Measurable cardinals and the Cofinality of the Symmetric Group

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Abstract

Assuming the existence of a hypermeasurable cardinal, we construct a model of Set Theory with a measurable cardinal κ such that $2^\kappa = \kappa^{++}$ and the group $Sym(\kappa)$ of all permutations of κ cannot be written as a union of a chain of proper subgroups of length $< \kappa^{++}$. The proof involves the iteration of a suitably defined uncountable version of the Miller forcing poset as well as the "tuning fork" argument introduced by the first author and K. Thompson in [9].

1 Introduction

A deep theorem of Macpherson and Neumann [13] states that if the symmetric group $Sym(\kappa)$ consisting of all permutations of a cardinal κ can be written as a union of an increasing chain $\langle G_i : i < \lambda\rangle$ of proper subgroups G_i, then $\lambda > \kappa$. Throughout this paper the minimal λ with this property will be denoted by $\mathrm{cf}(Sym(\kappa))$. It was proven in [18] that for $\kappa = \kappa^{<\kappa}$ the pair $(\mathrm{cf}(Sym(\kappa)), 2^\kappa)$ can be anything not obviously wrong. More precisely, for every regular $\lambda > \kappa$ and θ such that $\mathrm{cf}(\theta) \geq \lambda$, there exist a cardinal preserving forcing extension V^P such that $\mathrm{cf}(Sym(\kappa)) = \lambda$ and $2^\kappa = \theta$ in V^P. Moreover, for inaccessible κ we can assume [14, § 1] that P is κ-directed closed. Therefore if κ is supercompact, then it remains so in V^P. The main result of this paper states that consistency of $\mathrm{cf}(Sym(\kappa)) > \kappa^+$ at a measurable κ can be obtained assuming much less than supercompactness.

Theorem 1. *Suppose GCH holds and there exists an elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$ and $(H(\kappa^{++}))^V = (H(\kappa^{++}))^M$. Then there exists a forcing extension V' of V such that κ is still measurable in V' and $V' \vDash \mathrm{cf}(Sym(\kappa)) = \kappa^{++}$.*

To the best knowledge of the authors, $\mathrm{cf}(Sym(\kappa)) = \kappa^+$ for measurable κ in all other known models of Set Theory constructed under assumptions weaker than (a certain degree of) supercompactness; see Remark 1 for a more detailed discussion.

The idea of the proof of Theorem 1 resembles that of the consistency of $\mathfrak{u} < \mathrm{cf}(Sym(\omega))$ established in [20]. In particular, in section 2 we introduce a variant of Miller forcing and a (slightly more general than in [11]) variant of Sacks forcing at an inaccessible cardinal κ.

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According to Theorem 10, iterated forcing constructions where at each stage we take any of these forcing notions do not collapse κ^+. In section 3 we introduce a new cardinal characteristic $\mathfrak{g}_{cl}(\kappa)$, which is a version for κ of the classical groupwise density number $\mathfrak{g}$. Section 4 is devoted to the proof of the fact that suitably arranged iterated forcing constructions considered in section 2 of length κ^{++} make $\mathrm{cf}(Sym(\kappa))$ equal to κ^{++}. More precisely, the Miller forcing is responsible for $\mathrm{cf}^*(Sym(\kappa)) = \kappa^{++}$, while the Sacks forcing makes $\mathrm{cf}(Sym(\kappa))$ and $\mathrm{cf}^*(Sym(\kappa))$ equal. (Here $\mathrm{cf}^*(Sym(\kappa))$ is the minimal length of a special chain of proper subgroups of $Sym(\kappa)$ introduced in Definition 19.) And finally, in section 5 we show how to extend elementary embeddings to forcing extensions considered in section 2, and thus prove Theorem 1. The idea of the proof in section 5 can be traced back to the work [9], where the "tuning fork" argument was introduced.

2 A variant of Miller forcing for uncountable cardinals. Basic properties. Alternation with Sacks.

In this section we suggest one of the possible ways to generalize the Miller forcing introduced in [15] to uncountable cardinals and study some basic properties of iterated forcing constructions, where at each stage we take either the Miller or Sacks forcing poset. The discussion is patterned after Kanamori [11]. It is worth mentioning here that there are other generalizations of Miller forcing, see e.g. [19].

Throughout this section κ denotes a strongly inaccessible cardinal.

Definition 2. Let $p \subset \kappa^{<\kappa}$. For $s \in p$ we denote by $C(p, s)$ (or simply by $C(s)$ if p is clear from the context) the set $\{\alpha \in \kappa : s\hat{\ }\alpha \in p\}$.

Miller(κ) denotes the following forcing. A condition is a subset p of $\kappa^{<\kappa}$ such that

(i) $s \in p, t \subset s \longrightarrow t \in p$.

(ii) Each $s \in p$ is increasing and has a proper extension in p.

(iii) For every $\alpha < \kappa$ limit, $s \in \kappa^\alpha$, if $s \restriction \beta \in p$ for arbitrary large $\beta < \alpha$, then $s \in p$.

(iv) For every $s \in p$ there is $t \in p$ with $s \subset t$ which splits in p (i.e., $C(p, t)$ has more than one element).

(v) If $s \in p$ splits in p, then the set $C(p, s)$ is club.

(vi) If α is a limit ordinal, $s \in \kappa^\alpha$, and $s \restriction \beta$ splits in p for arbitrary large $\beta < \alpha$, then s splits in p and $C(s)$ is the intersection of $C(s \restriction \beta)$ for all β such that $s \restriction \beta$ splits in p.

We order Miller(κ) by declaring p to be stronger than q (and write $p \le q$) iff $p \subset q$. $\Box$

It is clear that Miller(κ) is κ-closed. For every subtree p of $\kappa^{<\kappa}$ we denote by Split(p) the family of all $s \in p$ which split in p. Given $s \in \kappa^{<\kappa}$, $\ell(s)$ denotes the length of s, i.e. the (unique) α such that $s \in \kappa^\alpha$. If $p \in$ Miller(κ) and $\alpha \in \kappa$, then we denote by $\mathrm{Split}_\alpha(p)$ the set

$$\{s \in p \ : o.t.(\{t \subsetneq s : t \in \mathrm{Split}(p)\}) \le \alpha, \ \forall t \subsetneq s(o.t.(s(\ell(t)) \cap C(p, t)) \le \alpha)\}.$$

In what follows we shall heavily apply *fusion argument* to Miller(κ) as well as to the Sacks forcing.

Definition 3. For $q \le p \in$ Miller(κ) and $\alpha \in \kappa$ the notation $q \le_\alpha p$ means that $\mathrm{Split}_\alpha(p) = \mathrm{Split}_\alpha(q)$. A sequence $\langle p_\alpha : \alpha \in \kappa\rangle$, where $p_\alpha \in$ Miller(κ), is called a *fusion sequence*, if

(i) If $\beta \le \alpha$, then $p_\alpha \le p_\beta$.

(*ii*) $p_{\alpha+1} \leq_\alpha p_\alpha$.
(*iii*) $p_\delta = \cap_{\alpha<\delta} p_\alpha$ for limit $\delta \in \kappa$. □

The following lemma is straightforward.

Lemma 4. *Let $\langle p_\alpha : \alpha \in \kappa\rangle$ be a fusion sequence. Then $q = \bigcap_{\alpha\in\kappa} p_\alpha \in \mathrm{Miller}(\kappa)$ and $q \leq_\alpha p_\alpha$ for all $\alpha \in \kappa$.*

Next, we recall the definition of the Sacks forcing for uncountable cardinals.

Definition 5. Let us fix a sequence $\vec{A} = \langle A_\alpha : \alpha < \kappa\rangle$ such that $|A_\alpha| < \kappa$ for all α. Let $\mathcal{T}$ be the set of all functions t which satisfy the following conditions.

(*i*) There exists α such that the domain of t equals α.

(*ii*) For all $\beta \in \mathrm{dom}(t)$, $t(\beta) \in A_\beta$.

$\mathrm{Sacks}(\vec{A})$ stands for the forcing whose conditions are subsets T of $\mathcal{T}$ such that:

(1) $s \in T, t \subset s \to t \in T$.

(2) Each t has a proper extension in T.

(3) If $t \in \mathcal{T}$ and the set of such β that $t \restriction \beta \in T$ is unbounded in $\ell(t)$, then $t \in T$.

(4) There exists a club $C(T)$ such that the set $\mathrm{succ}_T(t)$ of immediate successors of an element $t \in T$ with domain α coincides with $\{t\hat{\ }a : a \in A_\alpha\}$ provided $\alpha \in C(T)$, and $|\mathrm{succ}_T(t)| = 1$ otherwise.

Extension is defined by $S \leq T$ iff S is a subset of T. □

When each A_α is $\{0, 1\}$ we get the usual Sacks forcing considered in [6, 9, 11]. Some other sequences $\vec{A}$ are employed in [7]. Yet another sequence will be used in Section 4. But the basic properties (e.g. chain condition, fusion) of $\mathrm{Sacks}(\vec{A})$ does not really depend on $\vec{A}$.

Given any $T \in \mathrm{Sacks}(\vec{A})$ and $i \in \kappa$, we denote by $\mathrm{Split}_i(T)$ the set $\{t \in T : (\exists j \leq i)\ell(t) = \alpha_j\}$, where $\langle \alpha_i : i \in \kappa\rangle$ is the increasing enumeration of $C(T)$. Now the notions of $\leq_\alpha$ and of a fusion sequence can be introduced for $\mathrm{Sacks}(\vec{A})$ in the same way as for $\mathrm{Miller}(\kappa)$.

If γ is an ordinal and S^0, S^1 are disjoint subsets of γ such that $S^0 \bigcup S^1 = \gamma$, then we denote by $ST_{S^0,S^1,\vec{A}}$ the forcing poset $\mathbb{P}_\gamma$ from the iterated forcing construction $\langle \mathbb{P}_\xi, \dot{\mathbb{Q}}_\eta : \xi \leq \gamma, \eta < \gamma\rangle$ with supports of size $\leq \kappa$ defined as follows:

$$\{\eta < \gamma : \mathbb{P}_\eta \Vdash \dot{\mathbb{Q}}_\eta = \mathrm{Miller}(\kappa)\} = S^0 \text{ and } \{\eta < \gamma : \mathbb{P}_\eta \Vdash \dot{\mathbb{Q}}_\eta = \mathrm{Sacks}(\vec{A})\} = S^1.$$

Definition 6. Suppose that $\alpha \leq \kappa$ and $\langle p_\beta : \beta \in \alpha\rangle$ is a decreasing sequence of elements of $ST_{S^0,S^1,\vec{A}}$. The "meet" $q = \wedge_{\beta\in\alpha} p_\beta \in ST_{S^0,S^1,\vec{A}}$ is defined as follows: $\mathrm{supp}(q) = \bigcup_{\beta\in\alpha} \mathrm{supp}(p_\beta)$ and for every $\xi \in \mathrm{supp}(q)$, $q \restriction \xi \Vdash q(\xi) = \bigcap_{\beta\in\alpha} p_\beta(\xi)$. (Note that in case $\alpha = \kappa$ there could be no such q.) □

In order to prove that κ^+ is preserved by $ST_{S^0,S^1,\vec{A}}$ and κ^{++} is preserved for $\gamma = \kappa^{++}$ we need to employ a suitable variant of fusion.

Definition 7. Suppose that $\alpha \in \kappa$, $F \in [\gamma]^{<\kappa}$, and $q, p \in ST_{S^0,S^1,\vec{A}}$. $q \leq_{F,\alpha} p$ means that $q \leq p$ and $q \restriction \xi \Vdash q(\xi) \leq_\alpha p(\xi)$ for all $\xi \in F$.[1]

A sequence $\langle (p_\alpha, F_\alpha) : \alpha \in \kappa\rangle$ is a *generalized fusion sequence (for $ST_{S^0,S^1,\vec{A}}$)*, iff

(*i*) $|F_\alpha| < \kappa$ for all $\alpha \in \kappa$.

[1]The preorder $\leq_\alpha$ here depends on whether $\xi \in S^0$ or $\xi \in S^1$.

(*ii*) $F_\alpha \supset F_\beta$ for all $\beta \leq \alpha < \kappa$.
(*iii*) $p_{\alpha+1} \leq_{F_\alpha,\alpha} p_\alpha$ for all α.
(*iv*) If δ is limit, then $F_\delta = \bigcup_{\beta<\alpha} F_\beta$ and $p_\delta = \wedge_{\alpha<\delta} p_\alpha$.
(*v*) $\bigcup\{F_\alpha : \alpha \in \kappa\} = \bigcup\{\mathrm{supp}(p_\alpha) : \alpha < \kappa\}$. □

The easy but technical proof of the following lemma is left to the reader.

Lemma 8. *Let $\langle (p_\alpha, F_\alpha) : \alpha \in \kappa\rangle$ be a generalized fusion sequence for $ST_{S^0,S^1,\vec{A}}$. Then $q = \wedge_{\alpha<\kappa} p_\alpha \in ST_{S^0,S^1,\vec{A}}$ and $q \leq_{F_\alpha,\alpha} p_\alpha$ for all $\alpha \in \kappa$.*

There is no loss of generality to assume that each A_α is an element of κ.

Definition 9. Suppose that $p \in ST_{S^0,S^1,\vec{A}}$, $F \subset \mathrm{supp}(p)$ with $|F| < \kappa$, and $\sigma : F \to \kappa^{<\kappa}$. Then $p|\sigma$ is a function with the same domain as p such that $(p|\sigma)(\xi)$ equals $p(\xi)$ if $\xi \notin F$ and $p(\xi)_{\sigma(\xi)}$ otherwise, where for $q \in \mathrm{Miller}(\kappa) \bigcup \mathrm{Sacks}(\vec{A})$ and $t \in \kappa^{<\kappa}$ q_t denotes the set of all elements of q compatible with t. □

It is clear that $p|\sigma \in ST_{S^0,S^1,\vec{A}}$ if and only if for every $\xi \in F$ we have $(p|\sigma) \restriction \xi \Vdash_\xi \sigma(\xi) \in p(\xi)$. If $p|\sigma \in ST_{S^0,S^1,\vec{A}}$, then we say that σ *lies on* p.

Theorem 10. *For every ordinal γ and decomposition $\gamma = S^0 \sqcup S^1$ the forcing $ST_{S^0,S^1,\vec{A}}$ preserves cardinals $\leq \kappa^+$.*

Suppose that $2^\kappa = \kappa^+$ in V. If $\gamma < \kappa^{++}$, then there exists a dense subset $W_\gamma \subset ST_{S^0,S^1,\vec{A}}$ of size $|W_\gamma| \leq \kappa^+$. If $\gamma = \kappa^{++}$, then $ST_{S^0,S^1,\vec{A}}$ has the κ^{++}-chain condition.

Similar results were discussed in [11] and [6] for the Sacks forcing. Nevertheless, we give complete proofs here. Our exposition follows [11]. The first part of Theorem 10 follows from the lemma below. The second part could be proved independently from the first one, i.e., without using fusion.

Lemma 11. (1) *Assume that $p \in ST_{S^0,S^1,\vec{A}}$ and $p \Vdash \dot{z} \in V$. Then for every $F \in [\gamma]^{<\kappa}$ and $\alpha_0 \in \kappa$ there exists $q \leq_{F,\alpha_0} p$ and $x \in V$ with $|x| \leq \kappa$ such that $q \Vdash \dot{z} \in x$.*

(2) *Assume that $p \in ST_{S^0,S^1,\vec{A}}$ and $p \Vdash$ "$\dot{z} \in V$ and $|\dot{z}| \leq \kappa$". Then for every $F \in [\gamma]^{<\kappa}$ and $\alpha_0 \in \kappa$ there exists $q \leq_{F,\alpha_0} p$ and $x \in V$ with $|x| \leq \kappa$ such that $q \Vdash \dot{z} \subset x$.*

Proof. It is well-known how to obtain the second item from the first one, see [11, Theorem 2.3].

In order to prove the first item we shall inductively construct a generalized fusion sequence $\langle (p_\alpha, F_\alpha) : \alpha \in \kappa\rangle$ with $(p_\beta, F_\beta) = (p, F)$ for all $\beta \leq \alpha_0$, and $x \in V$ of size $|x| \leq \kappa$ such that $q = \wedge_{\alpha\in\kappa} p_\alpha$ and x are as required. The trivial description of how to construct F_α's is omitted. The limit step of the construction is obvious, so we concentrate on the successor case.

Let us enumerate as $\{\sigma_{\alpha,i} : i \in \eta\}$ all ground model functions $\sigma : F_\alpha \to \kappa^{\alpha+1}$ which lie on some $r \leq p_\alpha$ so that $r \restriction \xi \Vdash \sigma(\xi) \restriction \alpha \in \max \mathrm{Split}_\alpha(p_\alpha(\xi))$ for all $\xi \in F_\alpha$, and $\sigma(\xi)(\alpha) = \alpha$ for all $\xi \in F \cap S^0$. (Here $\eta < \kappa$ is a cardinal.) We shall construct a sequence $\langle p_{\alpha,i} : i \in \eta\rangle$ as follows. Set $p_{\alpha,-1} = p_\alpha$ and suppose that we have already constructed a decreasing sequence $\langle p_{\alpha,j} : j < i\rangle$ such that $p_{\alpha,j} \leq_{F_\alpha,\alpha} p_{\alpha,k}$ for all $k \leq j < i$. If i is limit, we set $p_{\alpha,i} = \wedge_{j<i} p_{\alpha,j}$. Suppose that $i = j + 1$. If there is no $r \leq p_{\alpha,j}$ such that $r = r|\sigma_{\alpha,j}$ and $r \restriction \xi \Vdash \sigma(\xi) \restriction \alpha \in \max \mathrm{Split}_\alpha(p_\alpha(\xi))$ for all $\xi \in F_\alpha$, we set $p_{\alpha,i} = p_{\alpha,j}$. And if there is such r, let $r_{\alpha,j} \leq r$ and $x_{\alpha,j} \in V$ be such that $r_{\alpha,j} \Vdash \dot{z} = x_{\alpha,j}$. Now, using the Maximal Principle we define $p_{\alpha,j+1}$ to be the amalgamation of $p_{\alpha,j}$ and $r_{\alpha,j}$ as in the proof of [11, Theorem 2.2]. More precisely,

(a) $\mathrm{supp}(p_{\alpha,j+1}) = \mathrm{supp}(r_{\alpha,j})$.

(b) If $\xi \in F_\alpha$, then $p_{\alpha,j+1}(\xi)$ is such that

$$r_{\alpha,j} \restriction \xi \Vdash p_{\alpha,j+1}(\xi) = (p_{\alpha,j}(\xi) \setminus p_{\alpha,j}(\xi)_{\sigma_{\alpha,j}(\xi)}) \bigcup r_{\alpha,j}(\xi),$$

and for any condition $c \leq p_{\alpha,j+1} \restriction \xi$ incompatible with $r_{\alpha,j} \restriction \xi$,

$$c \Vdash_\xi p_{\alpha,j+1}(\xi) = p_{\alpha,j}(\xi).$$

(c) if $\xi \notin F_\alpha$, then $p_{\alpha,j+1}(\xi)$ is such that

$$r_{\alpha,j} \restriction \xi \Vdash p_{\alpha,j+1}(\xi) = r_{\alpha,j}(\xi),$$

and for any condition $c \leq p_{\alpha,j+1} \restriction \xi$ incompatible with $r_{\alpha,j} \restriction \xi$,

$$c \Vdash_\xi p_{\alpha,j+1}(\xi) = p_{\alpha,j}(\xi).$$

Now we let $p_{\alpha+1} = \wedge_{i\in\eta} p_{\alpha,i}$. It follows that $p_{\alpha+1} \leq_{F_\alpha,\alpha} p_\alpha$. This completes our construction of $\langle (p_\alpha, F_\alpha) : \alpha \in \kappa \rangle$. Set $x = \{x_{\alpha,i}\}$.

Claim 12. *Suppose that $r \leq q$. Then there exists a sequence $\langle r_\alpha : \alpha \in \kappa \rangle$ of elements of $ST_{S^0,S^1,\vec{A}}$ with $r_0 = r$, a sequence $\langle \sigma_\alpha : F_\alpha \to \kappa^{<\kappa} | \alpha < \kappa \rangle$, and sequences $\langle \mu_{\alpha,\xi}, \nu_{\alpha,\xi} : \alpha \in \kappa, \xi \in F_\alpha \rangle$ of ordinals less than κ such that*

(i) If $\beta < \alpha$, then $r_\alpha \leq r_\beta$.

(ii) If $\xi \in F_\alpha$, then $\ell(\sigma_\alpha(\xi)) = \mu_{\alpha,\xi} + 1$.

(iii) If $\beta < \alpha$, then $\sigma_\beta(\xi) \subset \sigma_\alpha(\xi)$ for all $\xi \in F_\beta$.

(iv) For every $\xi \in F_{\alpha+1}$ we have $r_{\alpha+1} \restriction \xi \Vdash$ "$r_{\alpha+1}(\xi) = r_{\alpha+1}(\xi)_{\sigma_{\alpha+1}(\xi)}$, $\sigma_{\alpha+1}(\xi) \restriction \mu_{\alpha+1,\xi}$ splits in $r_\alpha(\xi)$, and $\sigma_{\alpha+1}(\xi) \restriction \mu_{\alpha+1,\xi} \in \max \mathrm{Split}_{\nu_{\alpha+1,\xi}}(q(\xi))$".

(v) If δ is limit, then

- *$\sigma_\delta(\xi) \restriction \mu_{\delta,\xi} = \bigcup_{\alpha<\delta} \sigma_\alpha(\xi)$;*
- *$\sigma_\delta(\xi)(\mu_{\delta,\xi}) = \sup\{\sigma_\alpha(\xi)(\mu_{\alpha,\xi}) : \alpha < \delta\}$ for all $\xi \in F_\delta \bigcap S^0$ (We assume that $\sigma_\alpha(\xi) = \emptyset$ for all $\xi \notin F_\alpha$);*
- *$\nu_{\delta,\xi} = \sup_{\alpha<\delta} \nu_{\alpha,\xi}$ for all $\xi \in F_\delta$;*
- *$r_\delta \restriction \xi \Vdash$ "$\sigma_\delta(\xi) \in r_\delta(\xi)$, $\sigma_\delta(\xi) \restriction \mu_{\delta,\xi}$ splits in $r_\delta(\xi)$, and $\sigma_\delta(\xi) \restriction \mu_{\delta,\xi} \in \max \mathrm{Split}_{\nu_{\delta,\xi}}(q(\xi))$" for all $\xi \in F_\delta$.*

Proof. The construction proceeds by induction. For limit δ we simply set $\sigma_\delta(\xi)$ and $\nu_{\delta,\xi}$ to be so as it is required in (v) and $r_\delta = \wedge_{\alpha<\delta} r_\alpha$. Thus $\mu_{\delta,\xi} = \sup_{\alpha<\delta} \mu_{\alpha,\xi}$. Let us fix any $\alpha < \delta$ and $\xi \in F_\alpha \cap S^0$. From the above it follows that $r_\delta \restriction \xi \Vdash$"$\sigma_\beta(\xi) \restriction \mu_{\beta,\xi}$ splits in $r_\alpha(\xi)$ for all $\alpha < \beta < \delta$", and hence $r_\delta \restriction \xi \Vdash$"$\sigma_\delta(\xi) \restriction \mu_{\delta,\xi}$ splits in $r_\alpha(\xi)$", and consequently $r_\delta \restriction \xi \Vdash$"$\sigma_\delta(\xi) \restriction \mu_{\delta,\xi}$ splits in $r_\delta(\xi) = \bigcap_{\beta<\delta} r_\beta(\xi)$". By the definition of Miller(κ), $r_\delta \restriction \xi \Vdash C(r_\alpha(\xi), \sigma_\delta(\xi) \restriction \mu_{\delta,\xi}) = \bigcap_{\alpha<\beta<\delta} C(r_\alpha(\xi), \sigma_\beta(\xi) \restriction \mu_{\beta,\xi})$, and hence $r_\delta \restriction \xi \Vdash \sigma_\delta(\xi)(\mu_{\delta,\xi}) = \sup_{\alpha<\beta<\xi} \sigma_\beta(\xi)(\mu_{\beta,\xi}) \in C(r_\alpha(\xi), \sigma_\delta(\xi) \restriction \mu_{\delta,\xi}))$, which implies that $r_\delta \restriction \xi \Vdash \sigma_\delta(\xi)(\mu_{\delta,\xi}) \in C(r_\delta(\xi), \sigma_\delta(\xi) \restriction \mu_{\delta,\xi}) = \bigcap_{\alpha<\xi} C(r_\alpha(\xi), \sigma_\delta(\xi) \restriction \mu_{\delta,\xi}))$, which gives us that $r_\delta \restriction \xi \Vdash \sigma_\delta(\xi) \in r_\delta(\xi)$. Finally, equalities $\sigma_\delta(\xi) \restriction \mu_{\delta,\xi} = \bigcup_{\alpha<\delta} \sigma_\alpha(\xi) \restriction \mu_{\alpha,\xi}$ and $\nu_{\delta,\xi} = \sup_{\alpha<\xi} \nu_{\alpha,\xi}$ combined with $r_\delta \restriction \xi \Vdash \sigma_\alpha(\xi) \restriction \mu_{\alpha,\xi} \in$

$\max \mathrm{Split}_{\nu_{\alpha,\xi}}(q(\xi))$ imply $r_\delta \upharpoonright \xi \Vdash \sigma_\delta(\xi) \upharpoonright \mu_{\delta,\xi} \in \max \mathrm{Split}_{\nu_{\delta,\xi}}(q(\xi))$, which completes the limit step.

At successor step $\alpha+1$ consider the increasing enumeration $\langle \xi_i : i < \eta \rangle$ of $F_{\alpha+1}$ and find a decreasing sequence $\langle u_i : i < \eta \rangle$ of elements of $ST_{S^0,S^1,\vec{A}}$ as follows: Set $u_i = \wedge_{j<i} u_j$ for limit i. Now given u_i, find $v \leq u_i \upharpoonright \xi_i$, $\pi \in \kappa^{\mu+1}$ for some $\mu \in \kappa$, and $\nu \in \kappa$ such that $\pi \supset \sigma_\alpha(\xi_i)$ if $\xi_i \in F_\alpha$ and

$$v \Vdash_{\xi_i} \pi \in r_\alpha(\xi_i),\ \pi \upharpoonright \mu \in \max \mathrm{Split}_\nu(q(\xi_i)) \bigcap \mathrm{Split}(r_\alpha(\xi_i)).$$

Then we set

$$u_{i+1} = v^\frown r_\alpha(\xi_i)_\pi{}^\frown r_\alpha \upharpoonright (\gamma \setminus (\xi_i + 1)),$$

$\sigma_{\alpha+1}(\xi_i) = \pi$. ($\mu_{\alpha+1,\xi_i}$ and $\nu_{\alpha+1,\xi_i}$ automatically become equal to μ and ν respectively.) With u_i's thus defined, we set $r_{\alpha+1} = \wedge_{i<\eta} u_i$. This completes the inductive construction, hence the proof of the claim. □

The following claim is obvious.

Claim 13. *There exists a club $C \subset \kappa$ such that $\mu_{\alpha,\xi} = \nu_{\alpha,\xi} = \alpha$ and $\sigma_\alpha(\xi)(\mu_{\alpha,\xi}) = \alpha$ for every $\alpha \in C$ and $\xi \in F_\alpha$. Consequently, $r_\alpha \upharpoonright \xi \Vdash \sigma_\alpha(\xi) \upharpoonright \alpha \in \max \mathrm{Split}_\alpha(q(\xi))$ for every such $\alpha \in C$ and $\xi \in F_\alpha$.*

We are in a position now to finish the proof of Lemma 11. Let C be such as in Claim 13 and $\alpha \in C$. Then $\sigma_\alpha = \sigma_{\alpha,i}$ for some $i < \eta$ (see the construction of $p_{\alpha+1}$ at the beginning of the proof of Lemma 11). Since $r_{\alpha+1} \leq q \leq p_{\alpha,i}$, Claim 12($iv$) implies that for every $\xi \in F_\alpha$ we have $r_{\alpha+1} \upharpoonright \xi \Vdash r_{\alpha+1}(\xi) = r_{\alpha+1}(\xi)_{\sigma_\alpha(\xi)}$. Therefore the construction of $p_{\alpha,i+1}$ is nontrivial. Since $r_{\alpha+1} \leq q \leq p_{\alpha,i+1}$, $r_{\alpha+1} = r_{\alpha+1}|\sigma_\alpha \leq p_{\alpha,i+1}|\sigma_{\alpha,i} = r_{\alpha,i}$, and hence $r_{\alpha+1} \Vdash \dot{z} = x_{\alpha,i}$. Therefore for every $r \leq q$ there exists $r' \leq r$ such that $r' \Vdash \dot{z} \in x$, which finishes our proof. □

Proof of Theorem 10. The proof is analogous to that of [11, Lemma 3.1]. Let W_γ be the set of those $q \in ST_{S^0,S^1,\vec{A}}$ such that:

(i) There is an increasing sequence $\langle F_\alpha : \alpha \in \kappa \rangle$ of subsets of γ such that $|F_\alpha| < \kappa$ for all α, $F_\delta = \bigcup_{\alpha\in\delta} F_\alpha$ for limit δ, and $\bigcup_{\alpha\in\kappa} F_\alpha = \mathrm{supp}(q)$.

(ii) For every α there exists a (possibly empty) collection Σ_α of ground model functions $\sigma : F_\alpha \to \kappa^{\alpha+1}$ of size $|\Sigma_\alpha| < \kappa$ such that $q|\sigma \in ST_{S^0,S^1,\vec{A}}$ for all $\sigma \in \bigcup_{\alpha\in\kappa} \Sigma_\alpha$, and whenever $\beta \in \kappa$ and $r \leq q$, there exists $\alpha > \beta$ and $\sigma \in \Sigma_\alpha$ so that r and $q|\sigma$ are compatible.

The proof of Lemma 11 gives that W_γ is dense in $ST_{S^0,S^1,\vec{A}}$. In addition, almost literal repetition of the proof of [11, Lemma 3.1] gives that if a pair of sequences

$$\langle \langle F_\alpha : \alpha \in \kappa \rangle, \langle \Sigma_\alpha : \alpha \in \kappa \rangle \rangle$$

is a witness for $q_i \in W_\gamma$, $i \in 2$, then $q_0 \leq q_1 \leq q_0$ in $ST_{S^0,S^1,\vec{A}}$. It suffices to note that there are at most κ^+-many such pairs.

Finally, the fact that $ST_{S^0,S^1,\vec{A}}$ has κ^{++}-chain condition provided $\gamma = \kappa^{++}$ is a direct consequence of [1, Theorem 2.2]. □

3 Miller(κ) and a variant of the groupwise density number

Throughout this section κ is strongly inaccessible, $2^\kappa = \kappa^+$ in V, $\kappa^{++} = S^0 \sqcup S^1$, $\vec{A} = \langle A_\alpha : \alpha \in \kappa \rangle$ is a sequence of ordinals below κ, and S^0 is κ^+-stationary. Here we define a (new?) cardinal characteristic of κ and show that iteration of Miller(κ) pushes it to κ^{++}.

Definition 14. We say that $\mathcal{G} \subset [\kappa]^\kappa$ is a *cgd-family* (abbreviated from club groupwise dense), if for every continuous increasing function $\phi : \kappa \to \kappa$ there exists a club $C \subset \kappa$ such that $\bigcup_{\alpha \in C} \phi(\alpha+1) \setminus \phi(\alpha) \in \mathcal{G}$, and for every $A \in \mathcal{G}$ and $B \in [\kappa]^\kappa$ such that $|B \setminus A| < \kappa$ we have $B \in \mathcal{G}$. In what follows the minimal size of a collection $\mathfrak{G}$ of cgd-families with empty intersection is denoted by $\mathfrak{g}_{cl}(\kappa)$. □

Theorem 15. *Suppose that G is a $ST_{S^0,S^1,\vec{A}}$-generic filter. Then $V[G] \models \mathfrak{g}_{cl}(\kappa) = \kappa^{++}$.*

The proof of Theorem 15 is divided into a sequence of lemmas.

Lemma 16. *Suppose that G is a $ST_{S^0,S^1,\vec{A}}$-generic filter. Then for every subset x of κ such that $x \in V[G]$ there exists $\gamma < \kappa^{++}$ such that $x \in V[G_\gamma]$, and the smallest such γ has cofinality $\leq \kappa$.*

Proof. Let $\dot{x}$ be a $ST_{S^0,S^1,\vec{A}}$-name of x. Note that the set $D \subset W_{\kappa^{++}}$ of all $q \in ST_{S^0,S^1,\vec{A}}$ such as in the proof of Theorem 10 with additional property that for every $\sigma \in \Sigma_\alpha$ the condition $q|\sigma$ decides $\dot{x}(\beta)$ for all $\beta < \alpha$, is dense in $ST_{S^0,S^1,\vec{A}}$. (Any q obtained along the lines of the proof of Lemma 11 with an extra requirement that $r_{\alpha,j}$ decides $\dot{x}(\beta)$ for all $\beta < \alpha$ belongs to D.) Item (ii) from the proof of Theorem 10 implies that $\{q|\sigma : \sigma \in \bigcup_{\alpha \in \kappa} \Sigma_\alpha\}$ is predense below q. Therefore for every $q \in D$ and $\beta \in \kappa$ there exists a subset $E_{q,\beta}$ predense below q of size $\leq \kappa$ and such that each element of $E_{q,\beta}$ decides $\dot{x}(\beta)$. From the above it follows that for every $q \in D$ we have $q \Vdash \dot{x} = \pi$, where $\pi = \{\langle \langle \beta, \check{i}_{\beta,r} \rangle, r \rangle : \beta \in \kappa, r \in E_{q,\beta}\}$ and $r \Vdash \dot{x}(\beta) = i_{\beta,r}$. The rest of the proof is straightforward. □

The following lemma resembles [3, Lemma 5.10].

Lemma 17. *Let G be a $ST_{S^0,S^1,\vec{A}}$-generic filter and $\mathcal{F} \in V[G]$ be a cgd-family. There is a κ^+-closed unbounded set of ordinals $\eta < \kappa^{++}$ for which $\mathcal{F} \cap V[G_\eta] \in V[G_\eta]$ and $\mathcal{F} \cap V[G_\eta]$ is cgd-family in $V[G_\eta]$.*

Proof. Let $\dot{\mathcal{F}}$ be a $ST_{S^0,S^1,\vec{A}}$-name for $\mathcal{F}$ and $p \in G$ be a condition which forces that $\dot{\mathcal{F}}$ is a cgd-family, and $\gamma < \kappa^{++}$ be such that $p \in \mathbb{P}_\gamma$. The proof of Lemma 16 yields a set Π_γ of $\mathbb{P}_\gamma$-names of size $|\Pi_\gamma| = \kappa^+$ such that for every $\mathbb{P}_\gamma$-generic filter H and $x \in \mathcal{P}(\kappa) \cap V[H]$ there exists $\pi \in \Pi_\gamma$ with the property $x = \pi^H$. For every $\pi \in \Pi_\gamma$ we denote by $B(\pi)$ a maximal antichain of conditions in $\mathbb{P}_{\kappa^{++}}$ that decide whether $\pi \in \dot{\mathcal{F}}$. Let η_1 be the supremum of the union of supports of all conditions appearing in some $B(\pi)$, $\pi \in \Pi_\gamma$. (Recall that $ST_{S^0,S^1,\vec{A}}$ has κ^{++}-c.c..) Then $\mathcal{F} \cap V[G_\gamma] \in V[G_{\eta_1}]$.

For every $\pi \in \Pi_\gamma$ we can find a maximal antichain $A(\pi)$ below p whose elements decide whether π is (the range of) a continuous increaing function, and if $q \in A(\pi)$ decides that π is such, then for some $\xi(\pi, q) > \gamma$ and $\theta_{\pi,q} \in \Pi_{\xi(\pi,q)}$, q forces $\theta_{\pi,q}$ to be a club and $\bigcup_{\alpha \in \theta_{\pi,q}} [\pi(\alpha), \pi(\alpha+1) \in \dot{\mathcal{F}}$. Let η_2 be the upper bound of the set

$$\{\theta_{\pi,q} : \pi \in \Pi_\gamma, q \in \bigcup_{\pi \in \Pi_\gamma} A(\pi)\} \bigcup \{\mathrm{supp}(q) : q \in \bigcup_{\pi \in \Pi_\gamma} A(\pi)\}.$$

Then $\eta(\gamma) := \max\{\eta_1, \eta_2\}$ has the properties $\dot{\mathcal{F}}^H \cap V[H_\gamma] \in V[H_{\eta(\gamma)}]$, and if $\psi \in V[H_\gamma]$ is any continuous increasing sequence, then there is a club $C \in V[H_{\eta(\gamma)}]$ so that $\bigcup_{\alpha \in C} [\psi(\alpha), \psi(\alpha+1)) \in \dot{\mathcal{F}}^H$, where H is any $ST_{S^0,S^1,\vec{A}}$-generic filter containing p.

Let $E \subset \kappa^{++}$ be the κ^+-closed unbounded set of those η that $\eta(\gamma) \leq \eta$ for all $\gamma < \eta$. A similar argument as in [3, Lemma 5.10] gives us that E is as required. □

Lemma 18. *For every $p \in \mathrm{Miller}(\kappa)$ there exists a continuous increasing sequence $\langle \nu_\alpha : \alpha \in \kappa \rangle$ such that for every club C there exists $q \leq p$ such that the range of every branch through q is almost (=modulo a subset of size $< \kappa$) contained in $\bigcup_{\alpha \in C}[\nu_\alpha, \nu_{\alpha+1})$.*

Proof. We inductively define a desired sequence $\langle \nu_\alpha \rangle_{\alpha \in \kappa}$. Choose ν_0 arbitrary. For limit $\delta \in \kappa$ we set $\nu_\delta = \sup_{\alpha \in \delta} \nu_\alpha$. After ν_α is defined, let $\beta > \nu_\alpha$ be such that for every $s \in p$ whose range is a subset of ν_α and $\xi \in [\nu_\alpha, \beta)$, if $s\hat{\ }\xi \in p$, then the range of the smallest extension $t \in \mathrm{Split}(p)$ of $s\hat{\ }\xi$ is contained in β. We set $\nu_{\alpha+1} = \beta$.

We claim that the sequence $\langle \nu_\alpha : \alpha \in \kappa \rangle$ is as required. Indeed, it is continuous by the construction. Let C and $D \subset \bigcup_{\alpha \in C}[\nu_\alpha, \nu_{\alpha+1})$ be clubs. (The role of D here is to ensure that the splitting nodes of the condition q constructed below split into clubs rather than into sets containing clubs. We could take, e.g., $D = \{\alpha \in C : \nu_\alpha = \alpha\}$.) Let q be the tree generated by the set of those $s = s_1\hat{\ }\xi \in p$ such that $s_1 \in \mathrm{Split}(p)$ and for every $t \leq s$, if $t \in \mathrm{Split}(p)$, then $s(\ell(t)) \in \bigcup_{\alpha \in C \setminus \mu(t)}[\nu_\alpha, \nu_{\alpha+1}) \bigcap D$, where $\mu(t)$ is the minimal ordinal μ such that ν_μ contains the range of t. Then $q \in \mathrm{Miller}(\kappa)$. It suffices to note that the range of each branch through q is a subset of $\bigcup_{\alpha \in C}[\nu_\alpha, \nu_{\alpha+1}) \bigcup \beta$, where β is the range of the stem (=smallest splitting element) of p. □

Proof of Theorem 15. The simple density argument based on Lemma 18 gives us that if G is a $\mathrm{Miller}(\kappa)$-generic filter and $\mathcal{F}$ is a cgd-family in V, then the range of $\bigcap G \in \kappa^\kappa$ is almost included into some $F \in \mathcal{F}$. Using Lemma 17, the proof can be completed in the same way as that of [2, Theorem 2]. $\square_{\text{Theorem 15.}}$

4 A new lower bound for the cofinality of symmetric group

In this section κ denotes a strongly inaccessible cardinal. The main result of this section says that for a certain sequence $\vec{A}$, if both S^0 and S^1 are κ^+-stationary and G is $ST_{S^0,S^1,\vec{A}}$-generic, then $V[G] \models \mathrm{cf}(Sym(\kappa)) = \kappa^{++}$. The motivation for this is given in section 5. We follow the strategy of the proof of [20, Theorem 2.2]. In its turn that proof relies upon the methods developed in [17, § 2].

Following [20] we give the following definition.

Definition 19. For a subset A of κ we shall identify the group $Sym(A)$ with the subgroup of $Sym(\kappa)$ consisting of permutations σ such that $\sigma \restriction (\kappa \setminus A) = \mathrm{id}_{\kappa \setminus A}$.

For every increasing $\psi \in \kappa^\kappa$ we denote by P_ψ the group $\prod_{\alpha \in \kappa} Sym(\psi(\alpha+1) \setminus \psi(\alpha))$, which will be identified with a subgroup of $Sym(\kappa)$. $\mathrm{cf}^*(Sym(\kappa))$ is the least cardinal λ such that it is possible to express $Sym(\kappa) = \bigcup_{i<\lambda} \Gamma_i$ as the union of a chain of proper subgroups such that for every increasing *continuous* $\phi \in \kappa^\kappa$ there exists $i \in \lambda$ such that P_ϕ is a subgroup of Γ_i.

For an increasing function $\theta : \kappa \to \kappa$ we set $\tilde{\theta}(\alpha) = \sup_{\xi \in \alpha} \theta(\xi)$ and $Q_\theta = P_{\tilde{\theta}}$. (Note that $\tilde{\theta}$ is continuous and $P_\theta \subset Q_\theta$.) □

The following lemma resembles [20, Theorem 2.6]. But the proofs of Lemma 20 and Theorem 2.6 from [20] are completely different.

Lemma 20. $\mathrm{cf}^*(Sym(\kappa)) \geq \mathfrak{g}_{cl}(\kappa)$.

Proof. The proof is divided into two steps.

Claim 21. *For every $\pi \in Sym(\kappa)$ there exists continuous increasing $\psi \in \kappa^\kappa$ such that $\pi \in P_\psi$.*

Proof. For any $\alpha \in \kappa$ we set $\beta(\alpha) = \min\{\pi(\xi) : \xi \geq \alpha\}$ and $\gamma(\alpha) = \sup\{\pi(\xi) : \xi \in \alpha\}$. Since π is a bijection, the Fodor's lemma implies that $\beta(\alpha) \geq \alpha$ for club many α's. Therefore there exists a club $C \subset \kappa$ such that $\gamma \restriction C = \mathrm{id}_C$ and $\beta(\alpha) \geq \alpha$ for all $\alpha \in C$. Now, the increasing bijective enumeration $\psi : \kappa \to C \cup \{0\}$ is as required. □

Given any $B \in [\kappa]^\kappa$, we denote by $e_B : \kappa \to B$ the increasing bijective enumeration of B. Note that continuous strictly increasing functions from κ to κ are exactly those of the form e_C for a club C.

Claim 22. *Let Γ be a subgroup of $Sym(\kappa)$ containing $Sym_0(\kappa) = \{\pi : \pi(\alpha) = \alpha$ for all but $< \kappa$ many α's$\}$ and such that $\langle \Gamma, g\rangle \neq Sym(\kappa)$ for all $g \in Sym(\kappa)$, and $\mathcal{G}_\Gamma = \{A \in [\kappa]^\kappa : \forall B\, (|B \setminus A| < \kappa \to Q_{e_B} \not\subset \Gamma)\}$. Then $\mathcal{G}_\Gamma$ is a cgd-family.*

Proof. Let $\phi : \kappa \to \kappa$ be a continuous increasing function. Since $Sym_0(\kappa) \subset \Gamma$, it is enough to show that there exists a club C such that, letting $C_\phi = \bigcup_{\alpha \in C}[\phi(\alpha), \phi(\alpha+1))$, we have $C_\phi \in \mathcal{G}_\Gamma$, which means $Q_{e_{C_\phi}} \not\subset \Gamma$. Assume to the contrary that $Q_{e_{C_\phi}} \subset \Gamma$ for every club $C \subset \kappa$. Set $O = \bigcup_{\alpha \text{ odd}}[\phi(\alpha), \phi(\alpha+1))$. We claim that $Sym(O) \subset \Gamma$. Once this is established, we get a contradiction with [13, Lemma 2.4]. Let us fix $\sigma \in Sym(O)$. Claim 21 yields a continuous increasing $\psi : \kappa \to \kappa$ such that $\sigma \in \prod_{\xi \in \kappa} Sym([e_O \circ \psi(\xi), e_O \circ \psi(\xi+1)) \bigcap O)$. Set $C = \{\alpha : \alpha$ is limit and $\phi(\alpha) = \sup_{\xi \in \alpha} e_O \circ \psi(\xi)\}$. It is clear that C is club. Since elements of C are limit ordinals, the choice of O ensures that $C_\phi \cap O = \emptyset$. We claim that

$$[\phi(\alpha), \phi(\alpha+1)) \bigcap [e_O \circ \psi(\xi), e_O \circ \psi(\xi+1)) = \emptyset \tag{1}$$

for every $\alpha \in C$ and $\xi \in \kappa$. Indeed, if $\xi < \alpha$, then $e_O \circ \psi(\xi+1) < \sup_{\eta<\alpha} e_O \circ \psi(\eta) = \phi(\alpha)$. Now suppose $\xi \geq \alpha$. Then $O \ni e_O \circ \psi(\xi) \geq \phi(\alpha)$, and therefore $[\phi(\alpha), \phi(\alpha+1)) \cap O = \emptyset$ implies $e_O \circ \psi(\xi) \geq \phi(\alpha+1)$, which proves (1). For any $\xi \in \kappa$ consider $\alpha(\xi), \beta(\xi) \in \kappa$ such that $\alpha(\xi) = \min\{\alpha \in C : \phi(\alpha) \geq e_O \circ \psi(\xi+1)\}$ and $\phi(\alpha(\xi))$ is the $\beta(\xi)$'s element of C_ϕ. Equation (1) gives

$$[e_O \circ \psi(\xi), e_O \circ \psi(\xi+1)) \subset [\sup_{\beta<\beta(\xi)} e_{C_\phi}(\beta), e_{C_\phi}(\beta(\xi))) = [\tilde{e}_{C_\phi}(\beta(\xi)), \tilde{e}_{C_\phi}(\beta(\xi)+1)),$$

and therefore

$$\prod_{\xi\in\kappa} Sym([e_O \circ \psi(\xi), e_O \circ \psi(\xi+1)) \bigcap O) \subset Q_{e_{C_\phi}} \subset \Gamma,$$

which implies $\sigma \in \Gamma$ and thus completes our proof. □

Let us express $Sym(\kappa) = \bigcup_{i<\lambda} \Gamma_i$ as a union of an increasing chain of proper subgroups such that each P_ψ is contained in some Γ_i. Since $|Sym_0(\kappa)| = \kappa$ and $\lambda > \kappa$, we can assume that $Sym_0(\kappa) \subset \Gamma_0$. For every $A \in [\kappa]^\kappa$ there exists $i \in \lambda$ such that $Q_{e_A} = P_{\tilde{e_A}} \subset \Gamma_i$, consequently $\bigcap_{i\in\lambda} \mathcal{G}_{G_i} = \emptyset$, and therefore $\mathfrak{g}_{cl}(\kappa) \leq \lambda$, which finishes our proof. $\square_{\text{Lemma 20}}$

Definition 23. Let us fix a continuous increasing function $\phi_0 : \kappa \to \kappa$ such that $\phi_0(\alpha+1) \geq \phi_0(\alpha) + \alpha$ for all $\alpha \in \kappa$. We set $N_\alpha = Sym(\phi_0(\alpha+1) \setminus \phi_0(\alpha))$, $\vec{N} = \langle N_\alpha : \alpha < \kappa\rangle$, and $ST_{S^0,S^1} \stackrel{\text{def}}{=} ST_{S^0,S^1,\vec{N}}$.

Each branch $\vec{t} = \langle t(\alpha)\rangle_{\alpha\in\kappa}$ of $T \in \mathrm{Sacks}(\vec{N})$ can be naturally identified with an element of $\sigma_{\vec{t}} \in P_{\phi_0}$ such that $\sigma_{\vec{t}} \upharpoonright (\phi(\alpha+1) \setminus \phi(\alpha)) = t(\alpha)$. We also need the following

Definition 24. $[\kappa]^{\kappa,\kappa}$ denotes the set $\{A \subset \kappa : |A| = |\kappa \setminus A| = \kappa\}$. If $A \in [\kappa]^{\kappa,\kappa}$ and $\sigma \in \mathit{Sym}(\kappa)$, then σ^A is defined by $\sigma^A(e_A(\alpha)) = e_A(\sigma(\alpha))$. If Γ is a subgroup of $\mathit{Sym}(\kappa)$, then $\Gamma^A = \{\sigma^A : \sigma \in \Gamma\}$ and $\Gamma(A) = \{\sigma \upharpoonright A : \sigma \in \Gamma, \sigma[A] = A\}$. □

The next lemma is of crucial importance for the proof of the equality $\mathrm{cf}^*(\mathit{Sym}(\kappa)) = \mathrm{cf}(\mathit{Sym}(\kappa))$ in $V^{ST_{S^0,S^1}}$ for κ^+-stationary subsets S^0, S^1 of κ^{++}.

Lemma 25. *Let $\psi : \kappa \to \kappa$ be a continuous increasing function. Then for every $T \in \mathrm{Sacks}(\vec{N})$ there exists $A \in [\kappa]^{\kappa,\kappa}$ such that for every $\pi \in P_\psi$ there exists $S \le T$ such that $\sigma_{\vec{s}} \upharpoonright A = \pi^A$ for all branches $\vec{s}$ of S.*

Proof. Consider $D \in [C(T)]^\kappa$ such that $C(T) \setminus D$ contains a club C' and $e_D(\alpha) \ge \psi(\alpha+1)$ for all α, and set $A = \bigcup_{\xi\in D}(\phi_0(\xi+1) \setminus \phi_0(\xi))$. By our choice of ϕ_0 we have $\phi_0(e_D(\alpha)+1) \ge \phi_0(e_D(\alpha)) + \psi(\alpha+1)$ for all α. A direct verification shows that $S \in \mathrm{Sacks}(\vec{N})$ such that $C(S) = C'$ and for every $\alpha \in \kappa$ and $s \in S$ we have

$$s(e_D(\alpha)) \upharpoonright [\phi_0 \circ e_D(\alpha), \phi_0 \circ e_D(\alpha) + \zeta(\alpha)) = h \circ \pi \upharpoonright (\psi(\alpha+1) \setminus \psi(\alpha)) \circ h^{-1},$$

where $\zeta(\alpha)$ is such that $\psi(\alpha+1) = \psi(\alpha) + \zeta(\alpha)$ and $h : \psi(\alpha+1) \setminus \psi(\alpha) \to [e_D(\alpha), e_D(\alpha) + \zeta(\alpha))$ is a monotone bijection, is as required. □

The following statement can be proven in the same way as [4, Lemma 2.7].

Lemma 26. *Suppose that $\lambda = \mathrm{cf}(\mathit{Sym}(\kappa)) < \mathrm{cf}^*(\mathit{Sym}(\kappa))$ and $\langle \Gamma_i : i \in \lambda\rangle$ is an increasing chain of proper subgroups of $\mathit{Sym}(\kappa)$ such that $\mathit{Sym}(\kappa) = \bigcup_{i<\lambda}\Gamma_i$. Then there exists a continuous increasing $\psi : \kappa \to \kappa$ such that $P_\psi^A \not\subset \Gamma_i(A)$ for all $i < \lambda$ and $A \in [\kappa]^{\kappa,\kappa}$.*

The next lemma can be proven by the same methods as Lemma 16.

Lemma 27. *Suppose that $2^\kappa = \kappa^+$ in V, $\kappa^{++} = S^0 \sqcup S^1$ is a decomposition into two κ^+-stationary subsets, and G is ST_{S^0,S^1}-generic filter. For every $\Pi \subset \mathit{Sym}(\kappa)$ of size $|\Pi| \le \kappa^+$ and every sequence $\langle \Gamma_i : i < \kappa^+\rangle \in V[G]$ of subgroups of $\mathit{Sym}(\kappa)$ there is a κ^+-closed unbounded set of ordinals $\eta < \kappa^{++}$ for which $\Pi \in V[G_\eta]$, $\langle \Gamma_i \cap V[G_\eta] : i < \kappa^+\rangle \in V[G_\eta]$, and for every $A \in [\kappa]^{\kappa,\kappa} \cap V[G_\eta]$ and $i < \kappa^+$ we have $\Gamma_i(A) \cap V[G_\eta] = (\Gamma_i \cap V[G_\eta])(A)$.*

Finally, we are in a position to prove the following theorem, which is the main result of this section.

Theorem 28. *Let S^0, S^1, and G be as in Lemma 27. Then $V[G] \vDash \mathrm{cf}(\mathit{Sym}(\kappa)) = \kappa^{++}$.*

Proof. Suppose to the contrary that $V[G] \vDash \mathit{Sym}(\kappa) = \kappa^+$. Let $\langle \Gamma_i : i < \kappa^+\rangle \in V[G]$ be an increasing chain of subgroups of $\mathit{Sym}(\kappa)$ such that $\mathit{Sym}(\kappa) = \bigcup_{i<\kappa^+}\Gamma_i$. By Theorem 15 and Lemma 20 we have $V[G] \vDash \mathrm{cf}^*(\mathit{Sym}(\kappa)) = \kappa^{++}$. Lemma 26 yields a continuous increasing $\psi : \kappa \to \kappa$ such that for every $A \in [\kappa]^{\kappa,\kappa}$ and $i < \kappa^+$ we have $P_\psi^A \not\subset \Gamma_i(A)$. Fix $A_* \in [\kappa]^{\kappa,\kappa}$ and for every $i < \kappa^+$ find $\pi_i \in P_\psi$ such that $\pi_i^{A_*} \in P_\psi^{A_*} \setminus \Gamma_i(A_*)$. Observe that $\Pi^A \not\subset \Gamma_i(A)$ for any $A \in [\kappa]^{\kappa,\kappa}$ and $i < \kappa^+$, where $\Pi = \{\pi_i : i < \kappa^+\}$. (The condition $\pi_i^A \notin \Gamma_i(A)$ holds at least starting from i such that Γ_i contains an extension of the order-preserving bijection between A_* and A.)

Let $\eta < \kappa^{++}$ be such an element of κ^+-closed unbounded subset provided by Lemma 27 for $\langle \Gamma_i : i < \kappa^+ \rangle$ and Π for which $\dot{\mathbb{Q}}_\eta = \mathrm{Sacks}(\vec{N})$, i.e. $\eta \in S^1$. We can additionally require $A_* \in V[G_\eta]$. Suppose that H is the $\mathrm{Sacks}(\vec{N})$-generic filter over $V[G_\eta]$ such that $G_{\eta+1} = G_\eta * H$ and $\vec{h}$ is the common branch of all trees in H. Applying Lemma 25 we conclude that the set

$$\{S \in \mathrm{Sacks}(\vec{N}) : \exists A \in [\kappa]^{\kappa,\kappa} \cap V[G_\eta]\, \exists \pi \in \Pi\, (\pi^A \notin (\Gamma_i \cap V[G_\eta])(A) \ \wedge\ S \Vdash \sigma_{\vec{h}} \upharpoonright A = \pi^A)\}$$

is dense for all $i < \kappa^+$. Therefore for every i there exists $A_i \in [\kappa]^{\kappa,\kappa} \cap V[G_\eta]$ and $j(i) < \kappa^+$ such that $\sigma = \sigma_{\vec{h}} \upharpoonright A_i = \pi^{A_i}_{j(i)} \notin (\Gamma_i \cap V[G_\eta])(A_i)$. Let $i < \kappa^+$ be such that $\sigma \in \Gamma_i$. Then

$$(\Gamma_i \cap V[G_\eta])(A_i) \not\ni \pi^{A_i}_{j(i)} = \sigma \upharpoonright A_i \in \Gamma_i(A_i) \cap V[G_\eta],$$

which contradicts our choice of η. $\square$

Now it is naturally to ask whether we needed to employ $\mathrm{Sacks}(\vec{N})$ at all.

Question 29. *Is* $\mathrm{cf}(Sym(\kappa)) \geq \mathfrak{g}_{cl}(\kappa)$*?*

The cardinal characteristic $\mathfrak{g}_{cl}(\kappa)$ seems to be a natural generalization of the classical groupwise density number $\mathfrak{g}$ introduced in [2] and it was proved in [4] that $\mathrm{cf}(Sym(\omega)) \geq \mathfrak{g}$. But the methods of [4] do not seem to be applicable to Question 29.

5 Proof of Theorem 1

Without loss of generality, j is given by a hyperextender ultrapower so that $M = \{j(f)(a) : f \in V, f : H(\kappa) \to V, \text{ and } a \in H(\kappa^{++})\}$.

Claim 30. *There exists a cardinal preserving forcing extension V' of V such that GCH holds in V' and j can be extended to an elementary embedding $j' : V' \to M'$ satisfying the following conditions:*

(i) $H(\kappa^{++})^{V'} = H(\kappa^{++})^{M'}$;

(ii) j' is given by a hyperextender ultrapower so that $M' = \{j(f)(a) : f \in V', f : H(\kappa)^{V'} \to V', \text{ and } a \in H(\kappa^{++})^{V'}\}$;[2]

(iii) There exist disjoint κ^+-stationary in V (and hence in M) subsets $S^0, S^1 \in M$ of κ^{++} such that $S^0 \cup S^1 = \kappa^{++}$, and a sequence $\langle (S^0_k, S^1_k) : k \in \kappa \rangle$, where S^0_k and S^1_k are disjoint ρ_k^+-stationary subsets of ρ_k^{++} for which $\rho_k^{++} = S^0_k \cup S^1_k$, such that $j\langle (S^0_k, S^1_k) : k \in \kappa \rangle(\kappa) = (S^0, S^1)$. (Here ρ_k denotes the k-th inaccessible cardinal below κ, $k < \kappa$.)

Proof. We define a forcing poset $\mathbb{R}$ as follows. Let $\mathbb{R}_0 = \{\mathbf{1}_0\}$. For $k \leq \kappa$ we denote by $\dot{\mathbb{S}}_k$ a $\mathbb{R}_k$-name for the poset $Fn(\rho_k^{++}, 2, \rho_k^{++})$ adding one Cohen subset to ρ_k^{++}, see [12]. Proceeding this way along all inaccessible cardinals $\leq \kappa$ and using reverse Easton supports we define $\mathbb{R}$. Let G be a $\mathbb{R}_\kappa$-generic over V, g be a $\mathbb{S}_\kappa = \dot{\mathbb{S}}^G_\kappa$-generic over $V[G]$, $G_k = G \cap \mathbb{R}_k$ and g_k be such that $G_{k+1} = G_k * g_k$ for all $k < \kappa$. Note that g_k is the characteristic function of some subset of S^0_k of ρ^{++}. It is clear that S^0_k as well as its complement meet all subsets of ρ_k^{++} of size ρ_k^{++} which appear in $V[G_k]$. Since $\mathbb{R}_{k+1}$ has ρ_k^{++}-c.c., each ρ_k^+-closed unbounded in ρ_k^{++} subset $C' \in V[G_{k+1}]$ contains a ρ_k^+-closed unbounded in ρ_k^{++} subset $C \in V$ (the proof of [10, Lemma 22.25] works in this case as well), and hence S^0_k as well as $\rho_k^{++} \setminus S^0_k$ are ρ_k^+-stationary

[2] We could assume here that the domain of f is still $H(\kappa)^V$ and $a \in H(\kappa^{++})^V$, but this is irrelevant.

subsets of ρ_k^{++} in $V[G_{k+1}]$. The rest of our forcing is ρ_k^{+++}-closed, and hence S_k^0 and $\rho_k^{++} \setminus S_k^0$ remain ρ_k^+-stationary in $V[G*g]$. Let S^0 be such that g is the characteristic function of S^0 and $S^1 = \kappa^{++} \setminus S^0$. Again, S^0 and S^1 are κ^+-stationary subsets of κ^{++} in $V[G*g]$.

$j(\mathbb{R})$ is the iteration with reverse Easton supports of length $j(\kappa)+1$. A standard argument gives us that $j(\mathbb{R})_\kappa = \mathbb{R}_\kappa$, and hence G is $j(\mathbb{R})_\kappa$-generic over M and $(H(\kappa)^{++})^{V[G]} = (H(\kappa)^{++})^{M[G]}$, see [6, Lemma 4.4]. From the above it follows that $Fn(\kappa^{++},2,\kappa^{++})^{V[G]} = Fn(\kappa^{++},2,\kappa^{++})^{M[G]}$, and therefore $\mathbb{R} = j(\mathbb{R})_{\kappa+1}$ and g is $Fn(\kappa^{++},2,\kappa^{++})$-generic over M as well.

Suppose that there exists a $j(\mathbb{R})$-generic filter $G' = G*g*H*h \in V[G*g]$ over M such that H is a $j(\mathbb{R})_{\kappa,j(\kappa)}$-generic over $M[G*g]$, h is $\mathbb{S}_{j(\kappa)} = j(\dot{\mathbb{S}}_\kappa)^{G*g*H}$-generic over $M[G*g*H]$, and $j[G*g] \subset G'$. Then j can be extended to an elementary embedding $j' : V[G*g] \to M[G']$ such that $j'(G*g) = G'$, see [5, Proposition 9.1]. Therefore $j'\langle S_k^0 : k \in \kappa\rangle(\kappa) = S^0$. In addition, conditions (i) and (ii) hold by [5, Proposition 9.3]. Thus j', $V' = V[G*g]$, and $M' = M[H]$ are as required.

It suffices to note that such H and h exist: the construction of H is standard, see, e.g., fourth, fifth and sixth paragraphs of the proof of [6, Theorem 4.2]; the existence of h follows from the κ^+-distributivity of $\mathbb{Q}_\kappa$ by virtue of [5, Proposition 15.1], which implies that the subfilter h of $\mathbb{S}_{j(\kappa)}$ generated by $j[g]$ is as required. □

There is no loss of generality in assuming $j = j'$, $V = V'$, and $M = M'$. We define a forcing poset $\mathbb{P}$ as follows. Let $\mathbb{P}_0 = \{\mathbf{1}_0\}$. For $k \leq \kappa$ we denote by $\dot{\mathbb{Q}}_k$ a $\mathbb{P}_k$-name for $ST_{S_k^0,S_k^1}$[3]. Proceeding this way along all inaccessible cardinals $\leq \kappa$ and using reverse Easton supports we define $\mathbb{P}$. Observe that $\mathbb{P}_k$ has ρ_k^+-c.c., and hence S_k^0, S_k^1 are still ρ_k^+-stationary in $V^{\mathbb{P}_k}$. From the above and Theorem 28 we have that $V^{\mathbb{P}} \vDash \mathrm{cf}(Sym(\kappa)) = \kappa^{++}$. Thus it suffices to prove that κ is measurable in $V^{\mathbb{P}}$. In order to do this we shall extend j to an elementary embedding from $V^{\mathbb{P}}$ into $M^{j(\mathbb{P})}$.

$j(\mathbb{P})$ is an iteration of length $j(\kappa)+1$ in M with reverse Easton support. It is clear that $j(\mathbb{P})_\kappa = \mathbb{P}_\kappa$. Let G be a $\mathbb{P}_\kappa$-generic filter over V. Since M and V have the same $H(\kappa^{++})$ and $j\langle (S_k^0, S_k^1) : k \in \kappa\rangle(\kappa) = (S^0, S^1)$, we have $(\kappa^{++})^{M[G]} = (\kappa^{++})^{V[G]}$ (see [6, Lemma 4.4]) and $j(\mathbb{P})_{\kappa+1} = \mathbb{P}$. Note that $j(\mathbb{P}) = j(\mathbb{P}_\kappa) * j(\dot{\mathbb{Q}}_\kappa)$. Let g be generic for $\dot{\mathbb{Q}}_\kappa^G$ over $V[G]$. We need to find a suitable $j(\mathbb{P})$-generic filter over M in order to lift j to $V[G*g]$. The following claim is analogous to [1, Lemma 6.4].

Claim 31. *If $x \subset M[G]$ (resp. $x \subset M[G*g]$), $x \in V[G]$ (resp. $x \in V[G*g]$), and $V[G] \vDash |x| \leq \kappa$ (resp. $V[G*g] \vDash |x| \leq \kappa$), then $x \in M[G]$ (resp. $x \in M[G*g]$).*

Proof. We present the proof of the $G*g$ part only. The other part is even simpler. Without loss of generality, x is a set of ordinals. Let $\dot{x}$ be a $\mathbb{P}$-name such that $\dot{x}^{G*g} = x$. The κ-c.c. of $\mathbb{P}_\kappa$ and Lemma 11(2) yield a set of ordinals $y \in V$ of size $|y| \leq \kappa$ in V and such that $\mathbb{P} \Vdash \dot{x} \subset y$. For every $\alpha \in y$ there exists a maximal antichain A_α of conditions $p \in \mathbb{P}$ such that $p \Vdash \alpha \in \dot{x}$ for every $p \in A_\alpha$. Applying Theorem 10, we conclude that $|A_\alpha| \leq \kappa^+$ for every $\alpha \in y$. It is clear that $\langle A_\alpha : \alpha \in y\rangle \in H(\kappa^{++})$, and hence $\langle A_\alpha : \alpha \in y\rangle \in (H(\kappa^{++}))^M$. It suffices to note that $x = \{\alpha \in D : G*g \cap A_\alpha \neq \emptyset\}$. □

In the same way as in the proof of [6, Theorem 4.2] (using Claim 31 instead of [9, Lemma 3]) we can find a $j(\mathbb{P}) \restriction (\kappa, j(\kappa))$-generic filter $H \in V[G*g]$ over $M[G*g]$. Thus $j[G] = G \subset G*g*H$,

[3] Here $S_\kappa^0 = S^0$ and $S_\kappa^1 = S^1$. Note that $ST_{S_k^0,S_k^1}$ is an iteration again, which will be denoted $\langle \mathbb{P}_\eta^k, \dot{\mathbb{Q}}_\xi^k : \eta \leq \rho_k^{++}, \xi < \rho_k^{++}\rangle$.

and hence j lifts to an embedding $j^* : V[G] \to M[G * g * H]$ definable in $V[G * g]$, see [5, Proposition 9.1]. Let M^* denote $M[G * g * H]$.

We give Definition 32 and Claim 33 in full generality for any iteration of Miller and Sacks forcings.

Definition 32. Let ρ be a strongly inaccessible cardinal and γ be an ordinal, S^0, S^1 be disjoint sets such that $S^0 \cup S^1 = \gamma$, and $\vec{A} = \langle A_\alpha : \alpha < \rho \rangle$ be a sequence of elements of ρ. Suppose that $\langle (p_\alpha, F_\alpha) : \alpha \in \rho \rangle$ is a generalized fusion sequence for $ST_{S^0,S^1,\vec{A}}$, $q = \wedge_{\alpha<\rho} p_\alpha$, and $i \in \rho$. We say that a function $\sigma : F \to \rho^{i+1}$ *is i-properly situated on q (with respect to the fusion sequence $\langle (p_\alpha, F_\alpha) : \alpha \in \rho \rangle$)*, if $F_i \subset F$, σ lies on some $r \leq q$ such that $r \restriction \xi \Vdash \sigma(\xi) \restriction i \in \max \mathrm{Split}_i(q(\xi))$ for all $\xi \in F$, and $\sigma(\xi)(i) = i$ for all $\xi \in F \cap S^0$.

Claim 33. *Let ρ, S^0, S^1, $\vec{A}$, $\langle (p_\alpha, F_\alpha) : \alpha \in \rho \rangle$, q, i be such as in Definition 32, $u \leq q$, $F, T \in [\gamma]^{<\rho}$ with $F \subset T$, and $C \subset \rho$ be a club. Then there exists $v \leq_{F,i} u$ satisfying the following conditions:*

For every $\sigma : F \to \rho^{i+1}$ which lies on v and has the property $\sigma(i) = i$ for all $i \in F \cap S^0$, ***there exist*** *$j \in C$ and $\pi : T \cup F_j \to \rho^{(j+1)}$ such that $\pi(\xi) \restriction (i+1) = \sigma(\xi)$ for all $\xi \in F$, π lies on v, $v|\sigma = v|\pi$, and $v|\pi$ is a witness for π being j-properly situated on q with respect to $\langle (p_\alpha, F_\alpha) : \alpha \in \rho \rangle$.*

Proof. Let us enumerate as $\{\sigma^\zeta : \zeta \in \eta\}$ all $\sigma : F \to \rho^{i+1}$ with the property $\sigma(\xi)(i) = i$ for all $\xi \in F \cap S^0$ and which lie on some $r \leq u$. Set $u_0 = u$ and suppose that for some $\zeta < \eta$ and all $\zeta' < \zeta$ we have already defined $u_{\zeta'} \in ST_{S^0,S^1,\vec{A}}$ such that $u_{\zeta'} \leq_{F,i} u_{\zeta''}$ for all $\zeta'' \leq \zeta' < \zeta$. If ζ is limit, we set $u_\zeta = \wedge_{\zeta' \in \zeta} u_{\zeta'}$.

Let us consider the case $\zeta = \zeta' + 1$. If there is no $r \leq u_{\zeta'}$ such that $\sigma^{\zeta'}$ lies on $r = r|\sigma^{\zeta'}$, then we set $u_\zeta = u_{\zeta'}$. Otherwise set $r_0^{\zeta'} = r$, $\sigma_0^{\zeta'} = \sigma^{\zeta'}$, and $F_\alpha^{\zeta'} = F_\alpha \bigcup T$. Repeating the same argument as in Claim 12, we can construct a sequence $\langle r_\alpha^{\xi'} : \alpha \in \rho \rangle$ of elements of $ST_{S^0,S^1,\vec{A}}$, a sequence $\langle \sigma_\alpha^{\zeta'} : F_\alpha^{\zeta'} \to \rho^{<\rho} | \alpha < \rho \rangle$, and sequences $\langle \mu_{\alpha,\xi}^{\zeta'}, \nu_{\alpha,\xi}^{\zeta'} : \alpha \in \rho, \xi \in F_\alpha^{\zeta'} \rangle$ of ordinals less than ρ fulfilling the items $(i)-(v)$ of Claim 12. Claim 13 yields a club $C^{\zeta'} \subset \rho$ such that $\mu_{\alpha,\xi}^{\zeta'} = \nu_{\alpha,\xi}^{\zeta'} = \alpha$ and $\sigma_\alpha^{\zeta'}(\xi)(\mu_{\alpha,\xi}^{\zeta'}) = \alpha$ for every $\alpha \in C^{\zeta'}$ and $\xi \in F_\alpha^{\zeta'} \cap S^0$. Let us fix $j^{\zeta'} \in C^{\zeta'} \cap C$ and set $\pi^{\zeta'} = \sigma_{j^{\zeta'}}^{\zeta'}$ and $r^{\zeta'} = r_{j^{\zeta'}+1}^{\zeta'}$. By Claim 12$(iii),(iv)$ we have $r^{\zeta'}|\pi^{\zeta'} = r^{\zeta'}$ and $r^{\zeta'}$ is a witness for $\pi^{\zeta'}$ being $j^{\zeta'}$-properly situated on q. Now let u_ζ be the amalgamation of $u_{\zeta'}$ and $r^{\zeta'}$ defined as follows:

(a) $\mathrm{supp}(u_\zeta) = \mathrm{supp}(r^{\zeta'})$.

(b) If $\xi \in F$, then $u_\zeta(\xi)$ is such that

$$r^{\zeta'} \restriction \xi \Vdash u_\zeta(\xi) = (u_{\zeta'}(\xi) \setminus u_{\zeta'}(\xi)_{\sigma^{\zeta'}(\xi)}) \bigcup r^{\zeta'}(\xi),$$

and for any condition $c \leq u_\zeta \restriction \xi$ incompatible with $r^{\zeta'} \restriction \xi$, $c \Vdash_\xi u_\zeta(\xi) = u_{\zeta'}(\xi)$.

(c) if $\xi \notin F$, then $u_\zeta(\xi)$ is such that $r^{\zeta'} \restriction \xi \Vdash u_\zeta(\xi) = r^{\zeta'}(\xi)$, and for any condition $c \leq u_\zeta \restriction \xi$ incompatible with $r^{\zeta'} \restriction \xi$, $c \Vdash_\xi u_\zeta(\xi) = u_{\zeta'}(\xi)$.

By the definition of u_ζ we have

$$u_\zeta|\sigma^{\zeta'} = r^{\zeta'} = r^{\zeta'}|\pi^{\zeta'} = (u_\zeta|\sigma^{\zeta'})|\pi^{\zeta'} = u_\zeta|\pi^{\zeta'}$$

and $u_\zeta \leq_{F,i} u_{\zeta'}$.

We claim that $v = \wedge_{\zeta<\eta} u_\zeta$ is as required. Indeed, let $\sigma : F \to \rho^{i+1}$ be such as in the formulation. Since $v \leq u$, $\sigma = \sigma^\zeta$ for some $\zeta \in \eta$ and the construction of $u_{\zeta+1}$ is nontrivial.

From the above it follows that $v|\sigma \leq u_\zeta|\sigma^\zeta = u_\zeta|\pi^\zeta$, consequently π^ζ lies on v and $v|\sigma = v|\pi^\zeta \leq u_\zeta|\pi^\zeta = r^\zeta$. Now it is easy to see that $j = j^\zeta$ and $\pi = \pi^\zeta$ are a as required. $\square$

Claim 34. *Let ρ, S^0, S^1, and $\vec{A}$ be such as in Definition 32, and $p \in ST_{S^0,S^1,\vec{A}}$. Then for every sequence $\langle D_\alpha : \alpha \in \rho\rangle$ of open dense subsets of $ST_{S^0,S^1,\vec{A}}$ there exists a generalized fusion sequence $\langle (p_\alpha, F_\alpha) : \alpha \in \rho\rangle$ with $p_0 = p$ and such that, letting $q = \wedge_{\alpha\in\rho} p_\alpha$, for every limit $i \in \rho$ and $\sigma : F_i \to \rho^{i+1}$ which is i-properly situated on q, σ lies on q and $q|\sigma \in D_i$.*

Proof. Take $r_{\alpha,j} \in D_\alpha$ in the construction of a fusion sequence from the proof of Lemma 11 (the part before Claim 12) instead of demanding that $r_{\alpha,j}$ decides $\dot{z}$ as a ground model object. The resulting fusion sequence is easily seen to be as required. $\square$

Let us come back to our main task, namely to extend j^* to an elementary embedding $j^{**} : V[G * g] \to M^*[h]$ for some $\mathbb{Q}_{j(\kappa)} := j^*(\dot{\mathbb{Q}}^G_\kappa) = ST^{M^*}_{j^*(S^0),j^*(S^1),j^*(\vec{N})}$-generic filter h over M^* so that j^{**} is definable in $V[G * g]$. By [5, Proposition 9.1] it is enough to find such a $\mathbb{Q}_{j(\kappa)}$-generic $h \in V[G * g]$ over M^* for which $j^*[g] \subset h$.

For every $\xi < \kappa^{++}$ we denote by $x(\xi) \in \kappa^\kappa \cap V[G * g]$ the (unique!) branch through all trees in $g(\xi)$ and let $a_\xi = \kappa$ (resp. $a_\xi = 0$) for all $\xi \in S^0$ (resp. $\xi \in S^1$). We claim that

$$h = \{j^*(p)|\sigma_I \ : \ p \in g, I \in M^*, I \subset j[\kappa^{++}], |I| = \kappa\},$$

where $\sigma_I(j(\xi)) = x(\xi)\hat{\ }a_\xi$ for all $j(\xi) \in I$, is $\mathbb{Q}_{j(\kappa)}$-generic over M^*. Let $\bar{D} \in M^*$ be an open dense subset of $\mathbb{Q}_{j(\kappa)}$. Write $\bar{D}$ as $j^*(f)(\bar{a})$, where f has domain $H(\kappa)^V$, $f \in V[G]$, and $a \in H(\kappa^{++})^V$. There is no loss of generality to assume that $f(a)$ is dense in $\mathbb{Q}_\kappa := \dot{\mathbb{Q}}^G_\kappa$ for all $a \in H(\kappa)^V$. Let us enumerate $H(\kappa)^V$ as $\langle a_k : k \in \kappa\rangle$ and set $D_k = \bigcap_{k'\leq k} f(a_{k'})$.

Let $p \in \mathbb{Q}_\kappa$ be arbitrary. Claim 34 yields a generalized fusion sequence $\langle (p_k, F_k) : k \in \kappa\rangle$ such that $p_0 = p$ and, letting $q = \wedge_{k\in\kappa} p_k$, for every limit $k \in \kappa$ and σ which is k-properly situated on q, σ lies on q and $q|\sigma \in D_k$.

Let $\langle \bar{F}_{\bar{k}} : \bar{k} \in j(\kappa)\rangle$ and $\langle \bar{p}_{\bar{k}} : \bar{k} \in j(\kappa)\rangle$ be the results of applying j^* to $\langle F_k : k \in \kappa\rangle$ and $\langle p_k : k \in \kappa\rangle$ respectively. By elementarity of j^*, $\langle (\bar{p}_{\bar{k}}, \bar{F}_{\bar{k}}) : \bar{k} \in j(\kappa)\rangle$ is a generalized fusion sequence for $\mathbb{Q}_{j(\kappa)}$, $\bar{q} := j^*(q) = \wedge_{\bar{k}<j(\kappa)} \bar{p}_{\bar{k}}$, and there exists $\bar{\beta} \in j(\kappa)$ so that for each limit $\bar{\alpha} \geq \bar{\beta}$ and $\bar{\sigma}$ which is $\bar{\alpha}$-properly situated on $\bar{q}$, $\bar{\sigma}$ lies on $\bar{q}$ and $\bar{q}|\bar{\sigma} \in \bar{D}$. We can additionally assume that $\bar{\beta} > \kappa$.

Fix $u \leq q$ and a club $C \subset \kappa$ such that $j(C) \cap (\kappa, \bar{\beta}] = \emptyset$ (its existence is established, e.g., in the proof of [9, Lemma 4]). Using Claim 33, we can construct a fusion sequence $\langle (u_k, T_k) : k \in \kappa\rangle$ with $u_0 = u$ satisfying the following conditions:

(i) $F_k \subset T_k$;

(ii) For every $\sigma : T_k \to \kappa^{k+1}$ which lies on u_k and has the property $\sigma(k) = k$ for all $k \in T_k \cap S^0$, there exist a limit ordinal $m \in C \setminus (k + 1)$ and $\pi : T_{k+1} \cup F_m \to \kappa^{(m+1)}$ such that $\pi(\xi) \upharpoonright (k + 1) = \sigma(\xi)$ for all $\xi \in T_k$, π lies on u_{k+1}, $u_{k+1}|\sigma = u_{k+1}|\pi$, and $u_{k+1}|\pi$ is a witness for π being m-properly situated on q with respect to $\langle (p_k, F_k) : k \in \kappa\rangle$.

Let $\langle \bar{T}_{\bar{k}} : \bar{k} \in j(\kappa)\rangle$ and $\langle \bar{u}_{\bar{k}} : \bar{k} \in j(\kappa)\rangle$ be the results of applying j^* to $\langle T_k : k \in \kappa\rangle$ and $\langle u_k : k \in \kappa\rangle$ respectively, $v = \wedge_{k<\kappa} u_k$, and $\bar{v} = j^*(v) = \wedge_{\bar{k}<j(\kappa)} \bar{u}_{\bar{k}}$. By elementarity of j^*, for every $\bar{\sigma} : \bar{T}_\kappa \to j(\kappa)^{\kappa+1}$ which lies on $\bar{u}_\kappa$ and has the property $\bar{\sigma}(\bar{\xi})(\kappa) = \kappa$ for all $\bar{\xi} \in \bar{T}_\kappa \cap j(S^0)$, there exist a limit ordinal $\bar{m} \in j(C) \setminus (\kappa + 1)$ and $\bar{\pi} : \bar{T}_{\kappa+1} \cup \bar{F}_{\bar{m}} \to j(\kappa)^{(\bar{m}+1)}$ such that $\bar{\pi}(\bar{\xi}) \upharpoonright (\kappa + 1) = \bar{\sigma}(\bar{\xi})$ for all $\bar{\xi} \in \bar{T}_\kappa$, $\bar{\pi}$ lies on $\bar{u}_{\kappa+1}$, $\bar{u}_{\kappa+1}|\bar{\sigma} = \bar{u}_{\kappa+1}|\bar{\pi}$, and $\bar{u}_{\kappa+1}|\bar{\sigma}$ is a witness for $\bar{\pi}$ being $\bar{m}$-properly situated on $\bar{q}$ with respect to $\langle (\bar{p}_{\bar{k}}, \bar{F}_{\bar{k}}) : \bar{k} \in j(\kappa)\rangle$.

Since p and $u \le q$ were chosen arbitrary, we can assume that $v \in g$. Observe that $\bar{T}_\kappa = \bigcup_{k\in\kappa} j[T_k] \subset j[\kappa^{++}]$, $|\bar{T}_\kappa| = \kappa$, and $\bar{T}_\kappa \in M^*$. The elementarity of j^* implies that $\bar{\sigma} := \sigma_{\bar{T}_\kappa}$ lies on $j^*(w)$ for any $w \in g$. In particular, $\bar{\sigma}$ lies on $\bar{u}_k = j^*(u_k)$ for all $k \in \kappa$, and hence it lies on $\bar{u}_\kappa = \wedge_{k\in\kappa}\bar{u}_k$ as well. Therefore we can find $\bar{m} \in j(C) \setminus (\kappa+1)$ and $\bar{\pi} : \bar{T}_{\kappa+1} \cup \bar{F}_{\bar{m}} \to j(\kappa)^{(\bar{m}+1)}$ as above, i.e. $\bar{u}_{\kappa+1}|\bar{\sigma}$ is a witness for $\bar{\pi}$ being $\bar{m}$-properly situated on $\bar{q}$ with respect to $\langle(\bar{p}_{\bar{k}}, \bar{F}_{\bar{k}}) : \bar{k} \in j(\kappa)\rangle$. By the construction of $\langle(p_k, F_k) : k \in \kappa\rangle$, elementarity of j^*, the equalities $j(C) \cap (\kappa, \bar{\beta}) = \emptyset$ and $\bar{m} \in j(C) \setminus (\kappa+1)$, and our choice of $\bar{\beta}$, we conclude that $\bar{\pi} \restriction \bar{F}_{\bar{m}}$ lies on $\bar{q}$ and $\bar{q}|(\bar{\pi} \restriction \bar{F}_{\bar{m}}) \in \bar{D}$. On the other hand,

$$\bar{q}|(\bar{\pi} \restriction \bar{F}_{\bar{m}}) \ge \bar{u}_{\kappa+1}|\bar{\pi} = \bar{u}_{\kappa+1}|\bar{\sigma} \ge \bar{v}|\bar{\sigma} = j^*(v)|\sigma_{\bar{T}_\kappa} \in h,$$

which means that $h \cap \bar{D} \neq \emptyset$ and thus finishes the proof of Theorem 1. □

Remark 1.

1. To the best knowledge of the authors there are essentially three other different forcing extensions V^P of V which preserve the measurability of κ and kill the GCH at κ under the assumption that κ is $P_2\kappa$-hypermeasurable, see [5, § 24], [9], and [6, § 4]. In all three cases we have $\mathrm{cf}(Sym(\kappa)) = \kappa^+$ in V^P. The historically first of them is due to Woodin [5, § 24]. His P can be written as $P_0 * P_1 * P_2$, where P_0 is iteration of Cohen posets below κ with reverse Easton support, and thus $|P_0| = \kappa$ and P_0 has κ-c.c.; P_1 is the poset adding κ^{++}-many Cohen subsets of κ, and P_2 adds no new subsets of κ. It is clear that $V^{P_0 * P_1} \vDash \mathrm{cf}(Sym(\kappa)) = \kappa^+$ (see the last paragraph on [18, p. 894]), and every forcing which does not add new subsets of κ cannot enlarge $\mathrm{cf}(Sym(\kappa))$.

In forcing extensions constructed in [9] and [6] the equality $\mathfrak{d}(\kappa) = \kappa^+$ holds, and it is well-known (the proof of [17, Proposition 1.4] works for every regular κ) that $\mathrm{cf}(Sym(\kappa)) \le \mathfrak{d}(\kappa)$ for every regular κ.

2. It is known [18] that the equality $\mathrm{cf}(Sym(\kappa)) = \kappa^{++}$ (and much more) is consistent for every inaccessible κ. But the authors were not able to lift elementary embeddings to forcing extensions used in [18][4] assuming considerably less than supercompactness. However, such possibility is not formally excluded. On the other hand, applying the methods developed in [8] to forcing extensions from [18] we could obtain the following result:

Suppose $0^\sharp$ exists. Then there is an inner model in which $\mathrm{cf}(Sym(\kappa)) = \kappa^{++}$ for every regular cardinal κ of the form $\aleph_{2\alpha}$.

It is worth mentioning here that for every cardinal κ the inequality $\mathrm{cf}(Sym(\kappa)) > \kappa^+$ implies $\mathrm{cf}(Sym(\kappa^+)) \le \mathrm{cf}(Sym(\kappa))$, and it is not known even how to obtain $\mathrm{cf}(Sym(\kappa)) > \kappa^+$ at two consecutive κ simultaneously, see [18].

3. In order to show that $j(\mathbb{P})_{\kappa+1} = \mathbb{P}$ in the proof of Theorem 1 we needed suitable stationary sets S^0, S^1 and $\langle S^0_k, S^1_k : k \in \kappa\rangle$. Instead of using the auxiliary forcing introducing such sets we could apply the same inner model argument as in the proof of [7, Theorem 11]. □

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