcing $Sacks^*(\alpha) * Code_\alpha(r_i, \langle S^\alpha_{j,2*i} \mid j < \alpha^+\rangle)$, where $r_i \subset \alpha$ is derived from the generic filter for $Sacks^*(\alpha)$. Q'_α is $< \alpha$-closed.[1]

P'_λ is the nonstationary support limit of the P'_α, $\alpha < \lambda$, for limit ordinals $\lambda \leq \kappa$. I.e., p belongs to P'_λ iff p belongs to the inverse limit of the P'_α, $\alpha < \lambda$, and if λ is inaccessible then the set of $\alpha < \lambda$ such that $p(\alpha)$ is nontrivial is a nonstationary subset of λ.

We quote the following useful fact regarding nonstationary support (Lemma 4 of [8]):

Fact 3.1. *Suppose that $\lambda \leq \kappa$ is inaccessible and $\langle \alpha_i \mid i < \lambda\rangle$ is the increasing enumeration of a closed unbounded subset of λ. Also suppose that $p_0 \geq p_1 \cdots$ is a λ-sequence of conditions in P'_λ where p_{i+1} agrees with p_i up to and including α_i for each $i < \lambda$, and p_γ is the greatest lower bound of the p_i, $i < \gamma$, for limit $\gamma < \lambda$. Then there is a condition p in P'_λ which extends each p_i.*

The following is an analogue of Lemma 13 of [8]:

Lemma 3.1. *P' preserves cofinalities.*

Proof. Suppose that α is an infinite regular cardinal. We show that any ordinal of cofinality greater than α in V also has cofinality greater than α in $V[G]$ for P'-generic G. Decompose P' as $P'_\alpha * Q'_\alpha * Q'_{(\alpha,\kappa+1)}$. Note that $Q'_{(\alpha,\kappa+1)}$ is $< \alpha^+$-closed in $V^{P'_{\alpha+1}}$. On the other hand, using the closure property of Q'_β for $\beta < \alpha$ and Fact 3.1, the proof of Lemma 5 of [8] shows that P'_α does not add a function mapping α cofinally into any ordinal of greater cofinality in V. It remains to deal with Q'_α. Via an argument almost identical to the proof of Lemma 2.1, Q'_α is cofinality preserving. □

Similar arguments indicate the following properties of P' which are parallel to Lemmas 14 and 15 of [8].

Lemma 3.2. *If G is P'-generic, then for any function $f^* : \kappa \to Ord$ in $V[G]$ there is a function $g : \kappa \to [Ord]^{<\kappa}$ in V such that for each $\alpha < \kappa$, $f^*(\alpha) \in g(\alpha)$ and $|g(\alpha)| = \alpha^{++}$.*

Proof. The proof is similar to the proof of Lemma 14 of [8]. But note that the definition of Q_κ is different. Factor P' as $P'_\kappa * Q'_\kappa$ and correspondingly write $V[G]$ as $V[G_\kappa][G^\kappa]$, where G_κ is P'_κ generic and G^κ is Q'_κ generic. If $\dot{f}$ is a name for a function from κ into the ordinals and p is a condition in Q'_κ, then there is $q < p$ forcing that for each $\alpha < \kappa$, $f(\alpha)$ takes one of at most card $((2^\alpha)^\alpha) = \alpha^+$-many values (corresponding to choices of nodes on the α-th splitting level of at most α-many trees). To be precise, q will satisfies the following statement: For every $\alpha < \kappa$, there is size $< \kappa$ subset F_α of $supp(q)$ such that if there are $\sigma : F_\alpha \to 2^{<\kappa}$ and $r <_{P'^\kappa_\gamma} q$ such that for all $\xi \in F_\alpha$, $r \restriction \xi \Vdash_{P'^\kappa_\xi} r(\xi)_0 < q(\xi)_0|\sigma(\xi)$, where $r \restriction \xi \Vdash_{P'^\kappa_\xi} \sigma(\xi) \in Split_\alpha(q(\xi)_0)$, then $q|\sigma$ is well-defined and $q|\sigma \Vdash_{P'^\kappa_\gamma}$ decides $\dot{f}(\alpha)$. The construction of such q is base on the proof of Lemma 2.1 and we omit it here.

[1] We only use the stationary set with even index for coding and preserve the odd part for the proof of Theorem 3.2.

Thus there is a function g as in the statement of the lemma which belongs to $V[G_\kappa]$. And the argument given in the proof of Fact 3.1 shows that if p belongs to P'_κ and $\dot{g}$ is a name for a function from κ into the $[\mathrm{Ord}]^{<\kappa}$ then there is $q < p$ and a closed unbounded subset C of κ such that for α in C, q forces $\dot{g}(\alpha)$ to take one of at most α^{++}-many values (corresponding to choices for conditions in $P'_\alpha * Q'_\alpha$). We may further extend q at nonstationary-many places to force such a bound on the number of possible values for $g(\alpha)$ for all $\alpha < \kappa$. Then putting these two approximation results together we obtain the lemma. □

Lemma 3.3. *For inaccessible $\alpha \leq \kappa$ and $\xi \leq \alpha^{++}$, $P'_\alpha * Q'^{\alpha}_\xi$ preserves the stationary of $S_{k,l}$ where the $P'_\alpha * Q'^{\alpha}_\xi$-generic does not specify that the stationarity of $S_{k,l}$ is to be killed. In particular, for any l satisfying $l = -1$, $l > 2\xi$ or $l = 2n + 1$ for some n, $S_{k,l}$ remains stationary.*

Proof. The proof is almost the same as the proof of Lemma 2.1. □

The following lemma replaces Lemma 16 of [8] and represents the key difference between the two forcings. For an inaccessible cardinal $\alpha \leq \kappa$, suppose that G^α is Q'_α-generic (over $V[G_\alpha]$). Let $\langle r_i \mid i < \alpha^{++}\rangle$ be the sequence of subsets of α derived from the $Sacks^*(\alpha)$ generics. Let $\langle C_i \mid i < \alpha^{++}\rangle$ be the sequence of clubs derived from the $Code_\alpha(r_i, \vec{S}^\alpha_{j,2*i})$ generics.

Lemma 3.4. *The following hold in $V[G_\alpha][G^\alpha]$.*

1. *For $i, j < \alpha^{++}$, i belongs to r_j iff $S^\alpha_{i,8j}$ is nonstationary and i does not belong to r_j iff $S^\alpha_{i,8j+2}$ is nonstationary.*
2. *For $i, j < \alpha^{++}$, i belongs to C_j iff $S^\alpha_{i,8j+4}$ is nonstationary and i does not belong to C_j iff $S^\alpha_{i,8j+6}$ is nonstationary.*
3. *There is a unique Q'_α-generic over $V[G_\alpha]$.*

Proof. (1) By the definition of extension for C_j, it follows that in $V[G_\alpha][G^\alpha \restriction j]$, for $i < \alpha$, $S^\alpha_{i,8j}$ is nonstationary if i belongs to r_j and $i < \alpha$, $S^\alpha_{i,8j+2}$ is nonstationary if i does not belong to r_j. Hence these remain valid in $V[G_\alpha][G^\alpha]$. On the other hand, via the same argument as in the proof of Lemma 2.1, if $S^\alpha_{i,j}$ is not explicitly killed by some $Code_\alpha(r_m, \vec{S}^\alpha_{m,n})$, then $S^\alpha_{i,j}$ is in fact stationary in $V[G_\alpha][G^\alpha]$.

(2) Just like (1).

(3) Using (1) and (2), suppose that we have another Q'_α-generic filter G'^α. Then for all $j < \alpha^{++}$, $V[G_\alpha][G'^\alpha \restriction j]$ is a inner model of $V[G_\alpha][G^\alpha]$. By induction on $j < \alpha^{++}$, we can show $G'^\alpha \restriction j = G^\alpha \restriction j$, as otherwise some stationary set of $V[G_\alpha][G^\alpha]$ is not stationary in $V[G_\alpha][G'^\alpha \restriction j]$. □

It can be verified that in $V[G]$, j can be lifted to $j^* : V[G] \to M[G * H]$. Moreover, the lifted embedding is unique in the sense that:

Lemma 3.5. *In $V[G]$ there is a unique $G * H \subset j(P')$ which is $j(P')$-generic over M and which contains $j[G]$ as a subset.*

The construction of the lifting and the proof of this lemma involve the fusion argument presented in the last section. We omit the construction here as we will need a more general version in the proof of the main theorem.

We quote the following lemma in [8] which completes the proof of Theorem 3.1. The proof actually does not rely on the exact definition of the forcing and can be applied to P'. We describe the form we need as follows:

Lemma 3.6. *Suppose $V = L[E]$ is the canonical core model for one κ^{++}-hypermeasurable cardinal and P is a forcing with the following properties:*

1. *P is cofinality preserving.*

(2) *P forces $2^\kappa = \kappa^{++}$.*
(3) *If G is P-generic, then for any function $f^* : \kappa \to Ord$ in $V[G]$ there is a function $g : \kappa \to [Ord]^{<\kappa}$ in V such that for each $\alpha < \kappa$, $f^*(\alpha) \in g(\alpha)$ and $|g(\alpha)| \leq \alpha^{++}$.*

Suppose moreover that U is a normal measure on κ in $V[G]$. Then U is the normal measure derived from an embedding $j^* : V[G] \to M^*$ where j^* extends j, where $j : V \to M$ is derived from the last extender of the E sequence.

Now we are in the position to present the proof of Theorem 3.2. We will define a forcing P which absorbs P' as a complete suborder. Let F be a booking-keeping function from κ^{++} to $\kappa^{++} * \kappa^+$. We define a length $\kappa + 1$ Easton support iterated forcing poset $P = \langle P_\alpha, \dot{Q}_\alpha \mid \alpha \leq \kappa \rangle$ as follows:

Case 1: $(\alpha < \kappa)$ P_α is equal to P'_α.

Case 2: $(\alpha = \kappa)$ $\dot{Q}_\kappa$ is defined as follows: $\dot{Q}_\kappa = \langle P^\kappa_\gamma, \dot{Q}^\kappa_\gamma \rangle$ is an iteration of length κ^{++} with supports of size at most κ. Let $P(\gamma) = P_\kappa * P^\kappa_\gamma$. Define $\dot{Q}^\kappa_\gamma$ to be a two step iteration $Q^\kappa_{\gamma 0} * Q^\kappa_{\gamma 1}$. Let $Q^\kappa_{\gamma 0}$ be Q'^κ_γ. Let A'_γ be the $F(\gamma)_1$-th subset of κ in $V[G'(F(\gamma)_0)]$. We say that A'_γ is *valid* if there is a $P(\gamma)$-condition p in $G(\gamma) \cap D(\gamma)$ such that the image p' of p under projection satisfies

$$p' \Vdash_{P'} A'_\gamma \in \dot{U}_{P'},$$

where $P' = P'(\kappa^{++})$ and $\dot{U}_{P'}$ denotes the canonical measure on κ in $V^{P'}$ (it suffices to take $\dot{U}_{P'}$ to be the $L[E]$-least P'-name for a measure on κ). If A'_γ is valid, let $Q^\kappa_{\gamma 1}$ be $Code_\kappa(\dot{A}_\gamma, \langle S^\kappa_{j,2*i+1} \mid j < \kappa^+ \rangle)$. Otherwise let $Q^\kappa_{\gamma 1}$ be the trivial forcing.

Case 3: (Otherwise) Let $\dot{Q}_\alpha$ be trivial forcing.

Using the same argument as in the last section, we can inductively prove the following lemma which justifies the definition of P:

Lemma 3.7. (1) *P^κ_α preserves the stationarity of S_β when the P^κ_α-generic does not specify that it be killed.*
(2) *There is a projection from a dense subset of P^κ_α onto P'^κ_α and $P^\kappa_\alpha / P'^\kappa_\alpha$ is $< \kappa^+$ distributive.*
(3) *$P_\kappa * P^\kappa_\alpha$ is cofinality preserving.*

Proof. The proof of (1) is identical to the proof of Lemma 2.1. The proof of (2) is identical to the proof of Lemma 2.2. The proof of (3) is identical to the discussion after Lemma 2.4. □

Lemma 3.8. (1) *P is cofinality preserving and forces $2^\kappa = \kappa^{++}$.*
(2) *P preserves the stationarity of S_β when it is not explicitly killed by the P-generic.*
(3) *If G is P-generic, then for any function $f^* : \kappa \to Ord$ in $V[G]$ there is a function $g : \kappa \to [Ord]^{<\kappa}$ in V such that for each $\alpha < \kappa$, $f^*(\alpha) \in g(\alpha)$ and $|g(\alpha)| = \alpha^{++}$.*
(4) *In $V[G_\kappa][G^\kappa]$, there is a unique Q_κ generic over $V[G_\kappa]$.*

Proof. (1) Cofinality preserving is proved in the last lemma. $2^\kappa = \kappa^{++}$ follows from the fact that $H(\kappa^+)^{V^{P'}} = H(\kappa^+)^{V^P}$.
(2) Just like Lemma 3.7 (2).
(3) Follow from Lemma 3.2 and Lemma 3.7 (3).
(4) Just like Lemma 3.4 (3). □

Let G' be the derived P' generic filter over V. As shown in Theorem 3.1, there is a unique normal measure $U_{G'}$ over κ in $V[G']$. Moreover, in $V[G]$, $U_{G'}$ is Σ_1 definable over $H(\kappa^{++})$ with parameters.

Lemma 3.9. *$U' = U_{G'}$ is Δ_1-definable over $\langle H(\kappa^{++}), \in\rangle$ with parameters.*

Proof. U' is coded by the following sentence:

$$x \in U' \leftrightarrow \exists\beta\forall\alpha < \kappa((\alpha \in x \to \kappa^+ \setminus S_{\kappa*\beta+8*\alpha+1} \cup S^{\kappa^+}_{<\kappa} \text{ contains a club })$$
$$\wedge(\alpha \notin x \to \kappa^+ \setminus S_{\kappa*\beta+8*\alpha+3} \cup S^{\kappa^+}_{<\kappa} \text{ contains a club }))$$

Also $\langle S_\alpha \mid \alpha < \kappa^{++}\rangle$ is Σ_1 definable over $H(\kappa^{++})$ with parameters and $S^{\kappa^+}_{<\kappa}$ is an element of $H(\kappa^{++})$. It follows that U' is Δ_1 definable over $H(\kappa^{++})$ with parameters. □

Since P/P' is $< \kappa^+$-distributive, U' remains a normal measure in $V[G]$. We show there exists only one normal measure in $V[G]$, which must be U'. Let U be any normal measure in $V[G]$. By properties of P and Lemma 3.6, U is the normal measure derived from a embedding $j^* : V[G] \to M^*$ which lifts j. Hence M^* is of the form $M[G^*]$ where G^* is $j(P)$-generic over M such that $j''G \subset G^*$. It suffices to show there is a unique choice of G^* in $V[G]$.

First we deal with the existence. Let G' be the derived P' generic fliter from G. Note $j(P) \restriction \kappa = P \restriction \kappa$ and $G_\kappa = G'_\kappa$. Hence we can lift j to $j_1 : V[G_\kappa] \to M[G_\kappa]$. It follows that $M[G_\kappa]_{\kappa^{++}} = V[G_\kappa]_{\kappa^{++}}$. Now by the definition of P, $j(P) \restriction j(\kappa) = j(P') \restriction j(\kappa) = P_\kappa * Q_\kappa * R$. Since G'^κ is generic over $V[G_\kappa]$, G'^κ is also generic over $M[G_\kappa]$, and thus we can lift j_1 to $j_2 : V[G_\kappa] \to M[G']$. Now we are in the position to construct a generic filter for R. The following fact, similar to Lemma 9 in [8], suggest that indeed we have only one possible choice.

Lemma 3.10. *For any dense open subset D of $j(P'_\kappa)$ in M, there is a condition $\bar{p} \in G_\kappa$ such that $j(\bar{p}) = p$ reduces D into $P'_{\kappa+1}$, in the sense that $\{q \in P_{\kappa+1} \mid q \cup p \restriction (\kappa, j(\kappa))$ meets $D\}$ is dense in $P_{\kappa+1}$ below $p \restriction \kappa + 1$.*

Proof. Let $g : \kappa \to V$ be such that $j(g)(\alpha) = D$ for some $\alpha < \kappa^{++}$. Without loss of generality, we can assume that for all $\beta < \kappa$, $g(\beta)$ is a dense subset of P'. Let C be a club of κ such that $j(C)(\kappa) > \alpha$ and for all $\beta < \kappa$, $C(\beta) < \beta^{++}$, where $C(\beta)$ is the least point in C greater than β.

For all inaccessible cardinal $\beta \in C$ and all P_κ condition r there is a $s \in P_{(\beta,\kappa)}$ such that for all $\gamma \in [\beta, C(\beta))$ the set $\{r' \in P_{\beta+1} \mid r' \cup s$ is in $g(\gamma)\}$ is dense below r. This can be done as there are fewer than β^{++} many such γ, $|P_{\beta+1}| = \beta^{++}$ and $P_{(\beta,\kappa)}$ is β^{+++}-closed.

We can then construct a decreasing sequence of condition $\langle p_i \mid i < \kappa\rangle$ starting with p_0 such that for all inaccessible $i \in C$, the above requirement is satisfied by q_i and $q_{i+1} \restriction (i, \kappa)$. The construction at successor steps is as in the last paragraph and at limit steps one uses Fact 3.1 to get greatest lower bounds. Let p' be the limit of the sequence.

Hence for any P'-generic filter G', there is a condition $\bar{p} \in G'$ such that whenever i is inaccessible and in C, for all $\gamma \in [i, C(i))$, the set $\{r' \in P_{i+1} \mid r' \cup \bar{p} \restriction (i, \kappa + 1)$ is in $g(\gamma)\}$ is dense below $p' \restriction \kappa$.

Since $\alpha \in [\kappa, C(\kappa))$, by elementarity, the set $\{r' \in P'_{\kappa+1} \mid r' \cup j(p') \restriction (\kappa, j(\kappa))$ is in $g(\alpha) = D\}$ is dense below $j(p') \restriction \kappa + 1$. □

It follows that $H = \{p \in j(P') \mid (\exists p_0 \in G_{<\kappa})(\exists p_1 \in G_\kappa)(j(p_0) \cup p_1 < p)\}$ is the unique $j(P')$-generic filter over $M[G_\kappa][G'_\kappa]$ compatible with $j[G_\kappa]$.

This allows us to lift j_2 to $j_3 : V[G_\kappa] \to M[H]$. Now we need to construct a $j(Q)_{j(\kappa)}$ generic filter g over $M[G' * H]$ in $V[G]$ such that $j[G_\kappa] \subset g$, which enable us to lift j_3 to $j^* : V[G] \to M[G' * H * g]$. Let $cl_\kappa(j[G_\kappa])$ be the set of all $j(Q)^{M[G'*H]}_{j(\kappa)}$-conditions p such that p is greater or equal to the great lower bound of a decreasing sequence of conditions belonging to $j[G_\kappa]$. It can be verified that each two elements of $cl_\kappa(j[G_\kappa])$ are compatible and $cl_\kappa(j[G_\kappa])$ is upward closed. We verify that $g = cl_\kappa(j[G_\kappa])$ is generic over $M[G' * H]$ in the next Lemma. This follows from the the fusion argument in the proof of Lemma 2.1 (also see the proof of Theorem 2.15 of [7] for a more general result). It also follows that g is the unique generic filter over $M[G' * H]$ containing $j[G^\kappa]$.

Lemma 3.11. *g is generic over $M[G' * H]$.*

Proof. Let D be an open dense in $M[G' * H]$. Suppose $f : \kappa \to V[G_\kappa]$ witnesses that $j^*(f)(\alpha) = D$ for some $\alpha \geq \kappa$. Without loss of generality, we can assume that $\alpha < \beta \to f(\alpha) \subseteq f(\beta)$. Suppose $C \subset \kappa$ is a club such that the least ordinal in $j(C)$ larger than κ is greater than α. Define $d : \kappa \to V[G_\kappa]$ by setting $d(\beta) = f(C(\beta))$, where $C(\beta)$ is the least element of C greater than β. Now $j(d)(\kappa) = j(f)(j(C)(\kappa)) \subset j(f)(\alpha) = D$. For any $p \in Q_\kappa$, we construct a $q < p$ such that the following holds:

For every $\beta < \kappa$, there is a $< \kappa$ size subset F_β of $supp(q)$ such that if $r < q$ is such that for all $\xi \in F_\beta$, $r \restriction \xi \Vdash r(\xi) < q|\sigma$, where σ satisfies $r \restriction \xi \Vdash \sigma \in Split_\beta(q(\xi))$, then $r \in d(\beta)$.

The construction is similar to the proof of Lemma 2.1 and we omit it here. We can assume that $q \in G_\kappa$. By construction, we can also fix a sequence $\vec{F} = \langle F_\eta \mid \eta < \kappa\rangle$ as witness such that $\vec{F}$ is continuous. Now in $M[G' * H]$, $j(q)$ is a condition such that:

For every $\beta < j(\kappa)$, if $r < j(q)$ is such that for all $\xi \in j(F)_\beta$, $r \restriction \xi \Vdash r(\xi) < j(q)|\sigma$, where σ satisfies $r \restriction \xi \Vdash \sigma \in Split_\beta(j(q)(\xi))$, then $r \in j(d)(\beta)$.

In particular, when $\beta = \kappa$, if $r < j(q)$ is such that for all $\xi \in j(F)_\kappa$, $r \restriction \xi \Vdash r(\xi) < j(q)|\sigma$, where σ satisfies $r \restriction \xi \Vdash \sigma \in Split_\kappa(j(q)(\xi))$, then $r \in j(d)(\kappa)$. Now choose a decreasing sequence of condition $\langle q_\beta \mid \beta < \kappa\rangle$ in G_κ such that for every $\beta < \kappa$, for all $\xi \in F_\beta$, $q_\beta \restriction \xi \Vdash r(\xi) < q|\sigma$, where σ satisfies $r \restriction \xi \Vdash \sigma \in Split_\beta(q(\xi))$. Consider the great lower bound q' of $\langle j(q_\beta) \mid \beta < \kappa\rangle$. If $\xi \in j(F)_\kappa$, then $\xi \in j(F)_\beta$ for some $\beta < \kappa$. Hence for all $\gamma > \beta$ and $\xi \in j(F)_\beta$, for all $q' \restriction \xi \Vdash q'(\xi) < j(q)|\sigma$, where σ satisfies $q' < j(q_\gamma) \restriction \xi \Vdash \sigma \in Split_\gamma(j(q)(\xi))$. It follows that $\sigma \in Split_\kappa(j(q)(\xi))$. Now for all $\xi \in j(F)_\kappa$, $q' \restriction \xi \Vdash q'(\xi) < j(q)|\sigma$, where $q' \restriction \xi \Vdash \sigma \in Split_\kappa(j(q)(\xi))$. Thus $q' \in j(d)(\kappa) \subset D$. □

Let $j_4 : V[G] \to M[G' * H * g]$ be the derived elementary embedding. Let g' be $M[G' * H]$-$j_3(Q'_\kappa)$ generic filter derived from g. Note it is clear that $j'_4 = j_4 \restriction V[G'] : V[G'] \to M[G' * H * g']$ is an elementary embedding which derives the normal measure $U_{G'}$.

Now we turn to the uniqueness of the lifting. It is clear that j_1 is determined by G_κ. For j_2, we use the following generalization of Lemma 3.4.

Lemma 3.12. *In $V[G_{\kappa+1}]$, G'^κ is the unique Q'_κ-generic over $M[G_\kappa]$.*

The proof is identical to the proof of Lemma 3.4 noting that G'^κ is uniquely determined by the stationarity pattern of $\langle S_{2*\beta} \mid \beta < \kappa^{++}\rangle$ and $M[G_\kappa]$, $V[G_\kappa]$ have the same $H(\kappa^{++})$.

By the discussion after the definition of H, it is clear that H and j_3 are uniquely defined. It follow that H is uniquely determined by G'. Since G' is uniquely derived from G', it follows that H is uniquely determined by G and hence j_3 is unique. Finally we deal with j_4.

Lemma 3.13. *g is uniquely determined by G^κ.*

Proof. We will argue by induction on $\gamma < j(\kappa^{++})$ that $g|\gamma$ is unique.

Case 1) $j(Q)^{j(\kappa)}_\gamma$ is trivial, nothing to show.

Case 2) $j(Q)^{j(\kappa)}_\gamma$ is Shooting Club forcing. It follows from the general argument for the fact that the lifting of a $< \kappa$-distributive forcing is unique. Let $d : \kappa \to \kappa^{++}$ and α represent γ. Now fix an $\epsilon < \kappa^{++}$. Let $h : \kappa \to \kappa^{++}$ and β represent ϵ. For any condition $p \in P^\kappa$, there is a $q < p$ such that for all $\delta < \delta' < \kappa$, $G^\kappa_{d(\delta)}$ decides $q(d(\delta))$ up to $f(\delta')$. By elementarity, $G^{j(\kappa)}_{j(d)(\alpha)} = G^{j(\kappa)}_\gamma$ decides $j(q)(j(d)(\alpha)) = j(q)(\gamma)$ up to $j(h)(\beta) = \epsilon$. Since p and ϵ is arbitrary, G^κ and $G^{j(\kappa)}_\gamma$ decides $G^{j(\alpha)}_{\gamma+1}$.

Case 3) $j(Q)^{j(\kappa)}_\gamma$ is $Sacks^*(\kappa)$ forcing. In [9] it is shown that if $j^* : V[G_\kappa] \to M[G]$ is the canonical extension of $j : V \to M$ and $Sacks(\kappa, \kappa^{++})$ denotes the κ-support product of κ^{++} copies of $Sacks(\kappa)$ then the range of j on the $Sacks(\kappa, \kappa^{++})$-generic specified by G_κ determines a unique $Sacks(j(\kappa))$-generic $S^*_0(i)$ for $i < j(\kappa^{++})$ not in the range of j and exactly two $Sacks(j(\kappa))$-generics $S^*_0(i), S^*_1(i)$ for i in the range of j. In [8], it is shown that if we replace $Sacks(\kappa, \kappa^{++})$ by $Sacks^*(\kappa, \kappa^{++})$, then the range of j on the $Sacks^*(\kappa, \kappa^{++})$-generic specified by G_κ determines a unique $Sacks^*(j(\kappa))$-generic $S^*_0(i)$ for all $i < j(\kappa^{++})$. A careful examination of the proof indicates that we can replace the product forcing $Sacks^*(\kappa, \kappa^{++})$ with the κ-support iteration of $Sacks^*(\kappa)$. We present a proof here for completeness.

Lemma 3.14. *In $M[H][g \restriction \gamma]$. Suppose T is the intersection of the trees $j(p)(\gamma)$, p in G^κ. Then T consists of exactly one cofinal branch b through $2^{<j(\kappa)}$.*

Proof. It is routine to verify that κ is the only ordinal which belongs to $j(C)$ for every $C \in V$ which is a club in κ. Let $h : \kappa \to \kappa^{++}$ and α represent γ. Let S be the range of h.

For any club subset C of κ and any condition p in Q_κ of $V[G'_\kappa]$ has an extension q such that for any $i \in S$,

$$\Vdash_{P^\kappa_i} s \in \mathrm{Split}(q(i)) \to \ \mathrm{length}(s) \in C \cap S^\kappa_\omega.$$

Choose such a q in G^κ. Then $j(q)$ has the property that for all $\beta < j(\kappa)$,

$$\Vdash_{P^\kappa_{h(\beta)}} s \in \mathrm{Split}(j(q)(j(h)(\beta))) \to \ \mathrm{length}(s) \in j(C) \cap S^{j(\kappa)}_\omega$$

In particular,

$$\Vdash_{j(P^\kappa_{h(\beta)})} s \in \mathrm{Split}(j(q)(\gamma)) \to \ \mathrm{length}(s) \in j(C) \cap S^{j(\kappa)}_\omega.$$

Now note that C can be chosen arbitrarily and the intersection of all such $j(C)$ with $S^{j(\kappa)}_\omega$ is $\emptyset$. It follows that the intersection T of the $j(p)(\gamma)$, $p \in G_\kappa$, is a subtree of $2^{<j(\kappa)}$ with has no splitting level and thus has a unique branch b. □

Hence, $g(\gamma)$ is uniquely determined by G^κ in the following sense. $t \in g(\gamma)$ if and only if the unique branch b goes through t. This completes Case 3) □

Now U_G is the unique normal measure derived from an embedding $j^* : V[G] \to M^*$ where j^* extends j, where $j : V \to M$ is derived from the last extender of the E sequence. Applying Lemma 3.6 to P, it follows that U_G is the unique normal measure in $V[G]$.

An easy generalization of the proof yields the following:

Theorem 3.3. *For all cardinals $\beta \leq \kappa^{++}$, it is consistent relative to a κ^{++}-hypermeasurable cardinal that κ is measurable, $2^\kappa = \kappa^{++}$ and there are exactly β many normal measures, each of which is definable with parameters over $H(\kappa^{++})$.*

4. Final remark

It is natural to ask whether we can generalize the result in this paper to a larger size of $H(\kappa^+)$.

Question 4.1. *Is it consistent to have ZFC + $2^\kappa = \kappa^{+++}$ + "there is a unique normal measure on κ which is definable in $H(\kappa^{++})$"?*

The current approach does not work as the iteration of Sacks forcing up to length κ^{+++} will automatically collapse κ^{++}. This question is also related to the general program of getting cardinal invariants on κ greater than κ^{++} together with κ being measurable.

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