

KILLING THE GCH EVERYWHERE WITH A SINGLE REAL

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Abstract. Shelah–Woodin [10] investigate the possibility of violating instances of GCH through the addition of a single real. In particular they show that it is possible to obtain a failure of CH by adding a single real to a model of GCH, preserving cofinalities. In this article we strengthen their result by showing that it is possible to violate GCH at all infinite cardinals by adding a single real to a model of GCH. Our assumption is the existence of an $H(\kappa^{+3})$ -strong cardinal; by work of Gitik and Mitchell [6] it is known that more than an $H(\kappa^{++})$ -strong cardinal is required.

§1. Introduction. Shelah–Woodin [10] investigate the possibility of violating instances of GCH through the addition of a single real. In particular they show that it is possible to obtain a failure of CH by adding a single real to a model of GCH, preserving cofinalities. In this article we bring this work to its natural conclusion by showing that it is possible to violate GCH at all infinite cardinals by adding a single real to a model of GCH.

THEOREM 1.1. *Assume the consistency of an $H(\kappa^{+3})$ -strong cardinal κ . Then there exists a pair (W, V) of models of ZFC such that:*

- (a) W and V have the same cardinals,
- (b) GCH holds in W ,
- (c) $V = W[R]$ for some real R ,
- (d) GCH fails at all infinite cardinals in V .

To achieve the result we add a generic for a Prikry product, code it by a real preserving $H(\kappa^{+3})$ -strength and then finish the proof by quoting a modified version of a result of Merimovich [9].

We also show that assuming the existence of a proper class of measurable cardinals, it is possible to force Easton’s theorem by adding a single real. More precisely:

THEOREM 1.2. *Let M be a model of ZFC + GCH + there exists a proper class of measurable cardinals. In M let $F: \text{REG} \rightarrow \text{CARD}$ be an Easton function, i.e., a definable class function such that*

- $\kappa \leq \lambda \rightarrow F(\kappa) \leq F(\lambda)$, and
- $\text{cf}(F(\kappa)) > \kappa$.

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Then there exists a pair (W, V) of cardinal preserving extensions of M such that

- (a) $W \models \ulcorner \text{GCH} \urcorner$,
- (b) $V = W[R]$ for some real R ,
- (c) $V \models \ulcorner \forall \kappa \in \text{REG}, 2^\kappa \geq F(\kappa) \urcorner$.

The reason that in (c) we do not require equality is that it might be possible that $F(\kappa)$ changes its cofinality in V to ω , and then clearly $2^\kappa \neq F(\kappa)$ in V . To achieve the result we define a class forcing version of the Prikry product, code its generic by a real using Jensen's coding and then finish the proof by applying Easton's theorem.

§2. Proof of Theorem 1.1.

2.1. Prikry products. Assume GCH and suppose that S is a set of measurable cardinals which is *discrete*, i.e., contains none of its limit points. Fix normal measures U_α on α for α in S . Then \mathbb{P}_S denotes the Prikry product of the forcings \mathbb{P}_α , $\alpha \in S$, where \mathbb{P}_α is the Prikry forcing associated with the measure U_α . Thus

$$\mathbb{P}_S = \{ \langle (s_\alpha, A_\alpha) : \alpha \in S \rangle \in \prod_{\alpha \in S} \mathbb{P}_\alpha : s_\alpha = \emptyset \text{ for all but finitely many } \alpha \in S \}.$$

For two conditions $p = \langle (s_\alpha, A_\alpha) : \alpha \in S \rangle$ and $q = \langle (t_\alpha, B_\alpha) : \alpha \in S \rangle$ in \mathbb{P}_S we define $p \leq q$ (p is stronger than q) if $(s_\alpha, A_\alpha) \leq (t_\alpha, B_\alpha)$ in \mathbb{P}_α for all $\alpha \in S$. We also define the auxiliary relation $p \leq^* q$ (p is a direct or a Prikry extension of q) if $p \leq q$ and $s_\alpha = t_\alpha$ for all $\alpha \in S$.¹

A \mathbb{P}_S -generic is uniquely determined by a sequence $\langle x_\alpha : \alpha \in S \rangle$, where each x_α is an ω -sequence cofinal in α . With a slight abuse of terminology, we say that $\langle x_\alpha : \alpha \in S \rangle$ is \mathbb{P}_S -generic.

LEMMA 2.1 (Fuchs [4], Magidor [8]). *Suppose that $\langle x_\alpha : \alpha \in S \rangle$ is \mathbb{P}_S -generic over V .*

- (a) V and $V[\langle x_\alpha : \alpha \in S \rangle]$ have the same cardinals.
- (b) *The sequence $\langle x_\alpha : \alpha \in S \rangle$ obeys the following "geometric property": If $\langle X_\alpha : \alpha \in S \rangle$ belongs to V and $X_\alpha \in U_\alpha$ for each $\alpha \in S$, then $\bigcup_{\alpha \in S} x_\alpha \setminus X_\alpha$ is finite.*
- (c) *Conversely, suppose that $\langle y_\alpha : \alpha \in S \rangle$ is a sequence (in any outer model of V) satisfying the geometric property stated above. Then $\langle y_\alpha : \alpha \in S \rangle$ is \mathbb{P}_S -generic over V .*

(d) *Suppose $\alpha \in S$, $p \in \mathbb{P}_S$ and $\langle \Phi_\gamma : \gamma < \eta \rangle$ is a sequence of statements of the forcing language for \mathbb{P}_S where $\eta < \alpha$. Then there exists $q \leq^* p$ such that $q \upharpoonright \alpha = p \upharpoonright \alpha$ and for each $\gamma < \eta$ if $r \leq q$ and r decides Φ_γ , then $(r \upharpoonright \alpha) \cup (q \upharpoonright [\alpha, \kappa))$ (where $\kappa = \sup(S)$) decides Φ_γ in the same way.*

THEOREM 2.2. *Suppose that κ is $H(\kappa^{+3})$ -strong and S is a discrete set of measurable cardinals less than κ . Then after forcing with \mathbb{P}_S , κ remains $H(\kappa^{+3})$ -strong.*

PROOF. Suppose that $j : V \rightarrow M \supseteq H(\kappa^{+3})$, $\text{crit}(j) = \kappa$ is an elementary embedding witnessing the $H(\kappa^{+3})$ -strength of κ . We can assume that j is derived from an extender $E = \langle E_a : a \in [\kappa^{+3}]^{<\omega} \rangle$. Then for each $a \in [\kappa^{+3}]^{<\omega}$, E_a is a κ -complete ultrafilter on $[\kappa]^{|a|}$ and if $j_a : V \rightarrow M_a \cong \text{Ult}(V, E_a)$ is the corresponding elementary embedding then for all $B \subseteq [\kappa]^{|a|}$, we have $B \in E_a \Leftrightarrow a \in j_a(B)$. We also have an embedding $k_a : M_a \rightarrow M$ such that $k_a \circ j_a = j$.

¹Thus \mathbb{P}_S is forcing equivalent to the Magidor iteration of the Prikry forcings \mathbb{P}_α , $\alpha \in S$.

We show that κ remains $H(\kappa^{+3})$ -strong in the generic extension by \mathbb{P}_S . The proof uses ideas from [7] and [8]. Let G be \mathbb{P}_S -generic over V . Also let $\delta = \min(j(S) - \kappa) > \kappa$.

Working in $V[G]$, we define for each $a \in [\kappa^{+3}]^{<\omega_1}$, E_a^* as follows: Let $\xi = o.t(a)$, and let \dot{a} be a \mathbb{P}_S -name for a such that

$$\Vdash^{-\Gamma} \dot{a} \subseteq \kappa^{+3} \text{ and } o.t(\dot{a}) = \xi^\neg$$

For $p \in \mathbb{P}_S$ define $p \Vdash^{-\Gamma} \dot{B} \in \dot{E}_a^*$ iff

- (1) $p \Vdash^{-\Gamma} \dot{B} \subseteq [\kappa]^\xi^\neg$,
- (2) there exists $q \leq^* j(p)$ in $j(\mathbb{P}_S)$ such that $q \upharpoonright \delta = j(p) \upharpoonright \delta = p$, and $q \Vdash^{-M^\Gamma} \dot{a} \in j(\dot{B})^\neg$.

Let $E_a^* = \dot{E}_a^*[G]$. It is easily seen that the above definition is well-defined.

LEMMA 2.3. (a) E_a^* is a κ -complete non-principal ultrafilter on $[\kappa]^\xi$,

(b) If $a \in V$ is finite, then E_a^* extends E_a .

PROOF. (a) We just prove that E_a^* is κ -complete. Suppose that $p \in \mathbb{P}_S$ and $p \Vdash^{-\Gamma} [\kappa]^\xi = \bigcup \{ \dot{B}_\gamma : \gamma < \eta \}^\neg$ where $\eta < \kappa$. Then $j(p) \Vdash^{-M^\Gamma} [j(\kappa)]^\xi = \bigcup \{ j(\dot{B}_\gamma) : \gamma < \eta \}^\neg$.

Working in M consider $\delta, j(p)$ and the sequence $(\Phi_\gamma : \gamma < \eta)$ of sentences where for each $\gamma < \eta$, Φ_γ is “ $\dot{a} \in j(\dot{B}_\gamma)$ ”. It then follows from Lemma 2.1.(d) that there is $q \leq^* j(p)$ in $j(\mathbb{P}_S)$ such that for each $\gamma < \eta$

- $q \upharpoonright \delta = j(p) \upharpoonright \delta = p$,
- if $r \leq q$ and r decides Φ_γ , then $(r \upharpoonright \delta) \cup (q \upharpoonright [\delta, j(\kappa)])$ decides Φ_γ in the same way.

Now $q \Vdash^{-M^\Gamma} \dot{a} \in [j(\kappa)]^\xi = \bigcup \{ j(\dot{B}_\gamma) : \gamma < \eta \}^\neg$ and hence we can find $r \leq q$ and $\gamma < \eta$ such that $r \Vdash^{-\Gamma} \Phi_\gamma^\neg$. Let $t = (r \upharpoonright \delta) \cup (q \upharpoonright [\delta, j(\kappa)])$. It is now easy to show that $t \upharpoonright \delta \leq p$ and $t \Vdash^{-\Gamma} \dot{B}_\gamma \in \dot{E}_a^*$. This completes the proof of the κ -completeness of E_a^* .

(b) Suppose $a \in V$ is finite. Let $B \in E_a$ and $p \in \mathbb{P}_S$. We show that $p \Vdash^{-\Gamma} B \in \dot{E}_a^*$. Let $q = j(p)$. Then q has the required properties in the definition above which gives the result. \dashv

In $V[G]$, for each $a \in [\kappa^{+3}]^{<\omega_1}$ let $j_a^* : V[G] \rightarrow M_a^* \simeq \text{Ult}(V[G], E_a^*)$ be the corresponding elementary embedding. Also for $a \subseteq b$ let $k_{a,b} : M_a^* \rightarrow M_b^*$ be the natural induced elementary embedding. Let

$$\langle M^*, \langle k_a^* : a \in [\kappa^{+3}]^{<\omega_1} \rangle \rangle = \text{dirlim} \langle \langle M_a^* : a \in [\kappa^{+3}]^{<\omega_1} \rangle, \langle k_{a,b}^* : a \subseteq b \rangle \rangle.$$

Also let $j^* : V[G] \rightarrow M^*$ be the induced embedding.

LEMMA 2.4. M^* is well-founded

PROOF. Suppose not. Then there is a sequence $(m_i : i < \omega)$ of elements of M^* such that

$$\cdots \in^* m_2 \in^* m_1 \in^* m_0$$

where $\in^* = \in_{M^*}$. For each $i < \omega$ choose a_i and f_i such that $m_i = k_{a_i}^*([f_i]_{E_{a_i}^*})$. Let $a = \bigcup \{ a_i : i < \omega \}$. Then $a \in [\kappa^{+3}]^{<\omega_1}$ and for some $g_i, m_i = k_a^*([g_i]_{E_a^*})$. It then follows from the elementarity of k_a^* that

$$\cdots \in [g_2]_{E_a^*} \in [g_1]_{E_a^*} \in [g_0]_{E_a^*}.$$

This is in contradiction with Lemma 2.3 which implies M_a^* is well-founded. Thus M^* is well-founded and the lemma follows. \dashv

If now we restrict ourself to E_a^* for finite a , then the smaller direct limit embeds into the full direct limit and is therefore well-founded. From now on, let M^* denote the smaller direct limit; accordingly each E_a^* is now given by the usual extender definition and j^* is the ultrapower embedding.

Note that $j^*: V[G] \rightarrow M^*$ is an elementary embedding with critical point κ . We show that it is an $H(\kappa^{+3})$ -strong embedding. For this it suffices to show that $H(\kappa^{+3})^{V[G]} \subseteq M^*$. But since $H(\kappa^{+3})^{V[G]} = H(\kappa^{+3})[G]$, it suffices to show that $H(\kappa^{+3}) \subseteq M^*$ and $G \in M^*$.

For this purpose we introduce some special functions in V . Let $F: \kappa \rightarrow \kappa$ be defined by $F(\alpha) = \alpha^{+3}$. Then $j(F)(\kappa) = \kappa^{+3}$. Now for each $a \in [\kappa^{+3}]^{<\omega}$ with $\kappa \in a$ and $|a| = n$ define the function $G_a: [\kappa]^n \rightarrow \kappa$ by $G_a(\alpha_1, \dots, \alpha_n) = \alpha_i^{+3}$ where κ is the i -th element of a . It is clear that $j(G_a)(a) = j(F)(\kappa) = \kappa^{+3}$. Also let $r: \kappa \rightarrow H(\kappa)$ be defined by $r(\alpha) = H(\alpha)$.

Suppose $f: [\kappa]^n \rightarrow H(\kappa)^{V[G]}$ is in $V[G]$ and a is a finite subset of κ^{+3} containing κ . We say the pair (f, a) has the property $(*)$ iff

$$\{\gamma: f(\gamma) \in r \circ G_a(\gamma)\} \in E_a^*.$$
²

We have the following easy lemma.

- LEMMA 2.5. (a) *If $j^*(f)(a) = j^*(g)(b)$ where κ is an element of both a and b , then (f, a) has the property $(*)$ iff (g, b) has the property $(*)$.*
 (b) *If (f, a) has the property $(*)$ and $j^*(g)(b) \in j^*(f)(a)$ for some b containing κ , then (g, b) has the property $(*)$.*

LEMMA 2.6. *If (f, a) has the property $(*)$, then there is a function $h: [\kappa]^m \rightarrow H(\kappa)$ in V and a finite set $b \subseteq \kappa^{+3}$ such that $j^*(f)(a) = j^*(h)(b)$.*

PROOF. Let $B = \{\gamma: f(\gamma) \in r \circ G_a(\gamma)\}$. Since (f, a) has the property $(*)$, $B \in E_a^*$. Let \dot{B} be a name for B and let $p \Vdash \dot{B} \in \dot{E}_a^*$. This means that there is some $q \leq^* j(p)$ such that $q \upharpoonright \delta = j(p) \upharpoonright \delta = p$ and $q \Vdash \dot{B} \in j(\dot{B})$. Hence we have $q \Vdash j(\dot{f})(a) \in j(r \circ G_a)(a) = H(\kappa^{+3})$.

For each $c \in H(\kappa^{+3})$ let Φ_c be the sentence " $j(\dot{f})(a) = c$ ". By applying Lemma 2.1.(d) we can find $r \leq^* q$ such that for every $c \in H(\kappa^{+3})$

- $r \upharpoonright \delta = q \upharpoonright \delta = p$,
- if $s \leq r$ and s decides Φ_c then $(s \upharpoonright \delta) \cup (r \upharpoonright [\delta, j(\kappa)])$ decides Φ_c in the same way.

Now $r \Vdash j(\dot{f})(a) \in j(r \circ G_a)(a) = H(\kappa^{+3})$, hence there are $s \leq r$ and $c \in H(\kappa^{+3})$ such that $s \Vdash \Phi_c$. Let $t = (s \upharpoonright \delta) \cup (r \upharpoonright [\delta, j(\kappa)])$. By above, $t \Vdash \Phi_c$.

Since $c \in H(\kappa^{+3})$, there is a function $h: [\kappa]^m \rightarrow H(\kappa)$ and a finite $b \subseteq \kappa^{+3}$ such that $c = j(h)(b)$. Thus $t \Vdash j(\dot{f})(a) = j(h)(b)$ and the result follows. \dashv

Define the sets X and X^* as follows

$$X = \{j(f)(a): (f, a) \text{ is in } V \text{ and has the property } (*),$$

$$X^* = \{j^*(f)(a): (f, a) \text{ is in } V[G] \text{ and has the property } (*).\}$$

²It can be shown that (f, a) has property $(*)$ iff $[f]_{E_a^*}$ represents an element of $H(\kappa^{+3})$ in M_a^* .

It follows from Lemma 2.5 that X and X^* are transitive.

LEMMA 2.7. *If (f, a) has the property $(*)$ and $f \in V$, then $j^*(f)(a) = j(f)(a)$.*

PROOF. Define $\Phi: X \rightarrow X^*$ by $\Phi(j(f)(a)) = j^*(f)(a)$. Then:

(1) Φ is well-defined: To see this suppose that $j(f)(a) = j(g)(b)$. We may further suppose that $a = b$. It then follows that $j(f)(a) = k_a([f]_{E_a}) = k_a([g]_{E_a}) = j(g)(b)$, and hence $B = \{x: f(x) = g(x)\} \in E_a$. By Lemma 2.3(b), $B \in E_a^*$ and hence $j^*(f)(a) = k_a^*([f]_{E_a^*}) = k_a^*([g]_{E_a^*}) = j^*(g)(b)$.

(2) Φ preserves the \in relation: As in (1).

Thus Φ is an isomorphism, and since both of X and X^* are transitive, it must be the identity. The lemma follows. \dashv

LEMMA 2.8. $H(\kappa^{+3}) \subseteq M^*$.

PROOF. We have $H(\kappa^{+3}) \subseteq X \subseteq X^* \subseteq M^*$. \square

LEMMA 2.9. $G \in M^*$

PROOF. First note that $\mathbb{P}_S \in H(\kappa^{+3}) \subseteq M^*$. Define $f: \kappa \rightarrow H(\kappa)^{V[G]}$ by $f(\alpha) = G_\alpha$, where $G_\alpha = G \cap H(\alpha)$ is $\mathbb{P}_S \cap H(\alpha)$ -generic over V . Show that $G = j^*(f)(\kappa)$, and hence $G \in M^*$. By maximality of G it suffices to show that $G \subseteq j^*(f)(\kappa)$.

Let $p \in G$. Choose $h: [\kappa]^n \rightarrow H(\kappa)$ in V and a finite set $a \subseteq \kappa^{+3}$ containing κ such that $p = j(h)(a)$. Then by Lemma 2.7 $p = j^*(h)(a)$. Define $f_a(\alpha_1, \dots, \alpha_n) = f(\alpha_i)$, where κ is the i -th element of a . Then $j^*(f_a)(a) = j^*(f)(\kappa)$. Now we have to prove that $j^*(h)(a) \in j^*(f_a)(a)$.

Let \dot{f}_a be a \mathbb{P}_S -name for f_a such that $\Vdash_{\mathbb{P}_S} \dot{f}_a(\alpha_1, \dots, \alpha_n) = \dot{G}_{\alpha_i}$. Then $\Vdash_{j(\mathbb{P}_S)} j(\dot{f}_a)(a) = \dot{G}$ and hence $\Vdash_{j(\mathbb{P}_S)} j(h)(a) \in j(\dot{f}_a)(a)$. The lemma follows. \dashv

2.2. Coding. Friedman [3] presents a method for creating reals which are class-generic (but not set-generic) over a sufficiently L -like model, preserving Woodin cardinals. A similar method can be used to preserve strong cardinals. However the general problem of coding a predicate into a real while preserving large cardinal properties is open; we show here that this is possible if the predicate is a sequence which is generic for a discrete Prikry product.

THEOREM 2.10. *Suppose that K is the canonical inner model for an $H(\kappa^{+3})$ -strong cardinal κ . Suppose that S is the discrete set consisting of those measurable cardinals less than κ in K which are not limits of measurable cardinals in K . Also let $(x_\alpha: \alpha \in S)$ be \mathbb{P}_S -generic over K for the measures $(U_\alpha: \alpha \in S)$, where U_α is the unique normal measure on α in K . Then there is a cofinality-preserving set-forcing \mathbb{P} for adding a real R over $K[(x_\alpha: \alpha \in S)]$ such that $K[(x_\alpha: \alpha \in S)][R] = K[R]$ and κ remains $H(\kappa^{+3})$ -strong in $K[R]$.*

PROOF. We will follow the proof of Jensen's coding theorem from [2], section 4.2, making use of Lemma 2.1 to argue that the relevant Σ_1 Skolem hulls taken with respect to certain initial segments of K are also Σ_1 elementary when the Prikry product generic is adjoined. We must impose some minor changes to the notion of "string s " and to the coding structures $\mathcal{A}^s, \tilde{\mathcal{A}}^s$, but for the most part the argument remains the same. The preservation of $H(\kappa^{+3})$ -strength is based on ideas from [3].

We work in $L[E][(x_\alpha: \alpha \in S)]$ where $K = L[E]$ is a fine-structural inner model built from the sequence E of (partial) extenders. Abbreviate $(x_\alpha: \alpha \in S)$ as \vec{x} and

for any β let $\vec{x}(\leq \beta)$ denote $(x_\alpha: \alpha \in S, \alpha \leq \beta)$. We may also assume that for α in S , the min of x_α is greater than the supremum of $S \cap \alpha$, using the discreteness of the set S . Let A denote the union of the x_α , $\alpha \in S$.

Card denotes the class of infinite cardinals. For α in Card we define the ordinals $\mu^{<\eta}$, μ^η by induction on $\eta \in [\alpha, \alpha^+)$. An ordinal μ is a *ZF⁻ ordinal* iff $L_\mu[E, \vec{x}(\leq \alpha)]$ is a model of ZF minus Power Set. Define: $\mu^{<\eta} = \cup\{\mu^\xi: \xi < \eta\} \cup \alpha$, $\mu^\eta =$ the least limit of ZF⁻ ordinals μ such that μ is greater than $\mu^{<\eta}$ and, setting $\mathcal{A}^\eta = L_\mu[E, \vec{x}(\leq \alpha)]$ we have that $\mathcal{A}^\eta \models \alpha$ is the largest cardinal.

S_α , the set of *strings* at α consists of all $s: [\alpha, |s|) \rightarrow 2$, $\alpha \leq |s| < \alpha^+$, such that $|s|$ is a multiple of α and s belongs to $\mathcal{A}^{|s|}$. We write $s \leq t$ when t extends s and $s < t$ when t properly extends s . For $s \in S_\alpha$ we write \mathcal{A}^s for $\mathcal{A}^{|s|}$ and μ^s for $\mu^{|s|}$.

For later use (see “Limit Precoding”) we also define $\tilde{\mu}^s < \mu^s$ to be the least ZF⁻ ordinal μ greater than $\mu^{<|s|}$ such that the structure $L_\mu[E, \vec{x}(\leq \alpha)]$ contains s and satisfies that α is the largest cardinal. The resulting structure $\tilde{\mathcal{A}}^s = L_{\tilde{\mu}^s}[E, x(\leq \alpha)]$ is a proper initial segment of \mathcal{A}^s and, like \mathcal{A}^s , each element of $\tilde{\mathcal{A}}^s$ is Σ_1 definable in $\tilde{\mathcal{A}}^s$ from parameters in $\alpha \cup \{\vec{x}(\leq \alpha), s\}$. (We say that $\tilde{\mathcal{A}}^s, \mathcal{A}^s$ are Σ_1 *projectible* to α with parameters $\vec{x}(\leq \alpha), s$.)

To set up the coding we need the functions f^s , defined as follows: For α an uncountable cardinal, s in S_α and $i < \alpha$ let $H^s(i)$ denote the Σ_1 Skolem hull of $i \cup \{\vec{x}(\leq \alpha), s\}$ in \mathcal{A}^s . Then $f^s(i)$ is the ordertype of $H^s(i) \cap \text{Ord}$. For α a successor cardinal we define the coding set b^s to be the range of $f^s \upharpoonright B^s$ where B^s consists of the successor elements of $\{i < \alpha: i \text{ is a limit of } j \text{ such that } j = H^s(j) \cap \alpha\}$.

We describe a cofinality-preserving forcing which codes $K[\vec{x}]$ into $K[X]$ for some $X \subseteq \omega_1$, preserving the $H(\kappa^{+3})$ -strength of κ . Then a simple *c.c.c* forcing can be used to code X into the desired real R .

We need a partition of the ordinals into four pieces: Let B, C, D, F denote the classes of ordinals which are congruent to 0, 1, 2, 3 mod 4, respectively (The letters A and E are already used for other purposes). For any ordinal α , α^B denotes the α -th element of B and for any set Y of ordinals, Y^B denotes the set of α^B for α in Y (similarly for C, D, F).

The successor coding: Suppose $\alpha \in \text{Card}$ and $s \in S_{\alpha^+}$. A condition in R^s is a pair (t, t^*) where $t \in S_\alpha$, $t^* \subseteq \{b^{s \upharpoonright \eta}: \eta \in [\alpha^+, |s|)\} \cup |t|$, $\text{card}(t^*) \leq \alpha$. Extension is defined by: $(t_0, t_0^*) \leq (t_1, t_1^*)$ iff t_0 extends t_1 , t_0^* contains t_1^* and:

- (1) If $|t_1| \leq \gamma^B < |t_0|$ and $\gamma \in b^{s \upharpoonright \eta} \in t_1^*$ then $t_0(\gamma^B) = 0$ or $s(\eta)$.
- (2) If $|t_1| \leq \gamma^C < |t_0|$ and $\gamma = \langle \gamma_0, \gamma_1 \rangle$ with $\gamma_0 \in A \cap t_1^*$ then $t_0(\gamma^C) = 0$ (where $\langle \cdot, \cdot \rangle$ is Gödel pairing of ordinals).

An R^s -generic over \mathcal{A}^s adds (and is uniquely determined by) a function $T: \alpha^+ \rightarrow 2$ such that $s(\eta) = 0$ iff $T(\gamma^B) = 0$ for sufficiently large $\gamma \in B^{s \upharpoonright \eta}$ and such that for $\gamma_0 < \alpha^+$, $\gamma_0 \in A$ iff $T(\langle \gamma_0, \gamma_1 \rangle^C) = 0$ for sufficiently large $\gamma_1 < \alpha^+$.

The limit precoding. Suppose that α is an infinite cardinal and s belongs to S_α . We say that $X \subseteq \alpha$ *precodes* s if X is the Σ_1 theory of $\tilde{\mathcal{A}}^s$ with parameters from $\alpha \cup \{\vec{x}(\leq \alpha), s\}$, viewed as a subset of α .

The limit coding. Suppose that α is an uncountable limit cardinal, $s \in S_\alpha$ and p is a sequence $((p_\beta, p_\beta^*): \beta \in \text{Card} \cap \alpha)$ where $p_\beta \in S_\beta$ for each $\beta \in \text{Card} \cap \alpha$. We will define what it means for p to “code s ”. First define the sequence $(s_\gamma: \gamma \leq \gamma_0)$

of elements of S_α as follows: Let $s_0 = \emptyset$. For limit $\gamma \leq \gamma_0$, s_γ is the union of the s_δ , $\delta < \gamma$. Now suppose that s_γ is defined and for successor cardinals β less than α let $f_p^{s_\gamma}(\beta)$ be the least $\delta \geq f^{s_\gamma}(\beta)$ such that $p_\beta(\delta^D) = 1$, if such a δ exists. If $f_p^{s_\gamma}(\beta)$ is undefined for cofinally many successor cardinals $\beta < \alpha$ then set $\gamma_0 = \gamma$. Otherwise define $X \subseteq \alpha$ by: $\delta \in X$ iff $p_\beta((f_p^{s_\gamma}(\beta) + 1 + \delta)^D) = 1$ for sufficiently large successor cardinals $\beta < \alpha$. If $\text{Even}(X) = \{\delta : 2\delta \in X\}$ precodes an element t of S_α extending s_γ such that \mathcal{A}^t contains X and the function $f_p^{s_\gamma}$, then set $s_{\gamma+1} = t$. Otherwise let $s_{\gamma+1}$ be $s_\gamma * X^F$ (i.e., the concatenation of s_γ with X^F viewed as a sequence of length α), provided $s_\gamma * X^F$ belongs to S_α and $f_p^{s_\gamma}$ belongs to $\mathcal{A}^{s_\gamma * X^F}$; if not, then again set $\gamma_0 = \gamma$. Now p exactly codes s if s equals one of the s_γ , $\gamma \leq \gamma_0$ and p codes s is an initial segment of some s_γ , $\gamma \leq \gamma_0$.

Finally we define the desired forcing. Let Card' denote the class of uncountable limit cardinals. Also fix an extender ultrapower embedding $j: V = K[\vec{x}] \rightarrow M = K^*[\vec{x}^*]$ witnessing that κ is $H(\kappa^{+3})$ -strong in $K[\vec{x}]$. I.e., j has critical point κ , $H(\kappa^{+3})$ of V is contained in M and every element of M is of the form $j(f)(\alpha)$ for some $f: \kappa \rightarrow V$ in V and $\alpha < \kappa^{+3}$.

The conditions. A condition in \mathbb{P} is a sequence $p = ((p_\alpha, p_\alpha^*): \alpha \in \text{Card}, \alpha \leq \alpha(p))$ where $\alpha(p) \leq \kappa^{+3}$ in Card and:

- (1) $p_{\alpha(p)}$ belongs to $S_{\alpha(p)}$ and $p_{\alpha(p)}^* = \emptyset$.
- (2) For $\alpha \in \text{Card} \cap \alpha(p)$, (p_α, p_α^*) belongs to $R^{p_{\alpha^+}}$.
- (3) For $\alpha \in \text{Card}'$, $\alpha \leq \alpha(p)$, $p \upharpoonright \alpha$ belongs to \mathcal{A}^{p_α} and exactly codes p_α .
- (4) For $\alpha \in \text{Card}'$, $\alpha \leq \alpha(p)$, if α is inaccessible in \mathcal{A}^{p_α} then there exists a closed unbounded subset C of α , $C \in \mathcal{A}^{p_\alpha}$, such that for $\beta \in C$, $p_\beta^* = p_{\beta^+}^* = p_{\beta^{++}}^* = p_{\beta^+} = p_{\beta^{++}} = \emptyset$.

Conditions are ordered by: $p \leq q$ iff:

- (a) $\alpha(p) \geq \alpha(q)$.
- (b) $p(\alpha) \leq q(\alpha)$ in $R^{p_{\alpha^+}}$ for $\alpha \in \text{Card} \cap \alpha(p) \cap (\alpha(q) + 1)$.
- (c) $p_{\alpha(p)}$ extends $q_{\alpha(q)}$ if $\alpha(p) = \alpha(q)$.
- (d) If $\alpha(q) \geq \kappa^{++}$, $|q_{\kappa^{++}}| \leq \gamma < |p_{\kappa^{++}}|$, $\xi < |j(q)_{\kappa^{+3}}|$ is of the form $j(f)(i)$ for some $i < |q_{\kappa^{++}}|$ and function f with domain κ , $j(q)_{\kappa^{+3}}(\xi) = 0$ and γ belongs to $b^{j(q)_{\kappa^{+3}} \upharpoonright \xi}$ (as defined in $K^*[\vec{x}^*]$, the ultrapower of $K[\vec{x}]$ by j) then $p_{\kappa^{++}}(\gamma^B) = 0$.

Clause (d) is to ensure that $G_{\kappa^{++}}$, the subset of κ^{+3} added by the generic G , codes the union of the $j(p)_{\kappa^{+3}}$ for p in G , a fact needed for the preservation of $H(\kappa^{+3})$ -strength (see below).

This completes the definition of \mathbb{P} . The verification of cofinality and GCH preservation for \mathbb{P} is as in [2], section 4.2, following the proofs of the Lemmas 4.3–4.6 found there. Here we only point out the added points to be made, taking into account that we are coding \vec{x} over $K = L[E]$ and not over L . For this verification, requirement (4) above can be weakened to only require that $p_\beta^* = \emptyset$ for $\beta \in C$; the stronger form of (4) above is needed for the preservation of $H(\kappa^{+3})$ -strength.

A general fact that is needed throughout the proof is the following.

LEMMA 2.11 (Condensation). *Suppose that α is an uncountable cardinal, $s \in S_\alpha$, $i < \alpha$ and as before let $H^s(i)$ denote the Σ_1 Skolem hull of $i \cup \{\vec{x}(\leq \alpha), s\}$ in \mathcal{A}^s .*

(a) If α is a successor cardinal then for sufficiently large $i < \alpha$, if i is a limit point of $\{j < \alpha : j = H^s(j) \cap \alpha\}$ then the transitive collapse of $H^s(i)$ is of the form $\bar{K}[\vec{x}]$ where \bar{K} is an initial segment of K .

(b) If α is a limit cardinal then for sufficiently large cardinals $i < \alpha$ the transitive collapse of $H^s(i)$ is of the form $\bar{K}[\vec{x}]$ where \bar{K} is an initial segment of K .

The same holds with \mathcal{A}^s replaced by any of its initial segments which contain s and have height equal to a ZF^- ordinal.

PROOF. Recall that s belongs to $\mathcal{A}^s = L_{\mu|s}[E, \vec{x}(\leq \alpha)]$. Now $x(\leq \alpha)$ is generic over K for the product $\mathbb{P}_{S(\leq \alpha)}$ of Prikry forcings at $\beta \leq \alpha$ in S . If α is in the closure of S then the intersection of $\mathbb{P}_{S(\leq \alpha)}$ with $L_\mu[E]$ is a class forcing in $L_\mu[E]$ whenever μ is a ZF^- ordinal of size α such that α is the largest cardinal in $L_\mu[E]$. Nevertheless, all definable antichains in this forcing are sets. An examination of the proof of Lemma 2.1 in [4] reveals that any sequence which satisfies the geometric property of that lemma with respect to $L_\mu[E]$ for the forcing $\mathbb{P}_{S(\leq \alpha)} \cap L_\mu[E]$ is in fact generic for this forcing over $L_\mu[E]$. It follows that $x(\leq \alpha)$, which satisfies the geometric property with respect to the entire $L[E]$, is generic over $L_\mu[E]$ for this forcing. From this we infer the Σ_1 definability of the forcing relation for Δ_0 formulas for the forcing $\mathbb{P}_{S(\leq \alpha)} \cap L_{\mu^s}[E]$ and therefore that for $i \leq \alpha$, $H_0^s(i)$ = the Σ_1 Skolem hull of $i \cup \{s\}$ in $\mathcal{A}_0^s (= L_{\mu^s|s}[E])$ is equal to the intersection with \mathcal{A}_0^s of $H^s(i)$ = the Σ_1 Skolem hull of $i \cup \{s\}$ in \mathcal{A}^s (where \dot{s} is a name for $s \in \mathcal{A}^s$). In particular, setting i equal to α , we see that \mathcal{A}_0^s is Σ_1 -projectible to α with parameter \dot{s} .

If i satisfies the requirements stated in (a) or (b) above, then the Σ_1 projectum of the transitive collapse of $H_0^s(i)$ is equal to i and if i is sufficiently large, then this transitive collapse is also sound. It follows that \bar{K} = the transitive collapse of $H_0^s(i)$ is an initial segment of K for such i . The last statement of the lemma follows by the same argument, as any initial segment of \mathcal{A}^s which contains s is Σ_1 projectible to α with parameter s . \dashv

Using Condensation as above, the proofs of Lemmas 4.3–4.6 from [2], section 4.2 can be carried out in the present setting:

In Lemma 4.3, one must take the α_i 's to enumerate the first α sufficiently large elements of $\{\beta < \alpha^+ : \beta \text{ is a limit of } \bar{\beta} \text{ such that } \bar{\beta} = \alpha^+ \cap \Sigma_1 \text{ Skolem hull of } (\bar{\beta} \cup \{x\}) \text{ in } \mathcal{A}\}$ which are sufficiently large so that Condensation (a) guarantees that the transitive collapse of the associated Σ_1 hull is of the form $\bar{K}[\vec{x}]$ with \bar{K} an initial segment of K . This facilitates the proof of the Claim in the proof of Lemma 4.3

In Lemma 4.4 one applies Condensation (b) to ensure that the Σ_1 Skolem hull H_β , when $\beta = \alpha \cap H_\beta$, transitively collapses to a structure built from an initial segment of K for sufficiently large cardinals $\beta < \alpha$; this is needed to argue that the resulting s_β is a string at β . The rest of the proof remains unchanged.

The proof of Lemma 4.5 (a) in the case of β inaccessible also uses Condensation (b) in the proof of the Claim, to verify that the p_γ^s are strings (in S_γ). Also note that Jensen's subtle use of the assumption that $0^\#$ does not exist (referred to in the Note) has no counterpart here, as our structures $\mathcal{A}_0^s = L_{\mu^s}[E]$, $s \in S_\alpha$ collapse $|s|$ to α without the use of s as an additional predicate (indeed, s is just a parameter in $L_{\mu^s}[E, \vec{x}(\leq \alpha)]$). The proofs of Lemma 4.5 in the case of singular β as well as Lemma 4.6 can be carried out as before.

We are left with the verification that κ remains $H(\kappa^{+3})$ -strong after forcing with \mathbb{P} . Recall that $j: V = K[\bar{x}] \rightarrow M = K^*[\bar{x}^*]$ is the extender ultrapower embedding witnessing that κ is $H(\kappa^{+3})$ -strong. Let G be \mathbb{P} -generic over V ; in $V[G]$ we must produce a G^M which is $j(\mathbb{P})$ -generic over M and which contains $j(p)$ for each p in G .

If $(D_i: i < \kappa)$ are dense subsets of \mathbb{P} and p belongs to \mathbb{P} then p has an extension q which “reduces each D_i below i^{+3} ”, i.e., any extension r of q can be further extended to meet D_i without changing $r(\beta)$ for $\beta \geq i^{+3}$. (This is a variant of Δ -distributivity, see page 30 of [2].) From this it follows that if we take the upward closure of $j[G]$, we obtain a compatible set of conditions which reduces each dense subset of $j(\mathbb{P})$ in M below κ^{+3} , using the ultrapower representation of M . Moreover, thanks to requirement (4) in the definition of \mathbb{P} , $j[G]$ contains no nontrivial information between κ and κ^{+3} (except for G_κ , the subset of κ^+ added by G), and therefore $j[G]$ is compatible with $G \cap H(\kappa^{+3})$. Moreover, thanks to condition (d) in the definition of extension of conditions, $G_{\kappa^{++}}$ will code the union of the $j(p)_{\kappa^{+3}}$, $p \in G$, and this coding is generic (using the fact that the $j(p)_{\kappa^{+3}}$ belong to \mathcal{A}^0 ; see Lemma 4.8 of [2]). So we can take G^M to be generated by the joins of conditions in $j[G]$ with those in $G \cap H(\kappa^{+3})$ to obtain the desired $j(\mathbb{P})$ -generic over M . \dashv

2.3. Killing the GCH everywhere by a cardinal preserving forcing. In [9] the following is proved.

THEOREM 2.12 (Merimovich [9]). *Suppose that GCH holds and κ is $H(\kappa^{+4})$ -strong. Then there exists a generic extension of the universe in which κ remains inaccessible and $\forall \lambda \leq \kappa, 2^\lambda = \lambda^{+3}$.*

Unfortunately in the Merimovich model a lot of cardinals are collapsed below κ . We show that a simple modification of his proof can give us the the total failure of the GCH below κ without collapsing any cardinals.

THEOREM 2.13. *Suppose that GCH holds and κ is $H(\kappa^{+3})$ -strong. Then there exists a cardinal preserving generic extension of the universe in which κ remains inaccessible and $\forall \lambda \leq \kappa, 2^\lambda > \lambda^+$.*

REMARK 2.14. *In fact it suffices to have a Mitchell increasing sequence of extenders of length κ^+ , each of them $(\kappa + 2)$ -strong. Thus the exact strength that we need is a measurable cardinal κ with $o(\kappa) = \kappa^{++} + \kappa^+$.*

The idea behind the proof is simple. We consider Merimovich’s proof of Theorem 2.12 and replace the collapsing functions introduced in his proof by suitable Cohen forcings for adding many new sets. We also need to replace the Cohen forcings used in the proof of Theorem 2.12 by new ones because of our weaker assumption. As a result we will get a model in which we have $2^\lambda = \lambda^{++}$ for a club of cardinals λ below κ . As requested by the referee, we now provide the details.

Extender Sequences

Suppose $j: V^* \rightarrow M^* \supseteq V_\lambda^*$, $\text{crit}(j) = \kappa$. Define an extender sequence (with projections)

$$E(0) = \langle \langle E_\alpha(0): \alpha \in A \rangle, \langle \pi_{\beta,\alpha}: \beta, \alpha \in A, \beta \geq_j \alpha \rangle \rangle$$

on κ by:

- $A = [\kappa, \lambda)$,

- $\forall \alpha \in A, E_\alpha(0)$ is the κ -complete ultrafilter on κ defined by

$$X \in E_\alpha(0) \Leftrightarrow \alpha \in j(X)$$

We write $E_\alpha(0)$ as U_α .

- $\forall \alpha, \beta \in A$

$$\beta \geq_j \alpha \Leftrightarrow \beta \geq \alpha \text{ and for some } f \in {}^\kappa \kappa, j(f)(\beta) = \alpha$$

- $\beta \geq_j \alpha \Rightarrow \pi_{\beta, \alpha}: \kappa \rightarrow \kappa$ is such that $j(\pi_{\beta, \alpha})(\beta) = \alpha$

For the basic properties $E(0)$ we refer to [5] where it is called ‘‘nice system’’ there. Now suppose that we have defined the sequence $\langle E(\tau'): \tau' < \tau \rangle$. If $\langle E(\tau'): \tau' < \tau \rangle \notin M^*$ we stop the construction and set

$$\forall \alpha \in A, \bar{E}_\alpha = \langle \alpha, E(0), \dots, E(\tau'), \dots : \tau' < \tau \rangle$$

and call \bar{E}_α an extender sequence of length τ ($l(\bar{E}_\alpha) = \tau$).

If $\langle E(\tau'): \tau' < \tau \rangle \in M^*$ then we define an extender sequence (with projections)

$$E(\tau) = \langle \langle E_{\langle \alpha, E(\tau'): \tau' < \tau \rangle}(\tau) : \alpha \in A \rangle, \langle \pi_{\langle \beta, E(\tau'): \tau' < \tau \rangle, \langle \alpha, E(\tau'): \tau' < \tau \rangle}(\langle v, d \rangle) : \beta, \alpha \in A, \beta \geq_j \alpha \rangle \rangle$$

on V_κ by:

- $X \in E_{\langle \alpha, E(\tau'): \tau' < \tau \rangle}(\tau) \Leftrightarrow \langle \alpha, E(\tau'): \tau' < \tau \rangle \in j(X)$,
- for $\beta \geq_j \alpha$ in A , $\pi_{\langle \beta, E(\tau'): \tau' < \tau \rangle, \langle \alpha, E(\tau'): \tau' < \tau \rangle}(\langle v, d \rangle) = \langle \pi_{\beta, \alpha}(v), d \rangle$

Note that $E_{\langle \alpha, E(\tau'): \tau' < \tau \rangle}(\tau)$ concentrates on pairs of the form $\langle v, d \rangle$ where $v < \kappa$ and d is an extender sequence. This makes the above definition well-defined.

We let the construction run until it stops due to the extender sequence not being in M^* .

DEFINITION 2.15. (1) $\bar{\mu}$ is an extender sequence if there are $j: V^* \rightarrow M^*$ and \bar{v} such that \bar{v} is an extender sequence derived from j as above (i.e., $\bar{v} = \bar{E}_\alpha$ for some α) and $\bar{\mu} = \bar{v} \upharpoonright \tau$ for some $\tau \leq l(\bar{v})$,

(2) $\kappa(\bar{\mu})$ is the ordinal of the beginning of the sequence (i.e., $\kappa(\bar{E}_\alpha) = \alpha$),

(3) $\kappa^0(\bar{\mu}) = (\kappa(\bar{\mu}))^0$ (i.e., $\kappa^0(\bar{E}_\alpha) = \kappa$),

(4) The sequence $\langle \bar{\mu}_1, \dots, \bar{\mu}_n \rangle$ of extender sequences is 0 -increasing if $\kappa^0(\bar{\mu}_1) < \dots < \kappa^0(\bar{\mu}_n)$,

(5) The extender sequence $\bar{\mu}$ is permitted to a 0 -increasing sequence $\langle \bar{\mu}_1, \dots, \bar{\mu}_n \rangle$ of extender sequences if $\kappa^0(\bar{\mu}_n) < \kappa^0(\bar{\mu})$,

(6) $X \in \bar{E}_\alpha \Leftrightarrow \forall \zeta < l(\bar{E}_\alpha), X \in E_\alpha(\zeta)$,

(7) $\bar{E} = \langle \bar{E}_\alpha : \alpha \in A \rangle$ is an extender sequence system if there is $j: V^* \rightarrow M^*$ such that each \bar{E}_α is derived from j as above and $\forall \alpha, \beta \in A, l(\bar{E}_\alpha) = l(\bar{E}_\beta)$. Call this common length, the length of \bar{E} , $l(\bar{E})$,

(8) For an extender sequence $\bar{\mu}$, we use $\bar{E}(\bar{\mu})$ for the extender sequence system containing $\bar{\mu}$ (i.e., $\bar{E}(\bar{E}_\alpha) = \bar{E}$),

(9) $\text{dom}(\bar{E}) = A$,

(10) $\bar{E}_\beta \geq_{\bar{E}} \bar{E}_\alpha \Leftrightarrow \beta \geq_j \alpha$.

Finding generic filters

Start with GCH and construct an extender sequence system $\bar{E} = \langle \bar{E}_\alpha : \alpha \in \text{dom } \bar{E} \rangle$ where $\text{dom } \bar{E} = [\kappa, \kappa^{++})$ and $l(\bar{E}) = \kappa^+$ such that $j_{\bar{E}}: V^* \rightarrow M_{\bar{E}}^* \supseteq V_{\kappa^{++}}^*$.

We may suppose that \bar{E} is derived from an elementary embedding $j: V^* \rightarrow M^*$. Consider the following elementary embeddings $\forall \tau' < \tau < l(\bar{E})$

$$\begin{aligned} j_\tau: V^* &\rightarrow M_\tau^* \simeq \text{Ult}(V^*, E(\tau)), \\ k_\tau(j_\tau(f)(\bar{E}_\alpha \upharpoonright \tau)) &= j(f)(\bar{E}_\alpha \upharpoonright \tau), \\ i_{\tau', \tau}(j_{\tau'}(f)(\bar{E}_\alpha \upharpoonright \tau')) &= j_\tau(f)(\bar{E}_\alpha \upharpoonright \tau'), \\ \langle M_{\bar{E}}^*, i_{\tau, \bar{E}} \rangle &= \lim \text{dir} \langle \langle M_\tau^* \mid \tau < l(\bar{E}) \rangle, \langle i_{\tau', \tau} \mid \tau' \leq \tau < l(\bar{E}) \rangle \rangle. \end{aligned}$$

We restrict $l(\bar{E})$ by demanding $\forall \tau < l(\bar{E}) \bar{E} \upharpoonright \tau \in M_\tau^*$.

Thus we get the following commutative diagram.

$$\begin{array}{ccccc} V^* & \xrightarrow{j_{\bar{E}}} & & \xrightarrow{j} & M^* \\ & \searrow j_{\tau'} & & \nearrow k_{\tau'} & \uparrow k_{\bar{E}} \\ & & M_{\tau'}^* & & M_{\bar{E}}^* \\ & \searrow j_\tau & & \nearrow k_\tau & \\ & & M_\tau^* = \text{Ult}(V^*, E(\tau)) & & \\ & & \xrightarrow[i_{\tau', \tau}]{} & & \\ & & \xrightarrow[i_{\tau, \bar{E}}]{} & & \end{array}$$

Note that

- the critical point of elementary embeddings originating in V^* is κ ,
- the critical point of elementary embeddings originating in other models is κ^{+3} as computed in that model.

Thus we get

$$\begin{aligned} \text{crit } i_{\tau', \tau} &= \text{crit } k_{\tau'} = \text{crit } i_{\tau', \bar{E}} = (\kappa^{+3})_{M_{\tau'}^*}, \\ \text{crit } k_\tau &= \text{crit}(i_{\tau, \bar{E}}) = (\kappa^{+3})_{M_\tau^*}, \\ \text{crit } k_{\bar{E}} &= (\kappa^{+3})_{M_{\bar{E}}^*}. \end{aligned}$$

Each of these models catches $V_{\kappa+2}^{M^*} = V_{\kappa+2}^*$ hence compute κ^{++} to be the same ordinal in all models. The larger τ is the more resemblance there is between M_τ^* and M^* , and hence with V^* towards $V_{\kappa+3}^*$. This can be observed by noting that

$$\kappa_{M_{\tau'}^*}^{+3} < j_{\tau'}(\kappa) < \kappa_{M_\tau^*}^{+3} < j_\tau(\kappa) < \kappa_{M_{\bar{E}}^*}^{+3} \leq \kappa_{M^*}^{+3} \leq \kappa^{+3}.$$

We also factor through the normal ultrafilter to get the following commutative diagram

$$\begin{array}{ccc} V^* & \xrightarrow{j_{\bar{E}}} & M_{\bar{E}}^* \\ \downarrow i_U & \nearrow j_\tau & \uparrow i_{\tau, \bar{E}} \\ N^* \simeq \text{Ult}(V^*, U) & \xrightarrow[i_{U, \tau}]{} & M_\tau^* \end{array} \quad \begin{aligned} U &= E_\kappa(0), \\ i_U: V^* &\rightarrow N^* \simeq \text{Ult}(V^*, U), \\ i_{U, \tau}(i_U(f)(\kappa)) &= j_\tau(f)(\kappa), \\ i_{U, \bar{E}}(i_U(f)(\kappa)) &= j_{\bar{E}}(f)(\kappa). \end{aligned}$$

N^* catches V^* only up to $V_{\kappa+1}^*$ and we have

$$\kappa^+ < \text{crit } i_{U, \tau} = \text{crit } i_{U, \bar{E}} = \kappa_{N^*}^{++} < i_U(\kappa) < \kappa^{++}.$$

Let $\langle \mathbb{P}_v, \mathbb{Q}_v : v \leq \kappa \rangle$ be the reverse Easton iteration such that for any $v \leq \kappa$:

- If v is accessible then $\Vdash_{-v} \mathbb{Q}_v$ is the trivial forcing $\mathbb{1}$,
- If v is inaccessible then

$$\Vdash_{-v} \mathbb{Q}_v = \text{Add}(v^+, v^{+3}) \times \text{Add}(v^{++}, v^{+4}) \times \text{Add}(v^{+3}, v^{+5}).$$

Then we can obtain the following lifting diagram for some suitable $\mathbb{P}_\kappa * \mathbb{Q}_\kappa$ -generic filter $G_{<\kappa} * H$:

$$\begin{array}{ccccc} V = V^*[G_{<\kappa}][H] & \xrightarrow{j_\tau} & M_{\bar{E}} = M_{\bar{E}}^*[G^{\bar{E}}][H^{\bar{E}}] & & \\ & \searrow^{j_{\tau'}} & \nearrow^{i_{\tau', \bar{E}}} & & \\ & & & & \\ N = N^*[G^U][H^U] & \xrightarrow{i_{U, \tau'}} & M_{\tau'} = M_{\tau'}^*[G^{\tau'}][H^{\tau'}] & \xrightarrow{i_{\tau', \tau}} & M_\tau = M_\tau^*[G^\tau][H^\tau] \\ & & & & \uparrow^{i_{\tau, \bar{E}}} \end{array}$$

Set

$$\begin{aligned} R_U &= (\text{Add}(\kappa^{+4}, i_U(\kappa^{++})))_{N^*[G_{<\kappa}]}, \\ R_\tau &= (\text{Add}(\kappa^{+4}, i_\tau(\kappa^{++})))_{M_\tau^*[G_\tau]}, \\ R_{\bar{E}} &= (\text{Add}(\kappa^{+4}, i_{\bar{E}}(\kappa^{++})))_{M_{\bar{E}}^*[G_{\bar{E}}]} \end{aligned}$$

LEMMA 2.16. *In V there are I_U, I_τ and $I_{\bar{E}}$ such that:*

- I_U is R_U -generic over N ,
- I_τ is R_τ -generic over M_τ ,
- $I_{\bar{E}}$ is $R_{\bar{E}}$ -generic over $M_{\bar{E}}$,
- The generics are so that we have the following lifting diagram

$$\begin{array}{ccccc} & & & & M_{\bar{E}}[I_{\bar{E}}] \\ & & & \nearrow^{i_{\tau', \bar{E}}^*} & \uparrow^{i_{\tau, \bar{E}}^*} \\ N[I_U] & \xrightarrow{i_{U, \tau'}^*} & M_{\tau'}[I_{\tau'}] & \xrightarrow{i_{\tau', \tau}^*} & M_\tau[I_\tau] \end{array}$$

Let i_U^2 be the iterate of i_U . We choose a function, $R(-, -)$, such that

$$R_U = i_U^2(R)(\kappa, i_U(\kappa)).$$

The following lemma gives us everything that we need about the model $N[I_U]$.

- LEMMA 2.17. (a) $N[I_U]$ and N have the same cardinals,
(b) The power function in $N[I_U]$ differs from the power function of N at the following point: $2^{\kappa^{+4}} = i_U(\kappa)^{++}$.

Also The following lifting says everything which we can possibly say about the models $M_\tau[I_\tau]$ and $M_{\bar{E}}[I_{\bar{E}}]$.

$$\begin{array}{ccccc} & & & & M_{\bar{E}}[I_{\bar{E}}] \\ & & & \nearrow^{i_{\tau', \bar{E}}^*} & \uparrow^{i_{\tau, \bar{E}}^*} \\ N[I_U] & \xrightarrow{i_{U, \tau'}^*} & M_{\tau'}[I_{\tau'}] & \xrightarrow{i_{\tau', \tau}^*} & M_\tau[I_\tau] \end{array}$$

The forcing notion $P_{\bar{E}}$ we define later, due to Merimovich, adds a club to κ . For each ν_1, ν_2 successive points in the club the cardinal structure and power function in the range $[\nu_1^+, \nu_2^{++}]$ of the generic extension is the same as the cardinal structure and power function in the range $[\kappa^+, j_{\bar{E}}(\kappa)^{++}]$ of $M_{\bar{E}}[I_{\bar{E}}]$.

Redefining extender Sequences

We define a new extender sequence system $\bar{F} = \langle \bar{F}_\alpha : \alpha \in \text{dom}(\bar{F}) \rangle$ by:

- $\text{dom}(\bar{F}) = \text{dom}(\bar{E})$,
- $1(\bar{F}) = 1(\bar{E})$
- $\leq_{\bar{F}} = \leq_{\bar{E}}$,
- $F(0) = E(0)$,
- $I(\tau) = I_\tau$,
- $\forall 0 < \tau < 1(\bar{F}), F(\tau) = \langle \langle F_\alpha(\tau) : \alpha \in \text{dom}(\bar{F}) \rangle, \langle \pi_{\beta,\alpha} : \beta, \alpha \in \text{dom}(\bar{F}), \beta \geq_{\bar{F}} \alpha \rangle \rangle$ is such that

$$X \in F_\alpha(\tau) \Leftrightarrow \langle \alpha, F(0), I(0), \dots, F(\tau'), I(\tau'), \dots : \tau' < \tau \rangle \in j_{\bar{E}}(X),$$

and

$$\pi_{\beta,\alpha}(\langle \zeta, d \rangle) = \langle \pi_{\beta,\alpha}(\zeta), d \rangle,$$

- $\forall \alpha \in \text{dom}(\bar{F}), \bar{F}_\alpha = \langle \alpha, F(\tau), I(\tau) : \tau < 1(\bar{F}) \rangle$.

Also let $I(\bar{F})$ be the filter generated by $\bigcup_{\tau < 1(\bar{F})} i''_{\tau, \bar{E}} I(\tau)$. Then $I(\bar{F})$ is $\mathbb{R}_{\bar{E}}$ -generic over $M_{\bar{E}}$.

From now on we work with this new definition of extender sequence system and use \bar{E} to denote it.

DEFINITION 2.18. (1) $T \in \bar{E}_\alpha \Leftrightarrow \forall \zeta < 1(\bar{E}_\alpha), T \in E_\alpha(\zeta)$,

(2) $T \restriction \bar{\nu} = T \setminus V_{\kappa^0(\bar{\nu})}^*$,

(3) $T \restriction \bar{\nu} = T \cap V_{\kappa^0(\bar{\nu})}^*$.

Definition of the forcing notion $\mathbb{P}_{\bar{E}}$

This forcing notion is the forcing notion of [9]. We give it in detail for completeness. First we define a forcing notion $\mathbb{P}_{\bar{E}}^*$.

DEFINITION 2.19. A condition p in $\mathbb{P}_{\bar{E}}^*$ is of the form

$$p = \{ \langle \bar{\gamma}, p^{\bar{\gamma}} \rangle : \bar{\gamma} \in s \} \cup \{ \langle \bar{E}_\alpha, T, f, F \rangle \}$$

where

(1) $s \in [\bar{E}]^{\leq \kappa}$, $\min \bar{E} = \bar{E}_\kappa \in s$,

(2) $p^{\bar{E}_\kappa} \in V_{\kappa^0(\bar{E})}^*$ is an extender sequence such that $\kappa(p^{\bar{E}_\kappa})$ is inaccessible (we allow $p^{\bar{E}_\kappa} = \emptyset$). Write p^0 for $p^{\bar{E}_\kappa}$,

(3) $\forall \bar{\gamma} \in s \setminus \{ \min(s) \}, p^{\bar{\gamma}} \in [V_{\kappa^0(\bar{E})}^*]^{\leq \omega}$ is a 0 -increasing sequence of extender sequences and $\max \kappa(p^{\bar{\gamma}})$ is inaccessible,

(4) $\forall \bar{\gamma} \in s, \kappa(p^0) \leq \max \kappa(p^{\bar{\gamma}})$,

(5) $\forall \bar{\gamma} \in s, \bar{E}_\alpha \geq \bar{\gamma}$,

(6) $T \in \bar{E}_\alpha$,

(7) $\forall \bar{\nu} \in T, |\{ \bar{\gamma} \in s : \bar{\nu} \text{ is permitted to } p^{\bar{\gamma}} \}| \leq \kappa^0(\bar{\nu})$,

(8) $\forall \bar{\beta}, \bar{\gamma} \in s, \forall \bar{\nu} \in T$, if $\bar{\beta} \neq \bar{\gamma}$ and $\bar{\nu}$ is permitted to $p^{\bar{\beta}}, p^{\bar{\gamma}}$, then $\pi_{\bar{E}_\alpha, \bar{\beta}}(\bar{\nu}) \neq \pi_{\bar{E}_\alpha, \bar{\gamma}}(\bar{\nu})$,

- (9) f is a function such that
- (9.1) $\text{dom}(f) = \{\bar{v} \in T : l(\bar{v}) = 0\}$,
 - (9.2) $f(v_1) \in R(\kappa(p^0), v_1^0)$. If $p^0 = \emptyset$, then $f(v_1) = \emptyset$,
- (10) F is a function such that
- (10.1) $\text{dom}(F) = \{\langle \bar{v}_1, \bar{v}_2 \rangle \in T^2 : l(\bar{v}_1) = l(\bar{v}_2) = 0\}$,
 - (10.2) $F(v_1, v_2) \in R(v_1^0, v_2^0)$,
 - (10.3) $j_{\bar{E}}^2(\alpha, j_{\bar{E}}(\alpha)) \in I(\bar{E})$.

We write $\text{mc}(p)$, $\text{supp}(p)$, T^p , f^p and F^p for \bar{E}_α , s , T , f and F respectively.

DEFINITION 2.20. For $p, q \in \mathbb{P}_{\bar{E}}^*$, we say p is a Prikry extension of q ($p \leq^* q$ or $p \leq^0 q$) iff

- (1) $\text{supp}(p) \supseteq \text{supp}(q)$,
- (2) $\forall \bar{v} \in \text{supp}(q)$, $p^{\bar{v}} = q^{\bar{v}}$,
- (3) $\text{mc}(p) \geq_{\bar{E}} \text{mc}(q)$,
- (4) $\text{mc}(p) >_{\bar{E}} \text{mc}(q) \Rightarrow \text{mc}(q) \in \text{supp}(p)$,
- (5) $\forall \bar{v} \in \text{supp}(p) \setminus \text{supp}(q)$, $\max \kappa^0(p^{\bar{v}}) > \bigcup \bigcup j_{\bar{E}}(f^q)(\kappa(\text{mc}(q)))$,
- (6) $T^p \leq \pi_{\text{mc}(p), \text{mc}(q)}^{-1} T^q$,
- (7) $\forall \bar{v} \in \text{supp}(q)$, $\forall \bar{v} \in T^p$, if \bar{v} is permitted to $p^{\bar{v}}$, then

$$\pi_{\text{mc}(p), \bar{v}}(\bar{v}) = \pi_{\text{mc}(q), \bar{v}}(\pi_{\text{mc}(p), \text{mc}(q)}(\bar{v})),$$

- (8) $\forall v_1 \in \text{dom}(f^p)$, $f^p(v_1) \leq f^q \circ \pi_{\text{mc}(p), \text{mc}(q)}(v_1)$,
- (9) $\forall \langle v_1, v_2 \rangle \in \text{dom}(F^p)$, $F^p(v_1, v_2) \leq F^q \circ \pi_{\text{mc}(p), \text{mc}(q)}(v_1, v_2)$.

We are now ready to define the forcing notion $\mathbb{P}_{\bar{E}}$.

DEFINITION 2.21. A condition p in $\mathbb{P}_{\bar{E}}$ is of the form

$$p = p_n \widehat{} \dots \widehat{} p_0$$

where

- $p_0 \in \mathbb{P}_{\bar{E}}^*$, $\kappa^0(p_0^0) \geq \kappa^0(\bar{\mu}_1)$,
- $p_1 \in \mathbb{P}_{\bar{\mu}_1}^*$, $\kappa^0(p_1^0) \geq \kappa^0(\bar{\mu}_2)$,
- \vdots
- $p_n \in \mathbb{P}_{\bar{\mu}_n}^*$.

and $\langle \bar{\mu}_n, \dots, \bar{\mu}_1, \bar{E} \rangle$ is a 0-increasing sequence of extender sequence systems, that is $\kappa^0(\bar{\mu}_n) < \dots < \kappa^0(\bar{\mu}_1) < \kappa^0(\bar{E})$.

DEFINITION 2.22. For $p, q \in \mathbb{P}_{\bar{E}}$, we say p is a Prikry extension of q ($p \leq^* q$ or $p \leq^0 q$) iff

$$p = p_n \widehat{} \dots \widehat{} p_0$$

$$q = q_n \widehat{} \dots \widehat{} q_0$$

where

- $p_0, q_0 \in \mathbb{P}_{\bar{E}}^*$, $p_0 \leq^* q_0$,
- $p_1, q_1 \in \mathbb{P}_{\bar{\mu}_1}^*$, $p_1 \leq^* q_1$,
- \vdots
- $p_n, q_n \in \mathbb{P}_{\bar{\mu}_n}^*$, $p_n \leq^* q_n$.

Now let $p \in \mathbb{P}_{\bar{E}}$ and $\bar{v} \in T^p$. We define $p_{(\bar{v})}$ a one element extension of p by \bar{v} .

DEFINITION 2.23. Let $p \in \mathbb{P}_{\bar{E}}$, $\bar{v} \in T^p$ and $\kappa^0(\bar{v}) > \bigcup \bigcup j_{\bar{E}}(f^{p, \text{Col}})(\kappa(\text{mc}(p)))$, where $f^{p, \text{Col}}$ is the collapsing part of f^p . Then $p_{\langle \bar{v} \rangle} = p_1 \hat{\ } p_0$ where

- (1) $\text{supp}(p_0) = \text{supp}(p)$,
- (2) $\forall \bar{\gamma} \in \text{supp}(p_0)$, $p_0^{\bar{\gamma}} = \begin{cases} \pi_{\text{mc}(p), \bar{\gamma}}(\bar{v}) & \text{if } \bar{v} \text{ is permitted to } p^{\bar{\gamma}} \\ & \text{and } 1(\bar{v}) > 0, \\ \pi_{\text{mc}(p), \bar{\gamma}}(\bar{v}) & \text{if } \bar{v} \text{ is permitted to } p^{\bar{\gamma}}, \\ & 1(\bar{v}) = 0 \text{ and } \bar{\gamma} = \bar{E}_{\kappa}, \\ p^{\bar{\gamma}} \hat{\ } \langle \pi_{\text{mc}(p), \bar{\gamma}}(\bar{v}) \rangle & \text{if } \bar{v} \text{ is permitted to } p^{\bar{\gamma}}, \\ & 1(\bar{v}) = 0 \text{ and } \bar{\gamma} \neq \bar{E}_{\kappa}, \\ p^{\bar{\gamma}} & \text{otherwise.} \end{cases}$
- (3) $\text{mc}(p_0) = \text{mc}(p)$,
- (4) $T^{p_0} = T^p \setminus \bar{v}$,
- (5) $\forall v_1 \in T^{p_0}$, $f^{p_0}(v_1) = F^p(\kappa(\bar{v}), v_1)$,
- (6) $F^{p_0} = F^p$,
- (7) if $1(\bar{v}) > 0$ then
 - (7.1) $\text{supp}(p_1) = \{\pi_{\text{mc}(p), \bar{\gamma}}(\bar{v}) : \bar{\gamma} \in \text{supp}(p) \text{ and } \bar{v} \text{ is permitted to } p^{\bar{\gamma}}\}$,
 - (7.2) $p_1^{\pi_{\text{mc}(p), \bar{\gamma}}(\bar{v})} = p^{\bar{\gamma}}$,
 - (7.3) $\text{mc}(p_1) = \bar{v}$,
 - (7.4) $T^{p_1} = T^p \upharpoonright \bar{v}$,
 - (7.5) $f^{p_1} = f^p \upharpoonright \bar{v}$,
 - (7.6) $F^{p_1} = F^p \upharpoonright \bar{v}$,
- (8) if $1(\bar{v}) = 0$ then
 - (8.1) $\text{supp}(p_1) = \{\pi_{\text{mc}(p), 0}(\bar{v})\}$,
 - (8.2) $p_1^{\pi_{\text{mc}(p), 0}(\bar{v})} = p^{\bar{E}_{\kappa}}$,
 - (8.3) $\text{mc}(p_1) = \bar{v}^0$,
 - (8.4) $T^{p_1} = \emptyset$,
 - (8.5) $f^{p_1} = f^p(\kappa(\bar{v}))$,
 - (8.6) $F^{p_1} = \emptyset$.

We use $(p_{\langle \bar{v} \rangle})_0$ and $(p_{\langle \bar{v} \rangle})_1$ for p_0 and p_1 respectively. We also let $p_{\langle \bar{v}_1, \bar{v}_2 \rangle} = (p_{\langle \bar{v}_1 \rangle})_1 \hat{\ } (p_{\langle \bar{v}_1 \rangle})_0 \langle \bar{v}_2 \rangle$ and so on.

The above definition is the key step in the definition of the forcing relation \leq .

DEFINITION 2.24. For $p, q \in \mathbb{P}_{\bar{E}}$, we say p is a 1-point extension of q ($p \leq^1 q$) iff

$$\begin{aligned} p &= p_{n+1} \hat{\ } \dots \hat{\ } p_0 \\ q &= q_n \hat{\ } \dots \hat{\ } q_0 \end{aligned}$$

and there is $0 \leq k \leq n$ such that

- $\forall i < k$, $p_i, q_i \in \mathbb{P}_{\bar{\mu}_i}^*$, $p_i \leq^* q_i$,
- $\exists \bar{v} \in T^{q_k}$, $(p_{k+1}) \hat{\ } p_k \leq^* (q_k)_{\langle \bar{v} \rangle}$
- $\forall i > k$, $p_{i+1}, q_i \in \mathbb{P}_{\bar{\mu}_i}^*$, $p_{i+1} \leq^* q_i$,

where $\bar{\mu}_0 = \bar{E}$.

DEFINITION 2.25. For $p, q \in \mathbb{P}_{\bar{E}}$, we say p is an n -point extension of q ($p \leq^n q$) iff there are p^n, \dots, p^0 such that

$$p = p^n \leq^1 \dots \leq^1 p^0 = q.$$

DEFINITION 2.26. For $p, q \in \mathbb{P}_{\bar{E}}$, we say p is an extension of q ($p \leq q$) iff there is some n such that $p \leq^n q$.

Suppose that H is $\mathbb{P}_{\bar{E}}$ -generic over V . Then all of the results in [9], except the following, work as well.

Replace Claim 10.3 with:

CLAIM 2.27. Assume $l(\bar{E}) > 0$. Then $\|_{-\mathbb{P}_{\bar{E}}} \uparrow 2^\kappa = \kappa^{++}$.

Replace Claim 10.4 with:

CLAIM 2.28. Assume $l(\bar{E}) > 0$. Then $\|_{-\mathbb{P}_{\bar{E}}} \uparrow 2^{\kappa^+} = \kappa^{+3}$, $2^{\kappa^{++}} = \kappa^{+4}$, $2^{\kappa^{+3}} = \kappa^{+5}$.

Replace Claim 10.6 with the following:

CLAIM 2.29. Let G be $\mathbb{P}_{\bar{E}}$ -generic with $p = p_l * \dots * p_k * \dots * p_0 \in G$ and \bar{e} be such that $p_{l,k} \in \mathbb{P}_{\bar{e}}$ and $l(\bar{e}) = 0$. Let $v = \kappa(p_k^0)$. Then, in $V[G]$, all cardinals in $[v^+, \kappa^0(\bar{e})^{++}]$ are preserved and $2^{v^+} = v^{+3}$, $2^{v^{++}} = v^{+4}$, $2^{v^{+3}} = v^{+5}$, $2^{v^{+4}} = \kappa^0(\bar{e})^{++}$.

Now the proof of our main theorem goes as follows: Let $p^* \in \mathbb{P}_{\bar{E}}^*$ such that $\kappa(p^{*0})$ is inaccessible and G be $\mathbb{P}_{\bar{E}}$ -generic with $p^* \in G$. Set

$$M = \bigcup \{p_0^{\bar{E}\kappa} : p \in G\},$$

$$C = \bigcup \{\kappa(p_0^{\bar{E}\kappa}) : p \in G\}.$$

Note that M is a Radin generic sequence for the extender sequence \bar{E}_κ , hence $C \subset \kappa$ is a club. Also the first ordinal in this club is $\lambda = \kappa(p^{*0})$. We first investigate the range (λ, κ) in $V[G]$. Note that, by 10.5 from [9], for $\bar{e} \in M$ it is enough to use $\mathbb{P}_{\bar{e}}$ in order to understand $V_{\kappa^0(\bar{e})}^{V[G]}$. So let $\mu \in (\lambda, \kappa)$.

- $\mu \in \lim C$: Then there is $\bar{e} \in M$ such that $l(\bar{e}) > 0$ and $\kappa(\bar{e}) = \mu$. By 10.7 from [9] μ remains a cardinal and by Claim 2.27, $2^\mu = \mu^{++}$,
- $\mu \in C \setminus \lim C$: Then there is $\bar{e} \in M$ such that $l(\bar{e}) = 0$ and $\kappa(\bar{e}) = \mu$. Let $\mu_2 \in C$ be the C -immediate predecessor of μ . By Claim 2.29 we have all cardinals in $[\mu_2^+, \mu^{++}]$ are preserved and $2^{\mu_2^+} = \mu_2^{+3}$, $2^{\mu_2^{++}} = \mu_2^{+4}$, $2^{\mu_2^{+3}} = \mu_2^{+5}$, $2^{\mu_2^{+4}} = \mu^{++}$. In particular $2^\mu \geq \mu^{++}$.
- $\mu \notin C$: Then there are μ_2 and μ_1 two successive points in C such that $\mu \in (\mu_2, \mu_1)$. By above, if $\mu \in \{\mu_2^+, \mu_2^{++}, \mu_2^{+3}\}$ then $2^\mu = \mu^{++}$, and if $\mu \in (\mu_2^{+3}, \mu_1)$ then $2^\mu \geq \mu_1^{++} > \mu^+$.

We may note that the above argument also shows that all cardinals $> \lambda$ are preserved in $V[G]$, and since forcing with $\mathbb{P}_{\bar{E}}$ adds no new bounded subsets to λ , hence all cardinals are preserved in $V[G]$. Finally let H be $\text{Add}(\aleph_0, \lambda^{++})_{V[G]}$ -generic over $V[G]$. It is then clear that in $V[G][H]$ all cardinals are preserved and that the GCH fails everywhere below (and at) κ .

2.4. Proof of Theorem 1.1. Suppose that K is the canonical inner model for a $H(\kappa^{+3})$ -strong cardinal κ . Let S be the discrete set of measurable cardinals below κ in K which are not limits of measurable cardinals in K and for each $\alpha \in S$ let U_α be the unique normal measure on α in K . Consider the forcing \mathbb{P}_S and let $(x_\alpha : \alpha \in S)$ be \mathbb{P}_S -generic over K . By Theorem 2.2, κ remains $H(\kappa^{+3})$ -strong in $K[(x_\alpha : \alpha \in S)]$, thus we can apply Theorem 2.10 to find a cofinality-preserving forcing \mathbb{P} which adds a real R over $K[(x_\alpha : \alpha \in S)]$ such that $K[(x_\alpha : \alpha \in S)][R] = K[R]$ and κ remains $H(\kappa^{+3})$ -strong in $K[R]$. By Theorem 2.13 there exists a

cardinal-preserving forcing \mathbb{Q} and a subset $C \subseteq S$, \mathbb{Q} -generic over $K[R]$ such that in $K[R][C]$, κ remains inaccessible and for every $\lambda < \kappa$, $2^\lambda > \lambda^+$. We now define a new sequence $(y_\alpha : \alpha \in S)$ by

$$y_\alpha = \begin{cases} x_\alpha & \text{if } \alpha \in C, \\ x_\alpha - \{\min(x_\alpha)\} & \text{otherwise.} \end{cases}$$

By Lemma 3, $(y_\alpha : \alpha \in S)$ is \mathbb{P}_S -generic over K . Let $W = V_\kappa^{K[(y_\alpha : \alpha \in S)]}$ and $V = W[R] = V_\kappa^{K[R][C]}$. Then the pair (W, V) is as required. Theorem 1.1 follows. \dashv

§3. Proof of Theorem 1.2.

3.1. A class version of the Prikry product. Let S be a class of measurable cardinals which is discrete. Fix normal measures U_α on α for α in S . We define a class version of the Prikry product as follows.

Conditions in \mathbb{P}_S are triples $p = (X^p, S^p, H^p)$ such that

- (1) X^p is a subset of S ,
- (2) $S^p \in \prod_{\alpha \in X^p} [\alpha \setminus \text{sup}(S \cap \alpha)]^{<\omega}$,
- (3) $H^p \in \prod_{\alpha \in X^p} U_\alpha$,
- (4) $\text{supp}(p) = \{\alpha : S^p(\alpha) \neq \emptyset\}$ is finite,
- (5) $\forall \alpha \in X^p, \max S^p(\alpha) < \min H^p(\alpha)$.

Let $p, q \in \mathbb{P}_S$. Then $p \leq q$ (p is an extension of q) iff

- (1) $X^p \supseteq X^q$,
- (2) $\forall \alpha \in X^q, S^p(\alpha)$ is an end extension of $S^q(\alpha)$,
- (3) $\forall \alpha \in X^q, S^p(\alpha) \setminus S^q(\alpha) \subseteq H^q(\alpha)$,
- (4) $\forall \alpha \in X^q, H^p(\alpha) \subseteq H^q(\alpha)$.

We also define an auxiliary relation \leq^* on \mathbb{P}_S as follows. Let $p, q \in \mathbb{P}_S$. Then $p \leq^* q$ (p is a direct or Prikry extension of q) iff

- (1) $X^p \supseteq X^q$,
- (2) $\forall \alpha \in X^q, S^p(\alpha) = S^q(\alpha)$,
- (3) $\forall \alpha \in X^q, H^p(\alpha) \subseteq H^q(\alpha)$.

For $p \leq q$ in \mathbb{P}_S we define the distance function $|p - q|$ to be a function on X^q so that for $\alpha \in X^q, |p - q|(\alpha) = l(S^p(\alpha)) - l(S^q(\alpha))$. Also let $\mathbb{P}_S \upharpoonright X = \{p \in \mathbb{P}_S : X^p \subseteq X\}$. It is clear that for any $X \subseteq S, \mathbb{P}_S \simeq (\mathbb{P}_S \upharpoonright X) \times (\mathbb{P}_S \upharpoonright S \setminus X)$.

LEMMA 3.1. \mathbb{P}_S is *pretame*: Given $p \in \mathbb{P}_S$ and a definable sequence $(D_i : i < \alpha)$ of dense classes below p there exist $q \leq p$ and a sequence $(d_i : i < \alpha) \in V$ such that each $d_i \subseteq D_i$ is *predense* below q .

PROOF. Let $p_0 = p$ and let $\delta_0 > \alpha, \delta_0 \notin S$ be such that $X^{p_0} \subseteq \delta_0$. By repeatedly thinning the measure one sets above δ_0 we can find $p_1 \leq p_0$ and $\delta_1 > \delta_0, \delta_1 \notin S$ such that:

- (1) $X^{p_1} \subseteq \delta_1$,
- (2) p_1 agrees with p_0 below δ_0 ,
- (3) for any $q \leq p_0, q \in \mathbb{P}_S \upharpoonright \delta_0$ and any $i < \alpha$ if q has an extension r meeting D_i which agrees with q below δ_0 , then there is such an $r \in \mathbb{P}_S \upharpoonright \delta_1$ whose measure one sets contain those of p_1 .

Now repeat this ω times, producing p_0, p_1, \dots and let q be the greatest lower bound, obtained in a natural way, so that $q \leq^* p_n$ for each $n \in \omega$. Also for each $i < \alpha$ set

$d_i = D_i \upharpoonright \delta_\omega = \{r \in D_i : X^r \subseteq \delta_\omega\}$, where $\delta_\omega = \sup_{n < \omega} \delta_n$. We show that q and the sequence $(d_i : i < \alpha)$ are as required.

Fix $i < \alpha$. Suppose $r \leq q$, $r \in D_i$. Let n be large enough so that $\text{supp}(r) \cap \delta_\omega \subseteq \delta_n$. At stage $n+1$ we considered $r \upharpoonright \delta_n$ and saw that it has an extension meeting D_i and agreeing with it below δ_n , so it must have such an extension whose measure one sets contain those of p_{n+1} and therefore those of q . This extension is compatible with r and therefore r has an extension which meets d_i , as required. \dashv

It follows from [2], Theorem 2.18, and the above Lemma that the forcing relation is definable. The proof of the following lemma uses ideas from [8].

LEMMA 3.2. $(\mathbb{P}_S, \leq, \leq^*)$ has the Prikry property, i.e., for each sentence ϕ of the forcing language of (\mathbb{P}_S, \leq) , and any $p \in \mathbb{P}_S$ there is $q \leq^* p$ which decides ϕ .

PROOF. Suppose ϕ is a sentence of the forcing language, $p \in \mathbb{P}_S$. Let $p = (X^p, S^p, H^p)$, let ϕ^0 denote $\neg\phi$ and ϕ^1 denote ϕ .

By reflection and by strengthening p in the sense of \leq^* , we may assume that $X^p = \gamma$, where it is dense in $\mathbb{P}_S \cap V_\gamma$ to decide ϕ .

For $\alpha < \gamma$, let \mathcal{S}_α denote the set of S^q where $q \in \mathbb{P}_{X^p \cap \alpha}$. For $s \in \mathcal{S}_\alpha$, set $F_{s,\alpha}(\delta_1, \dots, \delta_n) = i$ iff there is $q \leq p$ such that $X^q = \gamma$, $S^q \upharpoonright (X^p \setminus \{\alpha\}) = s$, $S^q(\alpha) = S^p(\alpha) * (\delta_1, \dots, \delta_n)$ and $q \Vdash \phi^i$. Set $F_{s,\alpha}(\delta_1, \dots, \delta_n) = 2$ iff no such q exists.

Let $H(s, \alpha) \subseteq H^p(\alpha)$, $H(s, \alpha) \in U_\alpha$ be homogeneous for $F_{s,\alpha}$, and let $H(\alpha) = \bigcap_{s \in \mathcal{S}_\alpha} H(s, \alpha)$. Then $H(\alpha) \in U_\alpha$ (as S is discrete) and we can set $q = (X^q, S^q, H^q)$, where $X^q = X^p$, $S^q = S^p$ and $H^q(\alpha) = H(\alpha)$ for $\alpha \in X^q$.

It is clear that $q \leq^* p$. We show that there is a \leq^* extension of q which decides ϕ . Suppose not. Let $r \leq q$ be such that r decides ϕ . Suppose for example that $r \Vdash \phi$. We may further suppose that r is so that $|r - q|$ is minimal, and that $X^r = \gamma$. We note that $|r - q|$ is not the 0-function.

Let $\alpha < \gamma$ be the maximum of $\text{supp}(r)$, and let r_0 be obtained from r by replacing $S^r(\alpha)$ with $S^p(\alpha)$. We claim that r_0 already decides ϕ . For let $w \leq r_0$, such that $w \Vdash \neg\phi$. Let n denote $|S^r(\alpha)|$; We may assume that $|S^w(\alpha)| \geq n$. Let s denote S^{r_0} and $\delta_1, \dots, \delta_k$ denote $S^w(\alpha)$. Then r witnesses that $F_{s,\alpha}$ has constant value 1 on $[H(s, \alpha)]^n$. Moreover, $\{\delta_1, \dots, \delta_n\} \in [H(s, \alpha)]^n$. So there is r_1 such that $r_1 \Vdash \phi$, $S^{r_1} \upharpoonright (X^p \setminus \{\alpha\}) = s$ and $S^{r_1}(\alpha) = \{\delta_1, \dots, \delta_n\}$. It is easily checked that S^{r_1} and $S^w \upharpoonright \gamma$ are compatible, so r_1 and w are compatible, contradicting that they decide ϕ differently. Thus, r_0 already decides ϕ , contradicting the minimality of r . \dashv

We can now easily show that \mathbb{P}_S preserves cardinals and the GCH. Also as in the usual Prikry product a \mathbb{P}_S -generic is uniquely determined by a sequence $\langle x_\alpha : \alpha \in S \rangle$ where each x_α is an ω -sequence cofinal in α . As before, with a slight abuse of terminology, we say that $\langle x_\alpha : \alpha \in S \rangle$ is \mathbb{P}_S -generic. The following is an analogue of Lemma 2.1 and its proof is essentially the same.

LEMMA 3.3. (a) *The sequence $\langle x_\alpha : \alpha \in S \rangle$ obeys the following ‘‘geometric property’’: if $\langle X_\alpha : \alpha \in S \rangle$ is a definable class (in V) and $X_\alpha \in U_\alpha$ for each $\alpha \in S$ then $\bigcup_{\alpha \in S} x_\alpha \setminus X_\alpha$ is finite.*

(b) *Conversely, suppose that $\langle y_\alpha : \alpha \in S \rangle$ is a sequence (in any outer model of V) satisfying the geometric property stated above. Then $\langle y_\alpha : \alpha \in S \rangle$ is \mathbb{P}_S -generic over V .*

3.2. Proof of Theorem 1.2. Suppose M is a model of $\text{ZFC} + \text{GCH} +$ there exists a proper class of measurable cardinals. Let S be a discrete class of measurable cardinals and for each $\alpha \in S$ fix a normal measure U_α over α . Consider the forcing \mathbb{P}_S and let $\langle x_\alpha : \alpha \in S \rangle$ be \mathbb{P}_S -generic over M . By Jensen's coding theorem (see [2]) there exists a cofinality-preserving forcing \mathbb{P} which adds a real R over $M[\langle x_\alpha : \alpha \in S \rangle]$ such that $M[\langle x_\alpha : \alpha \in S \rangle][R] = L[R]$. In $L[R]$ define the function $F^*: \text{REG} \rightarrow \text{CARD}$ by

$$F^*(\kappa) = \begin{cases} F(\kappa) & \text{if } cf F(\kappa) \neq \omega, \\ F(\kappa)^+ & \text{if } cf F(\kappa) = \omega. \end{cases}$$

Let \mathbb{R} be the Easton forcing corresponding to F^* for blowing up the power of each regular cardinal κ to $F^*(\kappa)$ and let $C \subseteq S$ be \mathbb{R} -generic over $L[R]$.

We now define a new sequence $\langle y_\alpha : \alpha \in S \rangle$ by

$$y_\alpha = \begin{cases} x_\alpha & \text{if } \alpha \in C, \\ x_\alpha - \{\min(x_\alpha)\} & \text{otherwise.} \end{cases}$$

Using lemma 3.3, $\langle y_\alpha : \alpha \in S \rangle$ is \mathbb{P}_S -generic over M . Let $W = M[\langle y_\alpha : \alpha \in S \rangle]$, and $V = M[\langle y_\alpha : \alpha \in S \rangle, R]$. Then the pair (W, V) is as required. This completes the proof of Theorem 1.2. \dashv

§4. A few more results. The following is proved in [1]. We give a proof for completeness.

LEMMA 4.1. *Suppose that R is a real in V . Then there are two reals a and b such that*

- (a) *a and b are Cohen generic over V ,*
- (b) *all of the models $V, V[a], V[b]$ and $V[a, b]$ have the same cardinals,*
- (c) *$R \in L[a, b]$.*

PROOF. Working in V , let a^* be $\text{Add}(\omega, 1)$ -generic over V and let b^* be $\text{Add}(\omega, 1)$ -generic over $V[a^*]$, where $\text{Add}(\omega, 1)$ is the Cohen forcing for adding a new real. Note that $V[a^*]$ and $V[a^*, b^*]$ are cardinal preserving generic extensions of V . Working in $V[a^*, b^*]$ let $\langle k_N : N < \omega \rangle$ be an increasing enumeration of $\{N : a^*(N) = 0\}$ and let $a = a^*$ and $b = \{N : b^*(N) = a^*(N) = 1\} \cup \{k_N : R(N) = 1\}$. Then clearly $R \in L[\langle k_N : N < \omega \rangle, b] \subseteq L[a, b]$ as $R = \{N : k_N \in b\}$.

We show that b is $\text{Add}(\omega, 1)$ -generic over V . It suffices to prove the following

- (*) For any $(p, q) \in \text{Add}(\omega, 1) * \text{Add}(\omega, 1)$ and any dense open subset $D \in V$ of $\text{Add}(\omega, 1)$ there is $(\bar{p}, \bar{q}) \leq (p, q)$ such that $(\bar{p}, \bar{q}) \Vdash \dot{b}$ extends some element of D .

Let (p, q) and D be as above. By extending one of p or q if necessary, we can assume that $lh(p) = lh(q)$. Let $\langle k_N : N < M \rangle$ be an increasing enumeration of $\{N < lh(p) : p(N) = 0\}$. Let $s : lh(p) \rightarrow 2$ be such that considered as a subset of ω ,

$$s = \{N < lh(p) : p(N) = q(N) = 1\} \cup \{k_N : N < M, R(N) = 1\}.$$

Let $t \in D$ be such that $t \leq s$. Extend p, q to \bar{p}, \bar{q} of length $lh(t)$ so that for i in the interval $[lh(s), lh(t))$

- $\bar{p}(i) = 1$,
- $\bar{q}(i) = 1$ iff $i \in t$.

Then

$$t = \{N < lh(t) : \bar{p}(N) = \bar{q}(N) = 1\} \cup \{k_N : N < M, R(N) = 1\}.$$

Thus $(\bar{p}, \bar{q}) \Vdash \neg \dot{b}$ extends $t \Vdash$ and $(*)$ follows. This completes the proof of the Lemma. \dashv

Using the above Lemma and Theorems 1.1 and 1.2 we can obtain the following.

THEOREM 4.2. *Assume the consistency of an $H(\kappa^{+3})$ -strong cardinal κ . Then there exist a model W of ZFC and two reals a and b such that*

- (a) *The models $W, W[a]$ and $W[b]$ have the same cardinals and satisfy the GCH,*
- (b) *GCH fails at all infinite cardinals in $W[a, b]$.*

THEOREM 4.3. *Let M be a model of ZFC + GCH + there exists a proper class of measurable cardinals. In M let $F: \text{REG} \rightarrow \text{CARD}$ be an Easton function. Then there exist a cardinal preserving generic extension W of M and two reals a and b such that*

- (a) *The models $W, W[a], W[b]$ and $W[a, b]$ have the same cardinals,*
- (b) *$W[a]$ and $W[b]$ satisfy GCH,*
- (c) *$W[a, b] \models \ulcorner \forall \kappa \in \text{REG}, 2^\kappa \geq F(\kappa) \urcorner$.*

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