INDEPENDENCE OF HIGHER KUREPA HYPOTHESES

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Abstract. We study the Generalized Kurepa Hypothesis introduced by Chang. We show that relative to the existence of an inaccessible cardinal the Gap-n-Kurepa hypothesis does not follow from the Gap-m-Kurepa hypothesis for m different from n. The use of an inaccessible is necessary for this result.

1. Introduction

In this paper we study the Generalized Kurepa Hypothesis introduced by Chang (see Chapter VII of [1]). We show that relative to the existence of an inaccessible cardinal the Gap$-n-$Kurepa hypothesis does not follow from the Gap$-m-$Kurepa hypothesis for m different from n. The use of an inaccessible is necessary for this result.

Definition 1.1. (a) For infinite cardinals $\lambda < \kappa, KH(\kappa, \lambda)$ is the following principle: There exists a family $\mathcal{F}$ of subsets of κ such that:

(i) $Card(\mathcal{F}) \geq \kappa^+$,

(ii) for all $x \in [\kappa]^\lambda$, $Card(\mathcal{F}|x) \leq \lambda$, where $\mathcal{F}|x = \{t \cap x : t \in \mathcal{F}\}$.

(b) For infinite cardinals $\lambda \leq \kappa, KH(\kappa, < \lambda)$ is the following principle: There exists a family $\mathcal{F}$ of subsets of κ such that:

(i) $Card(\mathcal{F}) \geq \kappa^+$,

(ii) for all $x \in [\kappa]^{<\lambda}$, $Card(\mathcal{F}|x) \leq Card(x) + \aleph_0$.

(c) Let $n \geq 1$. By the Gap$-n-$Kurepa hypothesis we mean the following statement: for all infinite cardinals κ, $KH(\kappa^{+n}, \kappa)$ holds

Remark 1.2. If $V = L$, then $KH(\kappa, < \lambda^+)$ (and hence $KH(\kappa, \lambda)$) holds for all infinite cardinals $\lambda < \kappa, \kappa$ regular (see [1], Chapter VII, Theorems 3.2 and 3.3).

In this paper we prove the following theorem:

Main Theorem. Let $n \geq 1$. The following are equiconsistent:

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(a) There exists an inaccessible cardinal,

(b) GCH+ the Gap$-m-$Kurepa hypothesis holds for all $m \neq n$, but the Gap$-n-$Kurepa hypothesis fails.

Remark 1.3. Our proof shows that if $\lambda < \kappa$ are infinite cardinals, κ regular and $KH(\kappa, \lambda)$ fails, then κ^+ is inaccessible in L (see Lemma 2.5).

Remark 1.4. (b) can be strengthened to the Gap$-m-$Kurepa hypothesis holds for all $m \neq n$, but $KH(\aleph_n, \aleph_0)$ fails (see Lemma 2.4).

2. Proof of the Main Theorem

Con(a) **implies Con**(b)**.** Fix some $n \geq 1$. we show that if there exists an inaccessible cardinal, then in a forcing extension of L, the Gap$-m-$Kurepa hypothesis holds for all $m \neq n$, but the Gap$-n-$Kurepa hypothesis fails.

From now on assume that $V = L$, and let κ be an inaccessible cardinal. We consider two cases:

Case 1. $n = 1$.

Let $\mathbb{P} = Col(\omega_1, < \kappa)$ be the Levy collapse with countable conditions which converts κ into ω_2, and let G be $\mathbb{P}$-generic over V.

Lemma 2.1. The following hold in $V[G]$:

(a) $KH(\aleph_1, \aleph_0)$ fails,

(b) The Gap$-m-$Kurepa hypothesis holds for all $m \geq 2$.

Proof. (a) is a well known result of Silver (see [7]).

(b) Let $m \geq 2$, and let λ be an infinite cardinal in $V[G]$. Let $\mu = (\lambda^{+m})^{V[G]}$. By remark 1.2, there is a $KH(\mu, \lambda)$ family $\mathcal{F}$ in V. We show that it remains a $KH(\lambda^{+m}, \lambda)$ family in $V[G]$. Clearly $Card(\mathcal{F}) = \mu^{+V} = (\lambda^{+m+1})^{V[G]}$. Suppose $x \in ([\mu]^{\lambda})^{V[G]}$. By Jensen‘s covering lemma if $\lambda > \aleph_0$, or by ω_1-closure of $\mathbb{P}$ if $\lambda = \aleph_0$, there is a set $y \subseteq \lambda$ in V such that $x \subseteq y$ and x and y have the same cardinality in $V[G]$. If $\lambda \neq \aleph_1$, then y has $V-$cardinality λ, hence in $V, Card(\mathcal{F} \mid y) \leq \lambda$. It follows that in $V[G], Card(\mathcal{F} \mid x) \leq Card(\mathcal{F} \mid y) \leq \lambda$. If $\lambda = \aleph_1$, then y has $V-$cardinality less than κ, hence in $V, Card(\mathcal{F} \mid y) < \kappa$. It follows that in $V[G], Card(\mathcal{F} \mid y) \leq \aleph_1$, and hence in $V[G], Card(\mathcal{F} \mid x) \leq Card(\mathcal{F} \mid y) \leq \aleph_1 = \lambda$. □

Case 2. $n \geq 2$.

For each i, $0 < i < n$, fix an injection $J_i : [\omega_n]^{\leq \omega_i} \longrightarrow \omega_n$. Let $\mathbb{R} = \mathbb{P} \times \prod_{0<i<n} \mathbb{Q}_i$, where the forcing notions $\mathbb{P}$ and $\mathbb{Q}_i$, $0 < i < n$, are defined as follows.

- $\mathbb{P} = Col(\omega_n, < \kappa)$ is the Levy collapse with conditions of size $< \omega_n$ which converts κ into ω_{n+1}.
- A condition p is in $\mathbb{Q}_i$, $0 < i < n$, iff $p = (X_p, F_p, g_p)$, where

(i-1) X_p is a subset of ω_n of size $\leq \omega_i$,

(i-2) F_p is a subset of ${}^{X_p}2$ of size $\leq \omega_i$,

(i-3) g_p is a 1-1 function from a subset of κ into F_p,

(i-4) F_p is ω_i-closed in the following sense: If $t \in {}^{X_p}2$ and $\langle X_\xi : \xi < \omega_{i-1} \rangle$ is a sequence of subsets of X_p with $X = \bigcup_{\xi < \omega_{i-1}} X_\xi$ such that for all $\xi < \omega_{i-1}$, $J_i(X_\xi) \in X_p$ and $t|X_\xi \in F_p|X_\xi$, then there is $s \in F_p$ such that $s|X = t|X$ and $s|(X_p - X) = o|(X_p - X)$ (=the zero function on $X_p - X$).

For $p, q \in \mathbb{Q}_i$, let $p \leq q$ (p is an extension of q) iff

(i-5) $X_p \supseteq X_q$,

(i-6) $F_q \subseteq F_p|X_q$,

(i-7) $dom(g_p) \supseteq dom(g_q)$,

(i-8) for all $\alpha \in dom(g_q)$, $g_q(\alpha) = g_p(\alpha)|X_q$.

We show that in the generic extension by $\mathbb{R}$, the Gap$-m-$Kurepa hypothesis holds for all $m \neq n$, but the Gap$-n-$Kurepa hypothesis fails.

Lemma 2.2. (a) $\mathbb{P}$ is ω_n-closed,

(b) $\mathbb{P}$ satisfies the κ-c.c,

(c) Let $0 < i < n$. Then $\mathbb{Q}_i$ is ω_{i+1}-closed modulo J_i in the following sense: If $\langle p_\xi : \xi < \lambda \rangle$, $\lambda \leq \omega_i$, is a descending sequence of condition in $\mathbb{Q}_i$ such that for all $\xi < \lambda$, $J_i(X_{p_\xi}) \in X_{p_{\xi+1}}$, then there is a condition $p \in \mathbb{Q}_i$ which extends all of the p_ξ's, $\xi < \lambda$. Furthermore if $\lambda < \omega_i$, then p can be chosen to be the greatest lower bound of the p_ξ's, $\xi < \lambda$.

(d) Let $0 < i < n$. Then $\mathbb{Q}_i$ has the ω_{i+2}-c.c.

Proof. (a) and (b) are well known results of Levy (see [2], Lemma 20.4). We prove (c) and (d).

(c) Fix $0 < i < n$, and let $\langle p_\xi : \xi < \lambda \rangle$ be as above. To simplify the notation let $p_\xi = (X_\xi, F_\xi, g_\xi)$, $\xi < \lambda$. We consider two cases.

Case 1. $\lambda < \omega_i$.

Let $p = (X, F, g)$, where

- $X = \bigcup_{\xi<\lambda} X_\xi$,
- F is the least subset of ${}^X 2$ such that:
  - if $t \in^X 2$ and for all $\xi < \lambda$, $t|X_\xi \in F_\xi$, then $t \in F$,
  - F is ω_i-closed in the sense of (i-4),
- $dom(g) = \bigcup_{\xi<\lambda} dom(g_\xi)$,
- for all $\alpha \in dom(g)$, $g(\alpha) = \cup\{g_\xi(\alpha) : \xi < \lambda, \alpha \in dom(g_\xi)\}$.

It is easy to show that $p \in \mathbb{Q}_i$ and that p is the greatest lower bound for the sequence $\langle p_\xi : \xi < \lambda \rangle$.

Case 2. $\lambda = \omega_i$.

Let $p = (X, F, g)$, where

- $X = \bigcup_{\xi<\lambda} X_\xi$,
- $dom(g) = \bigcup_{\xi<\lambda} dom(g_\xi)$,
- for all $\alpha \in dom(g)$, $g(\alpha) = \cup\{g_\xi(\alpha) : \xi < \lambda, \alpha \in dom(g_\xi)\}$,
- $F = ran(g)$.

Then it is easy to show that $p \in \mathbb{Q}_i$ and that p is a lower bound for the sequence $\langle p_\xi : \xi < \lambda \rangle$.

(d) Fix $0 < i < n$. Suppose that $\mathbb{Q}_i$ does not satisfy the ω_{i+2}-c.c. Let A be a maximal antichain in $\mathbb{Q}_i$ of size ω_{i+2}. By a Δ-system argument we can assume that:

- The sequence $\langle X_p : p \in A \rangle$ forms a Δ-system with root X.
- The sequence $\langle dom(g_p) : p \in A \rangle$ forms a Δ-system with root D.

Let θ be large regular, and let $\mathcal{M}$ be an elementary submodel of $H(\theta)$ of size ω_{i+1} which is closed under ω_i-sequences and such that $X, D, A \in \mathcal{M}$. Pick $q \in A - \mathcal{M}$ and let $q|\mathcal{M} = (X_q|\mathcal{M}, F_q|\mathcal{M}, g_q|\mathcal{M})$, where

- $X_q|\mathcal{M} = X_q \cap \mathcal{M}$,
- $F_q|\mathcal{M} = \{t|(X_q \cap \mathcal{M}) : t \in F_q\}$,
- $dom(g_q|\mathcal{M}) = dom(g_q) \cap \mathcal{M}$,
- for all $\alpha \in dom(g_q|\mathcal{M}), (g_q|\mathcal{M})(\alpha) = g_q(\alpha)|(X_q \cap \mathcal{M})$.

Then $q|M \in \mathbb{Q}_i \cap \mathcal{M}$. Extend this condition to a condition $p \in \mathbb{Q}_i \cap \mathcal{M}$ which extends an element $r \in A$ and such that $F_p - ran(g_p)$ has size ω_i. It is easy to see that p and q and hence r and q are compatible. But this is impossible since $r, q \in A$. It follows that $\mathbb{Q}_i$ satisfies the ω_{i+2}-c.c. □

Let $K = G \times \prod_{0<i<n} H_i$ be $\mathbb{R} = \mathbb{P} \times \prod_{0<i<n} \mathbb{Q}_i$ generic over V. It follows from the above lemma that:

- $\omega_i^{V[K]} = \omega_i^V$ for all $i \leq n$.
- $\omega_{n+1}^{V[K]} = \kappa^V$.

Lemma 2.3. In $V[K]$, the Gap$-m-$Kurepa hypothesis holds for all $m \neq n$.

Proof. Using remark 1.2 it suffices to show that $KH(\aleph_n, \aleph_i)$ holds in $V[K]$, for all $0 < i < n$. This follows from the next claim:

Claim 2.3.1. Let $0 < i < n$. Forcing with $\mathbb{Q}_i$ adds a family $\mathcal{F} \subseteq {}^{\omega_n}2$ such that:

(a) $Card(\mathcal{F}) = \kappa$

(b) for all $X \in [\omega_n]^{\omega_i}$, $Card(\mathcal{F}|X) \leq \aleph_i$

Proof of the claim. By Lemma 2.2, $\mathbb{Q}_i$ is a cardinal preserving forcing notion. It is easy to prove the following (where H_i is assumed to be a $\mathbb{Q}_i$-generic filter over V):

- $\cup\{X_p : p \in H_i\} = \omega_n$,
- $\cup\{dom(g_p) : p \in H_i\} = \kappa$,
- for all $X \in [\omega_n]^{\omega_i}$, there is some $p \in H_i$ with $X_p \supseteq X$,
- if $\alpha < \kappa$, then $g(\alpha) : \omega_n \longrightarrow 2$, where

$$g(\alpha) = \cup\{g_p(\alpha) : p \in H_i, \alpha \in dom(g_p)\}$$

- if $\alpha < \beta < \kappa$, then $g(\alpha) \neq g(\beta)$.

Then $\mathcal{F} = \{g(\alpha) : \alpha < \kappa\}$ is as required. This completes the proof of the claim and hence of lemma 2.3. □

Lemma 2.4. $KH(\aleph_n, \aleph_0)$ fails in $V[K]$.

Before going into the details of the proof of lemma 2.4, we introduce some notions. Let λ be a regular cardinal, $\aleph_n < \lambda < \kappa$. Define the following forcing notions:

- $\mathbb{P}_\lambda = Col(\omega_n, < \lambda)$,
- $\mathbb{Q}_{i,\lambda}$ =the set of all $p \in \mathbb{Q}_i$ such that $dom(g_p) \subseteq \lambda$,

- $\mathbb{R}_\lambda = \mathbb{P}_\lambda \times \prod_{0<i<n} \mathbb{Q}_{i,\lambda}$

Also let $K_\lambda = G_\lambda \times \prod_{0<i<n} H_{i,\lambda}$ be $\mathbb{R}_\lambda$-generic over V. Define $\pi_\lambda : \mathbb{R} \longrightarrow \mathbb{R}_\lambda$ by

$$\pi_\lambda(\langle p, \langle (X_i, F_i, g_i) : 0 < i < n \rangle \rangle) = \langle p|\lambda, \langle (X_i, F_i, g_i|\lambda) : 0 < i < n \rangle \rangle$$

Then it is easy to prove the following

Claim 2.4.1. π_λ is a projection, i.e.

(a) $\pi_\lambda(1_{\mathbb{R}}) = 1_{\mathbb{R}_\lambda}$,

(b) π_λ is order preserving,

(c) if $p \in \mathbb{R}_\lambda$, $q \in \mathbb{R}$ and $p \leq \pi_\lambda(q)$, then there is some $r \leq q$ in $\mathbb{R}$ such that $\pi_\lambda(r) \leq p$. □

Let

$$(\mathbb{R} : \mathbb{R}_\lambda) = \{ \langle p, \langle (X_i, F_i, g_i) : 0 < i < n \rangle \rangle \in \mathbb{R} : \pi_\lambda(\langle p, \langle (X_i, F_i, g_i) : 0 < i < n \rangle \rangle) \in K_\lambda \}.$$

It follows from Lemma 2.2 (c) that

Claim 2.4.2. $(\mathbb{R} : \mathbb{R}_\lambda)$ is countably closed modulo the J_i's, $0 < i < n$, in the following sense: if $\langle \langle p_m, \langle (X_{i,m}, F_{i,m}, g_{i,m}) : 0 < i < n \rangle \rangle : m < \omega \rangle$ is a descending sequence of conditions in $(\mathbb{R} : \mathbb{R}_\lambda)$ such that for all $0 < i < n$ and $m < \omega$, $J_i(X_{i,m}) \in X_{i,m+1}$, then this sequence has a lower bound in $(\mathbb{R} : \mathbb{R}_\lambda)$. □

Now, we are ready to prove Lemma 2.4.

Proof of Lemma 2.4. Assume on the contrary that $KH(\aleph_n, \aleph_0)$ holds in $V[K]$. Suppose for simplicity that $1_{\mathbb{R}} \Vdash \ulcorner \dot{\mathcal{F}}$ is a $KH(\aleph_n, \aleph_0)$-family $\urcorner$.

Let $\mathcal{F} = \dot{\mathcal{F}}[K]$, and let $A = \langle \mathcal{F}|X : X \in [\omega_n]^\omega \rangle$. Choose $\lambda < \kappa$ regular such that $A \in V[K_\lambda]$. Let $b \in \mathcal{F}$ be such that $b \notin V[K_\lambda]$.

From now on we work in $V[K_\lambda]$ and force with $(\mathbb{R} : \mathbb{R}_\lambda)$. Let $\dot{b}$ be an $(\mathbb{R} : \mathbb{R}_\lambda)$-name for b, and let $r_0 \in (\mathbb{R} : \mathbb{R}_\lambda)$, $r_0 = \langle p_0, \langle (X_{i,0}, F_{i,0}, g_{i,0}) : 0 < i < n \rangle \rangle$, be such that

$$r_0 \Vdash \ulcorner \dot{b} \in \dot{\mathcal{F}} \ \textit{ and } \ \dot{b} \notin V \urcorner$$

It is easy to prove the following:

Claim 2.4.3. For each $r \leq r_0$, $r = \langle p, \langle (X_i, F_i, g_i) : 0 < i < n \rangle \rangle$, there are two conditions $r_1 = \langle p_1, \langle (X_{i,1}, F_{i,1}, g_{i,1}) : 0 < i < n \rangle \rangle$, $r_2 = \langle p_2, \langle (X_{i,2}, F_{i,2}, g_{i,2}) : 0 < i < n \rangle \rangle$ and some $\xi < \omega_n$ such that:

(a) $r_1, r_2 \leq r$,

(b) $J_i(X_i) \in X_{i,m}$ for all $0 < i < n$ and $m = 1, 2$,

(c) $r_1 \Vdash \ulcorner \check{\xi} \in \dot{b} \urcorner$ iff $r_2 \Vdash \ulcorner \check{\xi} \notin \dot{b} \urcorner$. $\square$

Using the above, we can construct a sequence $\langle r_s = \langle p_s, \langle (X_{i,s}, F_{i,s}, g_{i,s}) : 0 < i < n \rangle \rangle : s \in {}^{<\omega}2 \rangle$ of conditions in $(\mathbb{R} : \mathbb{R}_\lambda)$ and a sequence $\langle \xi_m : m < \omega \rangle$ of elements of ω_n such that the following hold:

- $r_{s*m} \leq r_s$, for each $s \in {}^{<\omega}2$ and $m < 2$,
- $J_i(X_{i,s}) \in X_{i,s*m}$ for each $s \in {}^{<\omega}2$, $m < 2$ and $0 < i < n$,
- $r_{s*0} \Vdash \ulcorner \check{\xi}_m \in \dot{b} \urcorner$ iff $r_{s*1} \Vdash \ulcorner \check{\xi}_m \notin \dot{b} \urcorner$, where m is the length of s.

Let $X = \{\xi_m : m < \omega\}$, and for each $f \in^{\omega} 2$, using claim 2.4.2, let $r_f \in (\mathbb{R} : \mathbb{R}_\lambda)$ be an extension of all of the $r_{f|m}$'s, $m < \omega$. For each f as above, we can find some $q_f \leq r_f$ and some $b_f \in V[K_\lambda]$ such that

$$q_f \Vdash \ulcorner \dot{b} \cap \check{X} = \check{b}_f \urcorner$$

Note that for $f \neq g$ in ${}^{\omega}2$, we have $b_f \neq b_g$, and hence $\mathcal{F}|X$ must have size at least $2^{\aleph_0}$ which is in contradiction with our assumption.

It follows that $KH(\aleph_n, \aleph_0)$ fails in $V[K]$. This completes the proof of Lemma 2.4. $\square$

Con(b) **implies Con**(a)**.** Now we show that if $n \geq 1$, and the Gap$-n-$Kurepa hypothesis fails, then there exists an inaccessible cardinal in L. In fact we will prove the following more general result:

Lemma 2.5. Suppose that $\lambda < \kappa$ are infinite cardinals such that $\kappa^\lambda = \kappa$ and $KH(\kappa, \lambda)$ fails. Then κ^+ is an inaccessible cardinal in L.

Proof. Assume not. Thus we can find $X \subseteq \kappa$ such that:

- V and $L[X]$ have the same cardinals up to κ^+,
- $([\kappa]^\lambda)^V = ([\kappa]^\lambda)^{L[X]}$.

It follows that a $KH(\kappa, \lambda)$-family in $L[X]$ is a real $KH(\kappa, \lambda)$-family, and hence $KH(\kappa, \lambda)$ fails in $L[X]$. The following lemma gives us the required contradiction.

Lemma 2.6. Suppose that $V = L[X]$, where $X \subseteq \kappa$. Then $KH(\kappa, \lambda)$ holds.

Proof. Our proof is very similar to the proof of theorem 2 in [3]. We give it for completeness. For each $x \in [\kappa]^\lambda$ let

$$M_x = \text{the smallest } M \prec L_\kappa[X] \text{ such that } x \cup \{x\} \cup (\lambda + 1) \subseteq M.$$

Let $\mathcal{F} = \{t \subseteq \kappa : \forall x \in [\kappa]^\lambda, t \cap x \in M_x\}$. We show that $\mathcal{F}$ is a $KH(\kappa, \lambda)$-family. It suffices to show that $Card(\mathcal{F}) \geq \kappa^+$. Suppose not. Let $C = \langle t_\nu : \nu < \kappa \rangle$ be an enumeration of $\mathcal{F}$ definable in $L_{\kappa^+}[X]$. By recursion on $\nu < \kappa$, define a chain $\langle N_\nu : \nu < \kappa \rangle$ of elementary submodules of $L_{\kappa^+}[X]$ as follows:

$N_0 =$ the smallest $N \prec L_{\kappa^+}[X]$ such that $N \cap \kappa \in \kappa$,

$N_{\nu+1} =$ the smallest $N \prec L_{\kappa^+}[X]$ such that $N \cap \kappa \in \kappa$ and $N_\nu \cup \{N_\nu\} \subseteq N$,

$N_\delta = \bigcup_{\nu < \delta} N_\nu$, if δ is a limit ordinal.

For each $\nu < \kappa$ set $\alpha_\nu = N_\nu \cap \kappa$. Using condensation lemma for $L[X]$, we obtain an ordinal β_ν and an isomorphism σ_ν such that

$$\sigma_\nu : \langle N_\nu, \varepsilon, N_\nu \cap X \rangle \simeq \langle L_{\beta_\nu}[X \cap \alpha_\nu], \varepsilon, X \cap \alpha_\nu \rangle.$$

Then

- $\alpha_\nu < \beta_\nu < \alpha_{\nu+1}$,
- $\sigma_\nu(\kappa) = \alpha_\nu$,
- $\sigma_\nu(X) = X \cap \alpha_\nu$,
- $\sigma_\nu | \alpha_\nu = id | \alpha_\nu$,
- $L_{\beta_\nu}[X \cap \alpha_\nu] \models \ulcorner \alpha_\nu$ is a regular cardinal, and α_ν is the largest cardinal $\urcorner$.

Let $t = \{\beta_\nu : \beta_\nu \notin t_\nu\}$. Clearly $t \neq t_\nu$ for all $\nu < \kappa$, and hence $t \notin \mathcal{F}$. Let $x \in [\kappa]^\lambda$ be such that:

- $t \cap x \notin M_x$,
- $\alpha = sup(x)$ is minimal.

It follows that $t \cap x$ is cofinal in α, and hence $\alpha = \alpha_\eta$ for some $\eta < \kappa$. We have

$$t \cap x = \{\beta_\nu \in x : \beta_\nu < \alpha_\eta \ \ and \ \ \beta_\nu \notin t_\nu \cap \alpha_\eta\}$$

and thus $t \cap x$ is definable from x, $\langle \beta_\nu : \nu < \eta \rangle$ and $\langle t_\nu \cap \alpha_\eta : \nu < \eta \rangle$. It is clear that:

- $x \in M_x$,
- $\langle \beta_\nu : \nu < \eta \rangle$ is definable in $L_{\beta_\eta}[X \cap \alpha_\eta]$.
- $\sigma_\eta(C) = \langle t_\nu \cap \alpha_\eta : \nu < \eta \rangle$, and hence $\langle t_\nu \cap \alpha_\eta : \nu < \eta \rangle$ is definable in $L_{\beta_\eta}[X \cap \alpha_\eta]$.

Clearly $X \cap \alpha_\eta \in M_x$. We show that $\beta_\eta \in M_x$. It will follow that $t \cap x \in M_x$ which is a contradiction. The proof is in a sequence of claims. Let $M = M_x$.

Claim 1. $\mathcal{P}(\alpha_\eta) \cap M \not\subseteq L_{\beta_\eta}[X \cap \alpha_\eta]$.

Proof. Suppose not. Since $cf(\alpha_\eta) = cf(x) = cf(\lambda) < \kappa$, there is $a \in M$ such that $a \subseteq \alpha_\eta$ is cofinal in α_η and has order type less than α_η. Then $a \in L_{\beta_\eta}[X \cap \alpha_\eta]$, and hence α_η is not a regular cardinal in $L_{\beta_\eta}[X \cap \alpha_\eta]$. A contradiction. □

For $l < \nu < \kappa$ set:

- $\alpha^{(\nu)} = \langle \alpha_\iota : \iota \leq \nu \rangle$,
- $\beta^{(\nu)} = \langle \beta_\iota : \iota \leq \nu \rangle$,
- $\sigma_{\iota\nu} = \sigma_\nu \sigma_\iota^{-1} : \langle L_{\beta_\iota}[X \cap \alpha_\iota], \varepsilon, X \cap \alpha_\iota \rangle \longrightarrow \langle L_{\beta_\nu}[X \cap \alpha_\nu], \varepsilon, X \cap \alpha_\nu \rangle$,
- $\sigma^{(\nu)} = \langle \sigma_{\iota\nu} : \iota < \tau \leq \nu \rangle$.

Then it is easy to prove that:

Claim 2. $\nu \in M \cap \eta$ implies $\alpha^{(\nu)}, \beta^{(\nu)}, \sigma^{(\nu)} \in M$. □

We note that

$$\langle \langle L_{\beta_\iota}[X \cap \alpha_\iota], \varepsilon, X \cap \alpha_\iota \rangle_{\iota < \eta}, \langle \sigma_{\iota\nu} \rangle_{\iota < \nu < \eta} \rangle$$

is a directed system of elementary embeddings, and if

$$\langle \langle U, E, Y \rangle, \langle g_\iota \rangle_{\iota < \eta} \rangle$$

is its direct limit, then

- $\langle U, E, Y \rangle \simeq \langle L_{\beta_\eta}[X \cap \alpha_\eta], \varepsilon, X \cap \alpha_\eta \rangle$,
- $g_\iota : \langle L_{\beta_\iota}[X \cap \alpha_\iota], \varepsilon, X \cap \alpha_\iota \rangle \longrightarrow \langle U, E, Y \rangle$,
- If $f : \langle U, E, Y \rangle \simeq \langle L_{\beta_\eta}[X \cap \alpha_\eta], \varepsilon, X \cap \alpha_\eta \rangle$, then $\sigma_{\iota\eta} = f g_\iota$.

Now let $\pi : \langle M, \varepsilon, M \cap X \rangle \simeq \langle L_\delta[\tilde{X}], \varepsilon, \tilde{X} \rangle$, where $\tilde{X} = \pi[M \cap X]$. Let

- $\tilde{\alpha}^{(\nu)} = \pi(\alpha^{(\nu)})$,
- $\tilde{\beta}^{(\nu)} = \pi(\beta^{(\nu)})$,
- $\tilde{\sigma}^{(\nu)} = \pi(\sigma^{(\nu)})$,
- $\tilde{\alpha} = \bigcup_{\nu \in M \cap \eta} \tilde{\alpha}^{(\nu)}$,
- $\tilde{\beta} = \bigcup_{\nu \in M \cap \eta} \tilde{\beta}^{(\nu)}$,
- $\tilde{\sigma} = \bigcup_{\nu \in M \cap \eta} \tilde{\sigma}^{(\nu)}$,

Then

- $\tilde{\alpha}_\iota = \pi(\alpha_{\pi^{-1}(\iota)})$,

- $\tilde{\beta}_\iota = \pi(\beta_{\pi^{-1}(\iota)})$,
- $\tilde{\sigma}_{\iota\nu} = \pi(\sigma_{\pi^{-1}(\iota),\pi^{-1}(\nu)})$.

Now

$$\langle\langle L_{\tilde{\beta}_\iota}[X \cap \tilde{\alpha}_\iota], \varepsilon, \tilde{X} \cap \tilde{\alpha}_\iota\rangle_{\iota<\pi(\eta)}, \langle\tilde{\sigma}_{\iota\nu}\rangle_{\iota<\nu<\pi(\eta)}\rangle$$

is a directed system of elementary embeddings. Let

$$\langle\langle\tilde{U}, \tilde{E}, \tilde{Y}\rangle, \langle\tilde{g}_\iota\rangle_{\iota<\pi(\eta)}\rangle$$

be its direct limit. Then

- $\tilde{g}_\iota : \langle L_{\tilde{\beta}_\iota}[\tilde{X} \cap \tilde{\alpha}_\iota], \varepsilon, \tilde{X} \cap \tilde{\alpha}_\iota\rangle \longrightarrow \langle\tilde{U}, \tilde{E}, \tilde{Y}\rangle$,
- There is an elementary embedding h such that the following diagram is commutative:

$$\begin{array}{ccc} \langle L_{\beta_{\pi^{-1}(\iota)}}[X \cap \alpha_{\pi^{-1}(\iota)}], \varepsilon, X \cap \alpha_{\pi^{-1}(\iota)}\rangle & \overset{g_{\pi^{-1}(\iota)}}{\longrightarrow} & \langle U, E, Y\rangle \\ \pi^{-1}\uparrow & & \uparrow h \\ \langle L_{\tilde{\beta}_\iota}[\tilde{X} \cap \tilde{\alpha}_\iota], \varepsilon, \tilde{X} \cap \tilde{\alpha}_\iota\rangle & \overset{\tilde{g}_\iota}{\longrightarrow} & \langle\tilde{U}, \tilde{E}, \tilde{Y}\rangle \end{array}$$

It follows that $\langle\tilde{U}, \tilde{E}\rangle$ is well founded. Let

$$\tilde{f} : \langle\tilde{U}, \tilde{E}, \tilde{Y}\rangle \simeq \langle L_{\overline{\beta}}[\overline{X}], \varepsilon, \overline{X}\rangle.$$

Also let

- $\overline{\sigma}_\iota = \tilde{f}\tilde{g}_\iota : \langle L_{\tilde{\beta}_\iota}[\tilde{X} \cap \tilde{\alpha}_\iota], \varepsilon, \tilde{X} \cap \tilde{\alpha}_\iota\rangle \longrightarrow \langle L_{\overline{\beta}}[\overline{X}], \varepsilon, \overline{X}\rangle$,
- $\pi^* = fh\tilde{f}^{-1} : \langle L_{\overline{\beta}}[\overline{X}], \varepsilon, \overline{X}\rangle \longrightarrow \langle L_{\tilde{\beta}_\eta}[X \cap \alpha_\eta], \varepsilon, X \cap \alpha_\eta\rangle$.

Then $\tilde{\sigma}_{\iota\tau} = \overline{\sigma}_\tau^{-1}\overline{\sigma}_\iota$ for $\iota < \tau < \pi(\eta)$, and the following diagram is commutative:

$$\begin{array}{ccc} \langle L_{\beta_{\pi^{-1}(\iota)}}[X \cap \alpha_{\pi^{-1}(\iota)}], \varepsilon, X \cap \alpha_{\pi^{-1}(\iota)}\rangle & \overset{\sigma_{\pi^{-1}(\iota),\eta}}{\longrightarrow} & \langle L_{\beta_\eta}[X \cap \alpha_\eta], \varepsilon, X \cap \alpha_\eta\rangle \\ \pi^{-1}\uparrow & & \uparrow \pi^* \\ \langle L_{\tilde{\beta}_\iota}[\tilde{X} \cap \tilde{\alpha}_\iota], \varepsilon, \tilde{X} \cap \tilde{\alpha}_\iota\rangle & \overset{\tilde{\sigma}_\iota}{\longrightarrow} & \langle L_{\overline{\beta}}[\overline{X}], \varepsilon, \overline{X}\rangle \end{array}$$

Let $\overline{\alpha}$ be such that $L_{\overline{\beta}}[\overline{X}] \models \ulcorner\overline{\alpha}$ is the largest cardinal $\urcorner$. Then it is easy to show that:

Claim 3. (a) $\pi(\alpha_\eta) = \overline{\alpha}$,

(b) $\pi^*(\overline{\alpha}) = \alpha_\eta$,

(c) $\pi^*|\overline{\alpha} = id|\overline{\alpha}$. □

Next we have

Claim 4. If $a \subseteq \overline{\alpha}$ and $a \in L_{\overline{\beta}}[\overline{X}] \cap L_\delta[\tilde{X}]$, then $\pi^*(a) = \pi^{-1}(a)$.

Proof. Since $a \subseteq \overline{\alpha}$, $\pi^*(a)$, $\pi^{-1}(a) \subseteq \alpha_\eta$, and hence $\pi^*(a) = \bigcup_{\nu \in M \cap \eta} \pi^*(a) \cap \nu = \bigcup_{\nu < \pi(\eta)} \pi^*(a \cap \nu) \overset{claim3}{=} \bigcup_{\nu < \pi(\eta)} \pi^{-1}(a \cap \nu) = \pi^{-1}(a)$. □

Claim 5. $\delta > \overline{\beta}$.

Proof. Suppose not. Then $\delta \leq \overline{\beta}$ and $\pi^*\pi$ maps M into $L_{\beta_\eta}[X \cap \alpha_\eta]$, and by claim 4, $\pi^*\pi(a) = a$ for $a \subseteq \alpha_\eta$, $a \in M$. It follows that $\mathcal{P}(\alpha_\eta) \cap M \subseteq L_{\beta_\eta}[X \cap \alpha_\eta]$, which is in contradiction with claim 1. □

It follows that $\overline{\beta} \in L_\delta[\tilde{X}]$ and hence $\tilde{\beta} \in L_\delta[\tilde{X}]$. Similarly $\tilde{\sigma} \in L_\delta[\tilde{X}]$. It is easily seen that:

Claim 6.(a) $\pi^{-1}(\tilde{\alpha}) = \langle \alpha_\iota : \iota < \eta \rangle$,

(b) $\pi^{-1}(\tilde{\beta}) = \langle \beta_\iota : \iota < \eta \rangle$,

(c) $\pi^{-1}(\tilde{\sigma}) = \langle \sigma_{\iota\nu} : \iota < \nu < \eta \rangle$. □

Now note that:

- $L_{\overline{\beta}}[\overline{X}]$ is the direct limit of $L_{\tilde{\beta}_\iota}[\tilde{X} \cap \tilde{\alpha}_\iota]$, $\tilde{\sigma}_{\iota\nu}, \iota < \nu < \pi(\eta)$,
- $\pi^{-1}[\overline{X}] = X \cap \alpha_\eta$,
- $\pi^{-1}[\tilde{X} \cap \tilde{\alpha}_\iota] = X \cap \alpha_\iota$,

and hence by elementarily of π^{-1}, $L_{\pi^{-1}(\overline{\beta})}[X \cap \alpha_\eta]$ is the direct limit of $L_{\beta_\iota}[X \cap \alpha_\iota]$, $\sigma_{\iota\nu}, \iota < \nu < \eta$.

It follows that $\pi^{-1}(\overline{\beta}) = \beta_\eta \in M$. We are done. □

3. Open problems

We close the paper with some remarks and open problems.

By the results of Vaught, Chang, Jensen (see [1], Chapter VIII) and Silver (see [7]), it is consistent, relative to the existence of an inaccessible cardinal, to have the Gap−1−transfer principle with the failure of the gap−2−transfer principle. The answer is unknown for $n > 1$.

Question 3.1. Let $n > 1$. Is it consistent to have the Gap−n−transfer principle with the failure of the Gap−$(n+1)$−transfer principle?

Another related question is

Question 3.2. Let $n > 1$. Is it consistent to have (κ, n)−morasses for each uncountable regular κ, but no $(\omega_1, n+1)$−morasses?

Remark 3.3. Assuming the existence of large cardinals, it is possible to build a model of set theory in which there exists a $(\kappa, 1)-$morass for each uncountable regular κ, but there are no $(\omega_1, 2)-$morasses.

In the literature the canonical counter-example to the Gap$-1-$transfer principle is the non-existence of Special Aronszajn trees (see [5]). T. Raesch, in his dissertation (see [6]), showed that this principle can fail in the presence of such trees. On the other hand the canonical counter-example to the Gap$-2-$transfer principle is the non-existence of Kurepa trees (see [7]). Inspired by the work of Raesch, Jensen produced, relative to the existence of a Mahlo cardinal, a model in which the Gap$-2-$transfer principle fails, while the Gap$-1-$Kurepa hypothesis holds (see [4]). However the following is open.

Question 3.4. Is it consistent relative to an inaccessible cardinal to have the Gap$-1-$Kurepa Hypothesis but a failure of the Gap $-2-$transfer principle?

Remark 3.5. In our model, for $n = 2$, the Gap$-1-$Kurepa hypothesis holds, while in it there are no $(\omega_2, 1)-$morasses.

Question 3.6. Let $n > 1$. Is it consistent with GCH to have $KH(\aleph_n, \aleph_0)$ but not $KH(\aleph_n, \aleph_1)$?

Question 3.7. Let $n > 1$. Is it consistent with GCH to have $KH(\aleph_n, \aleph_i)$ for all $i < n$, but not $KH(\aleph_n, < \aleph_n)$?

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