The internal consistency of Easton’s theorem

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Let Card denote the class of infinite cardinals and Reg the class of infinite regular cardinals. The continuum function on regulars is the function $\kappa \mapsto 2^\kappa$, defined on Reg. This function $C$ has the following two properties: $\alpha \leq \beta \rightarrow C(\alpha) \leq C(\beta)$ and $\alpha < \text{cof}\,(C(\alpha))$. Easton [2] showed that, assuming GCH, any function $F : \text{Reg} \rightarrow \text{Card}$ with these two properties (any “Easton function”) is the continuum function on regulars of a cofinality-preserving generic extension of the universe. We say that this generic extension “realises” the Easton function $F$. In particular, the statement “$2^\kappa = \kappa^{++}$ for all regular $\kappa$” is consistent, as by Easton’s result it can be forced over Gödel’s universe $L$.

The concept of internal consistency was introduced in [5], where one demands that consistency be witnessed in an inner model, under the assumption of large cardinals. The first result of this article is that any Easton function definable in $L$ without parameters (or with parameters that are countable in $L[0^\#]$) can be realised in an inner model of $L[0^\#]$ with the same cofinalities as $L$. Thus the statement “$2^\kappa = \kappa^{++}$ for all regular $\kappa$” is not only consistent, but also internally consistent, under the assumption that $0^\#$ exists. The proof of this result makes us of a technique of “generic modification”.

One can also consider Easton functions which are $L$-definable using parameters which are not necessarily countable in $L[0^\#]$. We show that such

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functions can also be realised by inner models of $L[0^\#]$ with the same cofinalities as $L$, provided these parameters are at most $\omega_1^V$. The proof uses a "generic stretching" technique to transfer a generic for a given product forcing to a larger one.

One cannot hope to realise an arbitrary $L$-definable Easton function with parameter $\omega_2^V$ in an inner model, as $2^{\omega_2} = \omega_2^V$ will fail in all inner models if CH holds in $V$. A reasonable conjecture would be that any $L$-definable Easton function $f$ with parameter $\omega_2^V$ satisfying $f(\alpha) < \omega_2^V$ for countable $\alpha \in \text{Reg}_L$ can be realised in an inner model of $L[0^\#]$ with same cofinalities as $L$. We take a step in this direction by showing that in some inner model of $L[0^\#]$ with same cofinalities as $L$, $\omega_1^V$ is a strong limit cardinal and $2^{\omega_1^V} = \omega_2^V$. The proof uses a gap 1 morass.

Some preliminaries

We begin with some observations about $I = \text{the class of Silver indiscernibles, Skolem hulls and nice names}.

The following is easily verified.

**Lemma 1** Let $G$ be $P$-generic over $L$ where $P$ is a set in $L$ and let $X$ be a subclass of $\text{Ord}$. Let $\text{Hull}_{L[G]}(X)$ denote the smallest elementary submodel of $L[G]$ containing $X \cup \{G\}$. Then:

(a) $\text{Hull}_{L[G]}(X) = \{\tau(\vec{x})^G \mid x_j \in X \text{ for each } j, \tau \text{ an } L\text{-term, } \tau(\vec{x}) \text{ a } P\text{-name}\}$.

(b) If $P$ belongs to $L_i$ where $i \in I$ then $L_i[G] \prec L[G]$.

Let $\langle i_\alpha \mid \alpha \in \text{Ord} \rangle$ be the increasing enumeration of $I$ and for each $i$ in $I$ let $i^*$ denote the least indiscernible greater than $i$. For any $\alpha$, $i^{*\alpha}$ is the $\alpha^{th}$ indiscernible greater than $i$.

**Lemma 2** Let $i$ be an indiscernible, $j = i^{*\omega}$ and let $G$ be $P$-generic over $L$ where $P \in L_j$. Then $\bigcup_{n \in \omega} \text{Hull}_{L[G]}(i \cup \{i, i^*, \ldots, i^{*n}\}) = L_j[G]$.

*Proof.* Define $X = \bigcup_{n \in \omega} \text{Hull}_{L[G]}(i \cup \{i, i^*, \ldots, i^{*n}\})$. Let $\sigma$ be a name in $L_j^P$ for an element of $L_j[G]$. Then $\sigma \in \text{Hull}(i \cup \{i, i^*, \ldots, i^{*n}\})$ for some $n$. Therefore $\sigma^G \in X$; i.e., we have shown $X \supseteq L_j[G]$. Conversely, $L_j[G]$ is an elementary submodel of $L[G]$, so we have $X \subseteq L_j[G]$. $\square$
Lemma 3 Let $H_n = i^* \cap \text{Hull}^L(i \cup \{i, i^*, \ldots, i^n\})$, $X_0 = H_0$ and $X_{n+1} = H_{n+1} \setminus H_n$. Then $\|X_n\|^L = i$.

Proof. For all $n$ we have $\|X_n\|^L \leq \|H_n\|^L \leq i$. For the reverse inequality we first show $X_{n+1} \neq \emptyset$. Let $S_n = i \cup \{i, i^*, \ldots, i^n\}$. We have $L_{\rho_{\text{fin}(n+1)}} \prec L$, so $Y_n \equiv \text{Hull}^L(S_n) = \text{Hull}^L_{\rho_{\text{fin}(n+1)}}(S_n)$. So $Y_n$ belongs to $Y_{n+1}$. Also $Y_n \cap i^+L$ is an ordinal $\alpha_n < i^+L$ and $\alpha_n$ is in $Y_{n+1}$ but not in $Y_n$. So $\alpha_n \in X_{n+1}$.

Any $\alpha_n + \beta, \beta < i$, belongs to $H_{n+1}$ and thus we have $i$-many elements in $X_{n+1}$. So $\|X_{n+1}\|^L = i$. As $X_0$ contains $i$, we also have $\|X_0\|^L = i$. □

Corollary 4 Let $\alpha \in (i, i^+L)$ and $H_n = \alpha \cap \text{Hull}^L(i \cup \{i, i^*, \ldots, i^n\})$, $X_0 = H_0$ and $X_{n+1} = H_{n+1} \setminus H_n$. Then there exists $m \in \omega$ such that

(a) For all $n < m$: $\|X_n\|^L = i$.

(b) $\|X_m\|^L \leq i$.

(c) For $n > m$: $X_n = \emptyset$.

Proof. From the previous proof we have: $H_n = Y_n \cap \alpha = \alpha_n \cap \alpha$. If $\alpha_n < \alpha$, then $H_n = \alpha_n$ and $\|X_n\|^L = i$. If $\alpha_n \geq \alpha$, then $H_n = \alpha$. So we can choose $m$ to be $\min\{n \mid \alpha_n \geq \alpha\}$. ($\{n \mid \alpha_n \geq \alpha\}$ is not empty because $\alpha < i^+L$.) □

Corollary 5 Let $\alpha \in [i^+L, i^*)$ and $H_n = \alpha \cap (\text{Hull}^L(i \cup \{i, i^*, \ldots, i^n\}))$, $X_0 = H_0$ and $X_{n+1} = H_{n+1} \setminus H_n$. Then $\|X_n\|^L = i$.

Proof. The interval $[\alpha_n, \alpha_{n+1})$ from the proof of Lemma 3 is a subset of $X_{n+1}$. □

A nice $P$-name is a $P$-name of the form $\bigcup\{\{\alpha\} \times A_n \mid \alpha \in S\}$, where $S$ is a set of ordinals and each $A_n$ is an antichain in $P$.

For a proof of the following, see [9], Lemma VII.5.12., page 208.

Lemma 6 Let $M$ be an inner model, $P \in M$ a partial ordering, $\sigma, \rho \in M^P$. Then there is a nice $P$-name $\xi \in M^P$ such that $1^P \Vdash (\rho \subseteq \sigma \rightarrow \rho = \xi)$.

Distributivity is important for the existence of generic sets for partial orderings. This is formulated in the next two lemmas.

Lemma 7 Suppose that $P$ is a forcing in $L$, $\|P\|^L \leq i^*$ and $P$ is $i^+L$-distributive in $L$. Then there exists a $P$-generic over $L$ in $L[0^\beta]$. 3
Proof. We may assume that $P$ is a subset of $i^*$. Let $D$ be $\{D \mid D$ is open dense on $P, D \in L\}$. Write $D$ as $\bigcup_{n \in \omega} X_n$, where $X_n = \text{Hull}(i \cup \{i, i^*, \ldots, i^{*n}\}) \cap D$. Then $X_n \in L$ and $\|X_n\|_L \leq i$ for each $n$.

Now choose $p_n, n \in \omega$, to meet all $D$ in $X_n$, $p_{n+1} \leq p_n$. Then $G = \{p \mid p_n \leq p$ for some $n\}$ is $P$-generic. And the construction of $G$ is possible in $L[0^\#]$. □

Corollary 8 Suppose $H \in L[0^\#]$ is $Q$-generic over $L$ for some $Q \in L_i^{\ast \omega}$, where $i$ is an indiscernible. Let $P$ be a forcing in $L[H]$ such that $\|P\|^{L[H]} < i^{\ast \omega}$ and $P$ is $i^{\ast L}$-distributive in $L[H]$. Then there exists a $P$-generic over $L[H]$ in $L[0^\#]$.

Proof. We can use the previous proof, using Lemma 2 to guarantee $D = \bigcup_{n \in \omega} X_n$. □

Lemma 9 If $P$ is constructible, $\|P\|_L \leq \omega_1^V$, $0^\#$ exists and $P$ preserves $\omega_1$ over $V$ then there exists a $P$-generic over $L$.

Proof. [4], Theorem 1. □

The next lemma explains why we cannot use a simple iterated forcing in this paper.

Lemma 10 Suppose that $P = \text{Add}^L(\omega_1^L, \omega_1^L)$ and $Q = \text{Add}^{L[G]}(\omega_2^L, \omega_1^L)$, with generics $G$ and $H$, respectively. Then $\text{Card}^{L[G][H]} \neq \text{Card}^L$.

Proof. In $L[G]$, $2^{\omega_1^L} = \omega_3^L$. Let $H_1$ be the restriction of $H$ to $\text{Add}^{L[G]}(\omega_2^L, 1)$, a set generic for the latter forcing over $L[G]$. In $L[G][H_1]$ there is a function $f$ from $\omega_2^L$ onto $(2^{\omega_1^L})^{L[G]}$, as any subset of $\omega_2^L$ in $L[G]$ gets “coded” into an interval of $H_1$. So in $L[G][H_1] \subseteq L[G][H]$ there is a surjection from $\omega_2^L$ onto $\omega_3^L$, showing that cardinals are not preserved by the forcing $P \ast Q$. □

The next lemma is standard.

Lemma 11 Assume the forcing $P$ is in the inner model $M$, $\lambda$ is a regular cardinal in $M$ and $P$ is $\lambda$-distributive. Let $G$ be $P$-generic over $M$ and $\mu$ a cardinal in $M$ less then $\lambda$. Then $\mathcal{P}(\mu)^M = \mathcal{P}(\mu)^{M[G]}$. 4
Lemma 12 (Jensen’s Covering Theorem) Suppose there is an uncountable set of ordinals which is not covered in \( L \), i.e., not a subset of a constructible set of the same \( V \)-cardinality. Then \( 0^\# \) exists.

Lemma 13 Let \( M \) be an inner model, \( P \) and \( Q \) set-forcings in \( M \) and suppose that for some cardinal \( \lambda \) of \( M \), \( P \) has the \( \lambda^+ \)-c.c. and \( Q \) is \( \lambda^+ \)-closed in \( M \). If \( G = G_0 \times G_1 \) is \( P \times Q \)-generic over \( M \) then \( P(\lambda) \cap M[G] = P(\lambda) \cap M[G_0] \).

Proof. [7], proof of Lemma 15.19, page 234. (But note that [7] uses a different definition for “closed”: \( \kappa^+ \)-closed here is \( \kappa \)-closed in [7].) \( \Box \)

Lemma 14 Suppose that \( P = P_0 \times P_1 \) where \( P_0 \) and \( P_1 \) are class forcings definable over the model \( M \) and \( P_0 \)-forcing is definable.

If \( G_0 \) is \( P_0 \)-generic over \( M \) and \( G_1 \) is \( P_1 \)-generic over \( M[G_0] \) then \( G_0 \times G_1 \) is \( P \)-generic over \( M \).

If \( G \) is \( P \)-generic over \( M \), then \( G = G_0 \times G_1 \) where \( G_0 \) is \( P_0 \)-generic over \( M \) and \( G_1 \) is \( P_1 \)-generic over \( M[G_0] \).

Proof. [3], proof of Product Lemma 2.27, page 40. \( \Box \)

Lemma 15 Suppose that \( P = P_0 * P_1 \) is a two-step iteration defined over the model \( M \) and \( P_0 \)-forcing is definable.

If \( G_0 \) is \( P_0 \)-generic over \( M \) and \( G_1 \) is \( P_1 \)-generic over \( M[G_0] \) then \( \{ (p_0, \dot{p}_1) \mid p_0 \in G_0 \text{ and } (\dot{p}_1)^{G_0} \in G_1 \} \) is \( P \)-generic over \( M \).

If \( G \) is \( P \)-generic over \( M \), then \( G = \{ (p_0, \dot{p}_1) \mid p_0 \in G_0 \text{ and } (\dot{p}_1)^{G_0} \in G_1 \} \), where \( G_0 \) is \( P_0 \)-generic over \( M \) and \( G_1 \) is \( P_1 \)-generic over \( M[G_0] \).

Proof. [3], proof of Product Lemma 2.30, page 44. \( \Box \)

Definition 16 A family \( A \) of sets is called a \( \Delta \)-system iff there is a fixed set \( r \), called the root, such that \( a \cap b = r \) whenever \( a \) and \( b \) are distinct members of \( A \).

Lemma 17 Let \( \kappa \) be any infinite cardinal. Let \( \phi > \kappa \) be regular and satisfy \( \forall \alpha < \phi(\|\alpha^{<\kappa}\| < \phi) \). Assume \( \|A\| \geq \phi \) and \( \forall x \in A(\|x\| < \kappa) \). Then there is a \( B \subseteq A \) such that \( \|B\| = \phi \) and \( B \) forms a \( \Delta \)-system.
Proof. [9], proof of theorem II.1.6, page 49 ∎

We next discuss the type of iterated forcings we will use in this paper.

**Definition 18** Assume that $M$ satisfies GCH and consider an $M$-definable sequence $\langle P(<\beta) \mid \beta \leq \alpha \rangle$, where $\alpha \in \text{Ord}$ or $\alpha = \text{Ord}$, with the following properties:

1. $P(<0)$ is the trivial forcing $\{\emptyset\}$ and each $P(<\beta)$ consists of functions $p : \beta \to M$ in $M$.

2. For $\gamma + 1 \leq \alpha$, $P(<\gamma + 1) \simeq P(<\gamma) \ast P(\gamma)$ via the isomorphism $p \to (p(<\gamma), p(\gamma))$, where $p(<\gamma) = p|\gamma$, $P(<\gamma)$ is a set in $M$ and $P(<\gamma) \Vdash P(\gamma)$ is a cofinality-preserving set-forcing.

3. We have a continuous increasing sequence $\{c_\gamma \mid \gamma < \alpha\}$ of limit cardinals with the following properties:
   (a) $P(<\gamma) \Vdash P(\gamma)$ is $c_\gamma$-closed
   (b) $P(\gamma)$ is a $P(<\gamma)$-name of cardinality at most $c^+_{\gamma+1}$.

4. For a singular limit $\lambda \leq \alpha$: $P(<\lambda) = \text{Inverse-Limit} \langle P(<\beta) \mid \beta < \lambda \rangle \equiv \{p : \lambda \to M \mid \forall \beta < \lambda (p(<\beta) \in P(<\beta))\}$. This is ordered by:
   $p \leq q$ iff $p(<\beta) \leq q(<\beta)$ for each $\beta < \lambda$.

5. For a regular limit $\lambda \leq \alpha$: $P(<\lambda) = \text{Direct-Limit} \langle P(<\beta) \mid \beta < \lambda \rangle \equiv \{p : \lambda \to M \mid \exists \gamma < \lambda (p(<\gamma) \in P(<\gamma) \land \forall \beta \in [\gamma, \lambda), p(\beta) = 1^{P(\beta)})\}$. The ordering is the same as in the previous case.

When $\alpha = \text{Ord}$, then $P = \text{Direct-Limit} \langle P(\beta) \mid \beta \in \text{Ord} \rangle$.

We also define:

$P(\leq \beta) = P(<\beta + 1)$

$P[\beta, \gamma) \equiv$ the iterated forcing in $M^{P(<\beta)}$ which starts from $P(\beta)$ and has length $\gamma - \beta$

$P(\geq \beta) \equiv P[\beta, \alpha)$

$P(\beta, \gamma) \equiv P[\beta + 1, \gamma)$

$P(> \beta) \equiv P(\geq (\beta + 1))$

For this iteration we have the Factoring property:
Lemma 19  \( P(< \beta) \) preserves cofinalities and for any \( \alpha < \beta \), \( P(< \beta) \) is isomorphic to \( P(< \alpha) \ast P[\alpha, \beta) \).

Proof. This is similar to [3], Lemma 2.34, page 46.

By induction on \( \beta \). The result is trivial for \( \beta = 0 \). If \( \beta = \gamma + 1 \geq \alpha \) and we have defined the isomorphism \( \theta : P(< \gamma) \simeq P(< \alpha) \ast P[\alpha, \gamma) \), then \( P(\leq \gamma) \simeq P(< \gamma) \ast P(\gamma) \simeq (P(< \alpha) \ast P[\alpha, \gamma)) \ast P(\gamma) \simeq P(< \alpha) \ast (P[\alpha, \gamma) \ast P(\gamma)) \simeq P(< \alpha) \ast P[\alpha, \gamma + 1) \). As by induction \( P(< \gamma) \) preserves cofinalities and by hypothesis \( P(< \gamma) \models P(\gamma) \) preserves cofinalities it follows that \( P(\leq \gamma) = P(< \beta) \) preserves cofinalities too.

Suppose that \( \beta \) is a limit. By induction \( P(< \beta) \) is canonically isomorphic to \( P(< \alpha) \ast P[\alpha, \gamma) \) for \( \gamma < \beta \). It follows that \( P(< \beta) \) is canonically isomorphic to \( P(< \alpha) \ast P[\alpha, \beta) \) provided we know that \( P(< \alpha) \) preserves the regularity of \( \beta \) (in case \( \beta \) is regular). The latter follows from the fact that by induction, \( P(< \alpha) \) preserves cofinalities. Finally, we show that \( P(< \beta) \) preserves cofinalities. Suppose that \( \gamma \) is a regular cardinal; we show that \( P(< \beta) \) preserves “cofinality greater than \( \gamma \)” (cof \( \gamma \)). If \( \gamma \) is less than \( c_\beta \) then factor \( P(< \beta) \) as \( P(< \alpha) \ast P[\alpha, \beta) \) where \( \gamma \) is less than \( c_\alpha \). By induction, \( P(< \alpha) \) preserves cof \( \gamma \) and by definition \( P[\alpha, \beta) \) is \( c_\alpha \)-closed and therefore \( \gamma^+ \)-closed; it follows that \( P(< \beta) \) preserves cof \( \gamma \). If \( \gamma \) is greater than \( c_\beta \) then as \( P(< \beta) \) has a dense subset of cardinality \( c_\beta \), \( P(< \beta) \) preserves cof \( \gamma \). Finally if \( \gamma \) equals \( c_\beta \) then \( P(< \beta) \) has a dense subset of cardinality \( c_\beta = \gamma \); so again \( P(< \beta) \) preserves cof \( \gamma \). \( \Box \)

Lemma 20  \( P \) preserves cofinalities and for any \( \alpha \in \text{Ord} \) is isomorphic to \( P(< \alpha) \ast P(\geq \alpha) \).

Proof. For \( \beta = \infty \), we can use the same argument as for regular \( \beta \) in the previous proof. For smaller \( \beta \), we have the previous proof. \( \Box \)

Lemma 21  Let \( P \) be the direct limit of an iterated forcing over \( M \) as above, \( M \) a model of ZFC, and let \( G \) be \( P \)-generic. Then \( M[G] \models ZFC \).

Proof. See [3]. From the comment on page 47 we know, \( P \) is tame. Then we can use Lemma 2.21, page 36. \( \Box \)

Definition 22  Let \( E \) be a subset of \( \text{Ord} \) and \( \mathcal{P} = \{ P_\alpha \mid \alpha \in E \} \) a family of posets. The Easton product \( \prod_{\alpha \in E}^E P_\alpha \) of \( \mathcal{P} \) consists of all \( \langle p_\alpha \mid \alpha \in E \rangle \) in \( \prod_{\alpha \in E} P_\alpha \) such that for all \( \kappa \in \text{Reg} \) there is a \( \beta < \kappa \) such that \( \alpha \in (\beta, \kappa) \cap E \rightarrow p_\alpha = 1^{P_\alpha} \).
1 Easton functions with countable parameters

Definition 23 An $L$-definable $f : \text{Reg}^L \to \text{Card}^L$ is an Easton function (for $L$) iff on its domain, $\text{cof}^L(f(\kappa)) > \kappa$ and $\mu < \lambda \rightarrow f(\mu) \leq f(\lambda)$.

We first show:

Theorem 24 Take any $L$-definable (without parameters) Easton function $f : \text{Reg}^L \to \text{Card}^L$. There is an inner model $M$ of $L[0^\#]$ with the same cofinalities as $L$ in which $2^\kappa = f(\kappa)$ for all $\kappa \in \text{Reg}^L$.

We begin with some lemmas:

Lemma 25 Let $M$ be an inner model with the same cofinalities as $L$, $\kappa$ a successor of a regular cardinal $\mu$ in $M$ and assume $\lambda < \mu \rightarrow (\mu^\lambda)_M = \mu$. Let $f : \text{Reg}^L \to \text{Card}^L$ be an $L$-definable Easton function and $P = \prod_{\lambda < \kappa, \lambda \in \text{Reg}^L} \text{Add}(\lambda, f(\lambda))$ in $M$. Then $P$ is $\kappa$-c.c. in $M$.

Proof. Each $a$ in $P$ is a function from an “Easton subset” of $\{\lambda < \kappa \mid \lambda \in \text{Reg}^L\}$ which assigns to each $\lambda$ in its domain a condition in $\text{Add}(\lambda, f(\lambda))$, i.e., a function from a subset of $\lambda \times f(\lambda)$ of cardinality less than $\lambda$ into 2. For each such $a$ let $\text{Dom}^*(a)$ denote the set of triples $(\lambda, i, j)$ such that $\lambda$ is in the domain of $a$ and $(i, j)$ is in the domain of $a(\lambda)$. Note that $\text{Dom}^*(a)$ is a set of cardinality less than $\mu$.

Now suppose $A = \{a_\beta \mid \beta < \kappa\} \in M$ were an antichain in $P$ of cardinality $\kappa$ (where the $a_\beta$’s are distinct). Then apply Lemma 17 to the $\text{Dom}^*(a_\beta)$’s (where $\kappa$ is the current $\mu$, $\phi$ is the current $\kappa$ and $A$ is $\{\text{Dom}^*(a) \mid a \in A\}$; here we use the assumption $(\mu^\kappa)_M = \mu$ for $\lambda < \mu$). Let $B \subseteq \{\text{Dom}^*(a) \mid a \in A\} \cup \text{Dom}^*(a)$ have cardinality $\kappa$ and core $b$. Then $\|b\|_M < \mu$ because for all $a \in A$, $\|\text{Dom}^*(a)\| < \mu$. The number of functions from $b$ to 2 is at most $2^{\|b\|} \leq \mu^{\|b\|} = \mu < \kappa$, so there are distinct $x, y \in A$ which are compatible (because $\text{Dom}^*(x) \cap \text{Dom}^*(y) = b$ and $x, y$ agree on $b$). This contradicts the assumption that $A$ is an antichain. □
Definition 26 Let $\alpha, \beta, \gamma$ be ordinals, $\alpha < \beta$ and $i_\gamma$ the $\gamma$th Silver indiscernible. Define:

\[
\pi_{i_\alpha,i_\beta}(i_\gamma) = i_\gamma \text{ for } \gamma < \alpha \\
\pi_{i_\alpha,i_\beta}(i_{\alpha+\delta}) = i_{\beta+\delta}.
\]

$\pi_{i_\alpha,i_\beta}$ extends uniquely to an elementary embedding $L \rightarrow L$, which we also denote by $\pi_{i_\alpha,i_\beta}$.

Lemma 27 Suppose $i < j$ belong to $I$. If $x \in L$, $\|x\|^L = j$ then $x \cap Rng(\pi_{ij})$ belongs to $L$, $\pi_{ij}^{-1}|(x \cap Rng(\pi_{ij})) \in L$ and $x \cap Rng(\pi_{ij})$ has $L$-cardinality at most $i$.

Proof. Note that $Rng(\pi_{ij}) = \text{Hull}(i \cup I(\geq j))$, an elementary submodel of $L$.

First suppose that $x = j$. Then $x \cap Rng(\pi_{ij}) = j \cap Rng(\pi_{ij}) = i \in L$, $\pi_{ij}^{-1}|(x \cap Rng(\pi_{ij})) = \pi_{ij}^{-1}|i = \text{id} \in L$.

Now write $x = \tau(\bar{\alpha}, \bar{\beta}, j, \bar{\gamma})$ where $\bar{\alpha} < i \leq \bar{\beta} < j < \bar{\gamma}$ are in $I$.

Suppose $\bar{\beta} = \emptyset$, an hypothesis equivalent to $x \in Rng(\pi_{ij})$. Let $\phi$ be a bijection between $x$ and $\|x\|^L = j$. As $Rng(\pi_{ij})$ is an elementary submodel of $L$, we can choose $\phi \in Rng(\pi_{ij})$. Then for $y \in x$, we have $y \in Rng(\pi_{ij})$ iff $\phi(y) \in Rng(\pi_{ij})$. So:

\[
x \cap Rng(\pi_{ij}) = \phi^{-1}[j \cap Rng(\pi_{ij})] = \phi^{-1}[i] \in L.
\]

For the second property, we use $\psi = \pi_{ij}^{-1}(\phi)$. $\psi$ is a bijection between $\pi_{ij}^{-1}[x]$ and $i$. Let $y \in x \cap Rng(\pi_{ij})$. From $\psi(y) = \pi_{ij}(\psi(y)) = (\pi_{ij}(\psi))(\pi_{ij}(y)) = \phi(\pi_{ij}(y))$ we have $\pi_{ij}(y) = \phi^{-1}(\psi(y))$. So $\pi_{ij}^{-1}|(x \cap Rng(\pi_{ij}))$ is a composition of two functions in $L$ and therefore belongs to $L$.

Now suppose $\bar{\beta} \neq \emptyset$. Define $x^* = \bigcup\{\tau(\bar{\delta}, j, \bar{\gamma}) \mid \bar{\delta} < j \land \bar{\delta} \in \text{Ord}^{\omega} \land \|\tau(\bar{\delta}, j, \bar{\gamma})\|^L = j\}$. Then $x \subseteq x^*$, $\|x^*\|^L = j$ and $x^* \in Rng(\pi_{ij})$. So by the above, $x^* \cap Rng(\pi_{ij}) \in L$. We then have:

$x \cap Rng(\pi_{ij}) = x \cap (x^* \cap Rng(\pi_{ij})) \in L$ and $\pi_{ij}^{-1}|(x \cap Rng(\pi_{ij})) = \pi_{ij}^{-1}|(x^* \cap Rng(\pi_{ij}))| x \in L$, as desired. The statement about $L$-cardinality is implicit in the above. □
Lemma 28 Take any \( L \)-definable Easton function \( f : \text{Reg}^L \to \text{Card}^L \). Let \( M \) be an inner model such that \( \text{Card}^L = \text{Card}^M \) and GCH holds in \( M \) for cardinals in \([\delta_1, \delta_2] \), where \( \delta_1 < \delta_2 \) are regular in \( M \). Let \( P \) be the forcing \( \prod_{\kappa \in [\delta_1, \delta_2]} \text{Add}(\kappa, f(\kappa)) \) in \( M \). Also assume \( \kappa \in \text{Reg}^L \cap \delta_2 \rightarrow f(\kappa) \leq \delta_2 \) and \( \kappa \in \text{Reg}^L \cap \delta_1 \rightarrow 2^\kappa \leq \delta_1 \). Let \( G \) be \( P \)-generic over \( M \).

Then \((2^\kappa)^M[G] = f(\kappa)\) for all \( \kappa \in \text{Reg}^L \cap [\delta_1, \delta_2] \), \((2^\kappa)^{M[G]} = (2^\kappa)^M\) for \( \kappa \in \text{Reg}^L \setminus [\delta_1, \delta_2] \) and \( G \) preserve cofinalities.

Proof. Let \( \kappa \) be a regular cardinal in \( L \) from \([\delta_1, \delta_2] \) which is either \( \delta_1 \) or the successor of a regular cardinal. \( P \simeq P(<\kappa) \times P(\geq \kappa) \). By Lemma 14 we have \( G(<\kappa) \) and \( G(\geq \kappa) \) such that \( M[G] = M[G(<\kappa)][G(\geq \kappa)] \). \( P(\geq \kappa)^M \) is \( \kappa \)-closed in \( M \). By Lemma 25, \( P(<\kappa) \) is \( \kappa \)-c.c. in \( M \) (\( \kappa = \delta_1 \) is a trivial case). From this we have: \( M \models \text{cof} (\alpha) \geq \kappa \rightarrow M[G(<\kappa)] \models \text{cof} (\alpha) \geq \kappa \rightarrow M[G(\geq \kappa)] \models \text{cof} (\alpha) \geq \kappa \) for successors of regular cardinals \( \kappa \), and therefore for all regular cardinals. So \( G \) preserves all cofinalities.

We want \( M[G] \models 2^\kappa = f(\kappa) \) for regular \( \kappa \in [\delta_1, \delta_2] \) and \((2^\kappa)^M = (2^\kappa)^{M[G]}\) for other regular \( \kappa \). For \( \kappa < \delta_1 \): \( P \) is \( \delta_1 \)-closed in \( M \), so by Lemma 11 we have \( \mathcal{P}(\kappa)^M = \mathcal{P}(\kappa)^{M[G]} \). \( P \) preserves cardinalities, so \((2^\kappa)^M = (2^\kappa)^{M[G]}\). For \( \kappa \geq \delta_2 \): \( \|P\|^M \leq \delta_2 \), so we have at most \((2^{\delta_2})^\kappa = 2^\kappa\) nice names for subsets of \( \kappa \). So \( \|2^\kappa\|^M = \|2^\kappa\|^{M[G]} \).

Suppose \( \kappa \in [\delta_1, \delta_2] \), \( \kappa \) regular in \( M \). We want \( 2^\kappa \geq f(\kappa) \) and \( 2^\kappa \leq f(\kappa) \).

In \( M[G] \) we have a generic for \( \text{Add}^M(\kappa, f(\kappa)) \). So in \( M[G] \) we have \( 2^\kappa \geq f(\kappa) \). For the other direction we use \( P(<\kappa^+) \) and \( P(\geq \kappa^+) \). \( P(<\kappa^+) \) is \( \kappa^+ \)-c.c. and \( P(\geq \kappa^+) \) is \( \kappa^+ \)-closed. Let \( \lambda \) denote \( \kappa^+ \). Then \( \lambda \leq f(\kappa) \) and \( \|P(<\lambda)\|^M = f(\kappa) \). So the number of antichains in \( P(<\lambda) \) is \( f(\kappa)^{<\lambda} \) and therefore we have only \( (f(\kappa)^{<\lambda})^\kappa = f(\kappa) \) nice names for subsets of \( \kappa \). From Lemma 13 we have \( \mathcal{P}(\kappa)^M[G(<\kappa^+)] = \mathcal{P}(\kappa)^{M[G]} \). So \( M[G] \models \|\mathcal{P}(\kappa)\| = 2^\kappa \leq f(\kappa) \). \( \Box \)

We turn now to the proof of the theorem.

Proof of theorem 24. We want to use reverse Easton forcing. But we cannot use iteration for \( f \)-crossings \((\kappa < \lambda < f(\kappa) < f(\lambda)) \), because \( \text{Add}(\kappa, f(\kappa)) \ast \text{Add}(\lambda, f(\lambda)) \) collapses \( f(\kappa) \) to \( \lambda \) (see the example in Lemma 10). So, we split the ordinals into the intervals determined by the closure points of \( f \) and
take an Easton product within those intervals. Then we join these product forcings together to one iterated forcing.

Let $C$ be $\{\alpha \in \text{Card}^L | \omega \leq \alpha \land (\kappa \in \text{Reg}^L \cap \alpha \rightarrow f(\kappa) < \alpha)\}$. We know that $I \subseteq C \subseteq \text{LimCard}^L$, because $f$ is $L$-definable without parameters and $L_i \prec L$ for $i \in I$. Let $\{c_\alpha\}$ be the increasing enumeration of $C$. Then $c_0 = \omega$ and $c_i = i$ for $i \in I$.

We define $P$:

1. For each $\beta$, $P(\beta) = \prod_{\alpha \in [c_\beta, c_{\beta+1}]} \text{Add}(\alpha, f(\alpha))$ over $L^{P(<\beta)}$.

The rest is determined by Definition 18:

2. $P(<0) = \emptyset$, $P(<(\alpha + 1)) = P(\leq \alpha)$

3. For regular $\lambda$ in $L$, $P(<\lambda) = \text{Direct-Limit} \{P(\leq \alpha) | \alpha < \lambda\}$

4. For singular $\lambda$ in $L$, $P(<\lambda) = \text{Inverse-Limit} \{P(\leq \alpha) | \alpha < \lambda\}$

5. $P(\leq \alpha) = P(<\alpha) * P(\alpha)$

6. $P = \text{Direct-Limit} \{P(\leq \alpha) | \alpha \in \text{Ord}\}$

Because this definition fulfills definition 18, we can use Lemmas 19 and 21. For each $\alpha$ we have $P \simeq P(<\alpha) * P(\geq \alpha)$.

Our goal is to show that $P$ preserves cofinalities, $P$ forces $2^\kappa = f(\kappa)$ for all $\kappa \in \text{Reg}^L$ and that there exists a $P$-generic over $L$ in $V = L[0^{\#}]$.

Cofinality preservation

By Lemma 20 it suffices to verify that for each $\alpha$, $P(<\alpha) \Vdash P(\alpha)$ preserves cofinalities. This follows from Lemma 28, setting $\delta_1 = c_\alpha$ if $c_\alpha$ is $L$-regular, $\delta_1 = c_\alpha^{+L}$ if $c_\alpha$ is $L$-singular and $\delta_2 = c_\alpha^{+L}$.

$f$ is realised

We want: $2^\kappa = f(\kappa)$ for each $L$-regular $\kappa$. Write $P \simeq P(<\alpha) * P(\alpha) * P(>\alpha)$ where $\kappa$ belongs to the interval $[c_\alpha, c_{\alpha+1})$. Then $P(<\alpha)$ has cardinality
at most $\kappa^+$, $P(\alpha)$ adds exactly $f(\kappa)$ subsets if $\kappa$ by Lemma 28 and $P(>\alpha)$ is $\kappa^+$-closed. It follows that $P \Vdash 2^\kappa = f(\kappa)$.

A $P$-generic class

We build a $P$-generic $H$ definably in $L[0^\#]$. By induction on $i \in I$ we define a $P(\leq i)$-generic $H(\leq i)$ in $L[0^\#]$ with the property that for $j < k$ in $I$, the generics $H_0(j)$, $H_0(k)$ “fit together”, where $H_0(j)$ is the restriction of $H(j) \subseteq P(j) = \prod_{\lambda \in [\kappa, \kappa^+]} \text{Reg}^\lambda \text{Add}(\lambda, f(\lambda))$ to $P_0(j) = \text{Add}(j, f(j))$.

To make this precise, extend the $\pi_{ij}$ of Definition 26 to an embedding $\tilde{\pi}_{ij} : L[H(\leq i)] \to L[H(\leq j)]$ by $\tilde{\pi}_{ij}(\sigma^{H(\leq i)}) = \pi_{ij}(\sigma^{H(\leq j)})$. $\tilde{\pi}_{ij}$ is a well-defined elementary embedding:

\[
\begin{align*}
L[H(\leq i)] & \models \psi(\sigma_1^{H(\leq i)}, \ldots, \sigma_n^{H(\leq i)}) \rightarrow \\
\exists p \in H(\leq i) (p \models \psi(\sigma_1, \ldots, \sigma_n)) & \rightarrow \\
\exists p \in H(\leq i) (\pi_{ij}(p) \models \psi(\pi_{ij}(\sigma_1), \ldots, \pi_{ij}(\sigma_n))) & \rightarrow \\
L[H(\leq j)] & \models \psi(\pi_{ij}(\sigma_1^{H(\leq j)}), \ldots, \pi_{ij}(\sigma_n^{H(\leq j)})) \rightarrow \\
L[H(\leq j)] & \models \psi(\tilde{\pi}_{ij}(\sigma_1^{H(\leq j)}), \ldots, \tilde{\pi}_{ij}(\sigma_n^{H(\leq j)})).
\end{align*}
\]

In the third implication we use $\pi_{ij}[H(\leq i)] = H(\leq i) \subseteq H(\leq j)$. The above implications are in fact equivalences, as they also hold for $\neg \psi$. $\tilde{\pi}_{ij}$ also extends $\pi_{ij}$, as for $x \in L$, if $y = \pi_{ij}(x)$, then: $\tilde{\pi}_{ij}(x) = \tilde{\pi}_{ij}(\hat{x})^{H(\leq i)} = (\pi_{ij}(\hat{x}))^{H(\leq j)} = y = \pi_{ij}(x)$, where $\hat{x} = \{(\hat{z}, 1) \mid z \in x\}$ is the canonical name for $x$.

For indiscernibles $i < j < k$ we have $\tilde{\pi}_{jk} \circ \tilde{\pi}_{ij} = \tilde{\pi}_{ik}$. This follows from the definition of $\tilde{\pi}_{ij}$ and $\pi_{jk} \circ \pi_{ij} = \pi_{ik}$. And Lemma 27 also holds for $\tilde{\pi}_{ij}$ (the same proof works, using Lemma 1).

Now we construct $H(\leq i)$ by induction on $i \in I$ so that: $j, k \in I$, $j < k \rightarrow \tilde{\pi}_{jk}[H_0(j)] \subseteq H_0(k)$.

We start the induction from $P(\leq i_0)$. As the set of constructible dense subsets of this forcing is countable in $L[0^\#]$, we may choose a generic $H(\leq i_0)$ for it.
Suppose that \( H(\leq i) \) has been defined, and we now wish to define \( H(\leq i^*) \), where \( i^* \) is the least indiscernible greater than \( i \). Write \( P(\leq i^*) \simeq \mathcal{P}(\leq i) \ast \mathcal{P}(i, i^*) \); we want to choose a generic for \( \mathcal{P}(i, i^*)^{H(\leq i)} \). The latter forcing is \( i^* \)-closed and has cardinality \( f(i^*) < i^{**} \); it follows from Corollary 8 that we may choose a generic \( H(i, i^*) \) for it in \( L[0^\#] \). However we must ensure that \( \pi_{ii^*}[H_0(i)] \subseteq H_0(i^*) \), where \( H_0(i^*) \) is the restriction of \( H(i^*) \) to \( P_0(i^*) = \text{Add}(i^*, f(i^*)) \) (of \( L[H(< i^*)] \)). This will be guaranteed by modifying \( H_0(i^*) \). Note that \( H(i^*) \) can be written as \( H_0(i^*) \times H_1(i^*) \), where \( H_1(i^*) \) is generic over \( L[H(< i^*)] \) for \( P_1(i^*) = \prod_{\alpha \in (i^*, c_{i^*} + 1)} \text{Add}(\alpha, f(\alpha)) \), and for any \( \text{Add}(i^*, f(i^*)) \)-generic \( H_0^*(i^*) \), the product \( H_0^*(i^*) \times H_1(i^*) \) will still be generic for \( P_0(i^*) \times P_1(i^*) \), as \( P_0(i^*) \) has the \( (i^*) \)-c.c. and \( P_1(i^*) \) is \( (i^*) \)-closed. So to define the desired \( H(i^*) \), it suffices to find any \( P_0(i^*) \)-generic \( H_0^*(i^*) \) such that \( \pi_{ii^*}[H_0(i)] \subseteq H_0^*(i^*) \).

Suppose that \( p \) is a condition in \( P_0(i^*) = \text{Add}(i^*, f(i^*)) \). Define a new condition \( \Psi(p) \) as follows:

\[
\Psi(p)(w^*) = \begin{cases} p(w^*) & \text{if } w^* \in \text{Dom}(p) \land w^* \notin \text{Rng}(\tilde{\pi}_{ii^*}), \\ H_0(i)(w) & \text{if } w^* \in \text{Dom}(p) \land \tilde{\pi}_{ii^*}(w) = w^*. \end{cases}
\]

\( \Psi(p) \) is indeed a condition in \( P_0(i^*) \) by Lemma 27 for \( \tilde{\pi}_{ii^*} \), as \( \|p\|_{L[H(< i^*)]} < i^* \) (actually we only need \( \|p\|_{L[H(< i^*)]} \leq i^* \) here).

Now set \( H_0^*(i^*) = \{ \Psi(p) \mid p \in H_0(i^*) \} \), our desired modification of \( H_0(i^*) \). We prove that \( H_0^*(i^*) \) is \( P_0(i^*) \)-generic over \( L[H(< i^*)] \). Clearly \( H_0^*(i^*) \) is an upward-closed filter. Let \( D \subseteq L[H(< i^*)] \) be dense on \( P_0(i^*) \). To show that \( H_0^*(i^*) \) meets \( D \), we define:

- \( p' \) is a small modification of \( p \) iff \( \text{Dom}(p) = \text{Dom}(p') \) and \( \{ x \mid p(x) \neq p'(x) \} \) has cardinality at most \( i \).
- \( p \) strongly meets \( D \) iff all small modifications \( p' \) of \( p \) meet \( D \).

Note that the collection of all small modifications \( p' \) of \( p \) is a set in \( L[H(< i^*)] \) of cardinality at most \( (\|p\|^i \cdot 2^i)^L[H(< i^*)] < i^* \).

**Claim.** \( \{ p \mid p \text{ strongly meets } D \} \) is dense in \( P_0(i^*) \).

**Proof of Claim.** Let \( p_0 = q_0 \) be some element of \( P_0(i^*) \). We create a descending sequence of \( q_0 \)'s as follows: \( q_1 \) meets \( D \) and \( q_1 \leq q_0 \). \( q_1^0 \) is a small
modification of \( q_1 \) on \( \text{Dom}(q_0) \), \( q''_1 \) is an extension of \( q'_1 \) meeting \( D \), and \( q_2 \) is \( q''_1 \) without the modification. We repeat this with another modification on \( \text{Dom}(p_0) \), etc., taking unions at limits, until all modifications on \( \text{Dom}(p_0) \) have been considered.

\( p_1 \), the union of all \( q_n \), is an element of \( P_0(i^*) \). For this \( p_1 \) we do the same as for \( p_0 \), obtaining \( p_2 \), and continue this for \( i^{+L} \) steps, taking unions at limits. Then \( p_{i^{+L}} \) is an element of \( P_0(i^*) \) which strongly meets \( D \). \( \square \) (Claim)

Choose \( s \in H_0(i^*) \) which strongly meets \( D \). \( s' = \Psi(s) \in H_0^*(i^*) \) is a small modification of \( s \), so we have \( d \in D \) s.t. \( s' \leq d \). As \( H_0^*(i^*) \) is upward-closed, \( d \in H_0^*(i^*) \), so we have shown that \( H_0^*(i^*) \) meets \( D \). This completes the construction of \( H(\leq i^*) \).

Suppose that \( i \) is a limit indiscernible. By induction we have the following property: If \( i_0 < i_1 \) are indiscernibles less than \( i \) then \( \pi_{i_0 i_1}[H(< i_0) \ast H_0(i_0)] \) is contained in \( H(< i_1) \ast H_0(i_1) \) (where as before \( H_0(j) \) is the restriction of \( H(j) \) to \( \text{Add}(j, f(j)) \)). We take \( H(< i) \ast H_0(i) \) to be the union of the \( \pi_{i_0 i_i}[H(< i) \ast H_0(i)], i \in I \cap i \). This is an upward-closed filter; we verify that it meets all dense sets in \( L \) for \( P(< i) \ast P_0(i) \): Let \( D = \tau(j, i, \infty) \) be dense in \( P_0(i) \). Choose an indiscernible \( k \) in \( (\max(j), i) \) and set \( D_k = \tau(j, k, \infty) \). Then \( D_k \) is dense in \( P(< k) \ast P_0(k), \) so we have some \( p_k \in D_k \cap H(< k) \ast H_0(k) \). Then \( p = \pi_{k,i}(p_k) \) belongs to \( H(< i) \ast H_0(i) \) and belongs to \( D \). So \( H(< i) \ast H_0(i) \) is generic, as desired.

Finally, as \( P_1(i) = \prod_{\alpha \in (i, i_{i+1})}^{\text{Easton}} \text{Add}(\alpha, f(\alpha)) \) is \( i^{+\text{c}} \)-closed and of cardinality less than \( i^{**} \) in \( L[H(< i)] \), we can use Corollary 8 to obtain a generic \( H_1(i) \) for this forcing. Then \( H(< i) \ast H_0(i) \ast H_1(i) = H(\leq i) \) is the desired \( P(\leq i) \)-generic. This completes the construction of the \( H(\leq i), i \in I, \) and therefore of the desired \( P \)-generic \( H = \bigcup_{i \in I} H(< i) \). Clearly \( H \) can be built definably in \( L[0^\#] \). This completes the proof of Theorem 24. \( \square \)

**Corollary 29** Suppose that \( i \in I \) and \( f \) is an Easton function which is \( L \)-definable with parameters \( \leq i \). Then there is an inner model \( M \) of \( L[0^\#] \) with the same cofinalities and the same subsets of \( i \) as \( L \) in which \( 2^\kappa = f(\kappa) \) for all \( \kappa \in \text{Reg}^{f^+} \) greater than \( i \).
Proof. Consider the following variant of the iteration used to prove Theorem 24: $P(\beta)$ is trivial for $\beta < i$, $P(i)$ is $\prod_{\alpha \in (i, i+1) \cap \text{Reg}} \text{Add}(\alpha, f(\alpha))$ over $L^{P(<i)}$ and $P(\beta)$ is $\prod_{\alpha \in [c_\beta, c_\beta+1) \cap \text{Reg}} \text{Add}(\alpha, f(\alpha))$ over $L^{P(<\beta)}$ for $\beta > i$. Then as in the proof of the Theorem, this forcing preserves cofinalities and forces $2^\kappa = f(\kappa)$ for $L$-regular $\kappa$ greater than $i$. Moreover, subsets of $i$ are not added. □

**Theorem 30** Let $f : \text{Reg}^L \to \text{Card}^L$ be an Easton function which is $L$-definable with $L[0^\#]$-countable parameters. Then there is an inner model $M$ of $L[0^\#]$ with the same cofinalities as $L$ in which $2^\kappa = f(\kappa)$ for all regular $\kappa$.

Proof. Let $i$ be an $L[0^\#]$-countable indiscernible so that the parameters used to define $f$ belong to $L_i$. By Corollary 29, we can preserve cofinalities over $L$, not add subsets of $i$ and force $2^\kappa = f(\kappa)$ for all $L$-regular $\kappa$ greater than $i$. Over this generic extension $M$, consider the forcing $Q = \prod_{\kappa \in \text{Reg} \cap i^+} \text{Add}(\kappa, f(\kappa))$. This forcing has only countably many dense subsets in $M$ as $i$ is countable, and therefore we can choose a $Q$-generic over $M$. The result is a model with the same cofinalities as $L$ in which $2^\kappa = f(\kappa)$ for all $L$-regular $\kappa$. □

2 The parameter $\omega_1^V$

Assume that $0^\#$ exists.

**Definition 31** Let $M$ be an inner model, $\kappa$ an $M$-cardinal and $\alpha$ an ordinal, $\kappa \leq \alpha < \kappa^{+V}$. A bijection $f : \alpha \to \kappa$ is $M$-good iff $f \upharpoonright x$ is in $M$ whenever $x \in M$, $x \subseteq \alpha$ and $\|x\|^M \leq \kappa$.

**Lemma 32** (L-good bijections) For any uncountable cardinal $\kappa$ in $V$ and ordinal $\alpha$, $\kappa \leq \alpha < \kappa^{+V}$, there exists an $L$-good bijection $f : \alpha \to \kappa$.

Proof. Let $I$ be the class of Silver indiscernibles and $i^*$ the $I$-successor to $i$. We prove by induction on $i \in I \cap [\kappa, \kappa^{+V})$ that there is an $L$-good bijection $f_\alpha : \alpha \to \kappa$ for any $\alpha \in [\kappa, i^*]$. $f_\kappa$ is the identity. If we have $f_i$, then we construct $f_{i^*}$ as follows: For $n \in \omega$, define $H_n = i^* \cap \text{Hull}^L((i + 1) \cup \{i^*, i^{**}, \ldots, i^{*n}\})$. Each $H_n$ has
The union of these good bijections is a good bijection $f$ constructible bijection between $\kappa \in L$. By Lemma 3, $||X_\kappa||^L = i$. From $f$, we create a bijection $f^* : \bigcup X_n \to (\kappa \times \omega)$ by choosing $g_n : X_n \to i$ to be a constructible bijection and setting $f^*(\gamma) = (f_i(g_n(\gamma)), n)$ for $\gamma$ in $X_n$. We claim that $f^*|x$ is constructible for any constructible $x \subseteq i^*$ with $||x||^L = \kappa$. Any such $x$ is a subset of some $H_n$, and therefore is contained in the union of finitely many $X_n$'s. It follows that $f^*|x$ is the finite union of constructible functions and therefore constructible. We let $f_x$ be the composition of $f^*$ and a constructible bijection between $\kappa \times \omega$ and $\kappa$.

For $\alpha \in (i, i^+)$ we let $f_\alpha$ be the composition of a constructible bijection $g : \alpha \to i$ with $f_i$. For $\alpha \in [i^+, i^+]$ we can use corollary 5 and the above argument for $i^*$.

Suppose that $i \in (\kappa, \kappa^+)$ is a limit indiscernible with $V$-cofinality $\gamma$. Then $\gamma \leq \kappa$. Let $S = \{s_\alpha \mid \alpha \in \gamma\}$ be increasing and continuous with $s_0 = 0$, $s_1 = \kappa^*$ and $\bigcup \alpha s_\alpha = i$. We split $i$ into the intervals $[s_\alpha, s_{\alpha+1})$ and for every interval we can use $f_{s_{\alpha+1}}$ to create a good bijection between $[s_\alpha, s_{\alpha+1})$ and $\kappa$. The union of these good bijections is a good bijection $f$ between $i$ and $\kappa \times \gamma$: Any $x \in L$, $x \subseteq i$, $||x||^L = \kappa$ intersects only finitely many intervals $[s_\alpha, s_{\alpha+1})$, as otherwise some initial segment of $x$ is cofinal in some indiscernible $> \kappa$, contradicting the $L$-regularity of indiscernibles. Now compose $f$ with a constructible bijection between $\kappa \times \gamma$ and $\kappa$ to obtain the desired $L$-good bijection $f_i$. □

**Corollary 33 (M-good bijections)** Let $M$ be any inner model with $\text{Card}^L = \text{Card}^M$. Then for any uncountable cardinal $\kappa$ and ordinal $\alpha$, $\kappa \leq \alpha < \kappa^+, V$, there exists an $M$-good bijection $f : \alpha \to \kappa$.

**Proof.** We can find some $L$-good bijection $f : \alpha \to \kappa$ using Lemma 32. This bijection is also $M$-good: Let $x \in M$, $x \subseteq \alpha$, $||x||^M = \kappa$. By the Covering theorem (Lemma 12) we have $y \in L$, $x \subseteq y$ and $||y||^M = ||y||^L = \kappa$. As $f$ is an $L$-good bijection, $f|y$ is constructible, and therefore $f|x = (f|y)|x \in M$, as desired. □

**Lemma 34 (Stretching at $\omega_1^V$)** Let $M$ be an inner model with the same cofinalities as $L$ such that $\alpha < \omega_1^V \to (2^\alpha)^M = \omega_1^V$ and let $\mu$ be an ordinal in the interval $[\omega_1^V, 2^\omega]$. Suppose that there exists a generic over $M$ for
\[ P = \text{Add}(\omega_1^V,\omega_1^V)^M \text{ in } V. \text{ Then in } V \text{ there is also a generic over } M \text{ for } Q = \text{Add}(\omega_1^V,\mu)^M. \]

**Proof.** From corollary 33 we have an \( M \)-good bijection \( \pi : \mu \to \omega_1^V \). We use \( \pi \) to construct \( \pi' : \text{Add}(\omega_1^V,\mu)^M \to \text{Add}(\omega_1^V,\omega_1^V)^M \) as follows: Define \( \tilde{\pi} : \omega_1^V \times \mu \to \omega_1^V \times \omega_1^V \) by \( \tilde{\pi}(a,b) = (a,\pi(b)) \). We know that \( \tilde{\pi} \upharpoonright x \in M \) for \( x \subseteq \omega_1^V \times \mu, \|x\|^M \leq \omega_1^V \). Now for \( p \in \text{Add}(\omega_1^V,\mu)^M \) set \( \text{Dom}(\pi'(p)) = \tilde{\pi}[\text{Dom}(p)] \) and \( \pi'(p)(a,b) = p(\tilde{\pi}^{-1}(a,b)). \) As \( \tilde{\pi} \upharpoonright \text{Dom}(p) \) belongs to \( M \), so does \( \tilde{\pi}^{-1}\upharpoonright \text{Dom}(\pi'(p)). \) It follows that \( \pi'(p) \) is in \( M \) and is therefore a condition in \( \text{Add}(\omega_1^V,\omega_1^V)^M \). (\( \pi' \) is not surjective, but this will not matter.)

Let \( G \) be \( \text{Add}(\omega_1^V,\omega_1^V)^M \)-generic over \( M \). Our candidate for a \( Q \)-generic over \( M \) is \( H = \{ q \in Q \mid \pi'(q) \in G \}. \) We need only check that \( H \) intersects maximal antichains on \( Q \) which belong to \( M \).

Let \( A \in M \) be a maximal antichain on \( \text{Add}(\omega_1^V,\mu)^M \) and set \( A' = \{ \pi'(a) \mid a \in A \}. \) \( A' \) is in \( M \) because \( Q \) is \( (\omega_1^V)^+ \)-c.c. and \( \pi \) is an \( M \)-good bijection. Clearly \( A' \) is an antichain; we want to show that \( A' \) is a maximal antichain in \( M \).

Let \( D(A) = \bigcup\{ \text{Dom}(a) \mid a \in A \} \) and \( D(A') = \bigcup\{ \text{Dom}(a) \mid a \in A' \}. \) The bijection \( \text{id} \times \pi \) maps \( D(A) \) onto \( D(A') \) and \( \text{id} \times \pi \upharpoonright D(A) \) belongs to \( M \). Therefore we have the following key property: For any \( q \in \text{Add}(\omega_1^V,\omega_1^V)^M \) with \( \text{Dom}(q) \subseteq D(A') \), there is \( p \in \text{Add}(\omega_1^V,\mu)^M \) with \( \pi'(p) = q. \) Now we can verify that \( A' \) is maximal: Let \( q \in \text{Add}(\omega_1^V,\omega_1^V)^M. \) We want to find some element in \( A' \) compatible with \( q. \) Set \( q_1 = q\upharpoonright D(A'). \) There is a \( p_1 \) with \( \pi'(p_1) = q_1 \) and \( p_1 \) is compatible with some \( a \in A \) (because \( A \) is a maximal antichain). But then \( \pi'(a) \) is in \( A' \) and compatible with \( q_1. \) As \( q_1 \) and \( q \) agree on \( D(A') \), \( \pi'(a) \) is also compatible with \( q. \)

As \( A' \) is a maximal antichain there exists \( g \in A' \cap G. \) But \( g \) is some \( \pi'(h), \) with \( h \in A \cap H, \) showing that \( H \) meets \( A \), as desired. \( \square \)

**Lemma 35 (Uniformly L-good bijections)** Let \( f \in L, \kappa \in \text{Card}^V, f : \kappa \to [\kappa,\kappa^+V). \) Then there exists a “uniformly \( L \)-good” bijection \( g : \bigcup_{\alpha \in \kappa} \{ \alpha \} \times f(\alpha) \to \kappa \times \kappa, \) i.e., for each \( \alpha \in \kappa, \) \( g \upharpoonright \{ \alpha \} \times f(\alpha) \) is a bijection between \( \{ \alpha \} \times f(\alpha) \) and \( \{ \alpha \} \times \kappa, \) and for \( x \subseteq \text{Dom}(g), \|x\|^L = \kappa, \) we have \( g\upharpoonright x \in L. \)
Proof. By induction on $\gamma = \sup \{ f(\alpha) \mid \alpha < \kappa \}$. If $\gamma = \kappa$ then we choose $g$ to be the identity.

Suppose that $\gamma \in (i, i^+L)$ for some $i \in I$. Let $f'(\alpha) = \min \{ f(\alpha), i \}$. For this $f'$ we have the desired $g'$ by induction. In $L$, canonically choose bijections $h_\beta : \beta \to i$ for $\beta \in [i, i^+L)$. Then define:

$$g(\alpha, y) = g'(\alpha, y) \text{ (if } f(\alpha) \leq i),$$
$$g(\alpha, y) = g'(\alpha, h_{f(\alpha)}(y)) \text{ (if } f(\alpha) > i).$$

As the sequence of $h_\alpha$’s is in $L$, this $g$ is uniformly $L$-good.

Next suppose $\gamma \in [i^+L, i^*)$ for some $i \in I$. In this case we use the disjoint splitting of $i^*$ into the $X_n$’s from Lemma 3. Set:

$$f'(\alpha) = f(\alpha) \text{ (if } f(\alpha) < i^+L),$$
$$f'(\alpha) = i \text{ (if } f(\alpha) \geq i^+L).$$

For this $f'$ we have the desired $g'$ by induction. For each $n \in \omega$, choose in $L$ canonical bijections $h^n_\beta : X_n \cap \beta \to i$, $\beta \in [i^+L, i^*)$. Then define:

$$g(\alpha, y) = g'(\alpha, y) \text{ (if } f(\alpha) < i^+L),$$
$$g(\alpha, y) = g'(\alpha, h^n_{f(\alpha)}(y)) \text{ (if } f(\alpha) \geq i^+L \text{ and } y \in X_n).$$

We verify that $g$ is uniformly $L$-good. Let $x \subseteq \text{Dom}(g), \|x\|^L = \kappa$. We need $g| x \in L$. $x$ is subset of $\kappa \times \gamma$ where $\gamma < i^*$. So $x \in L_{i^*}$ and therefore a subset of the union of finitely many $X_n$. So $g| x$ is the union of finitely many constructible functions and is therefore in $L$.

Finally, suppose that $\gamma$ is an indiscernible. We have that $\beta = \sup \{ f(\alpha) \mid \alpha \in \text{Dom}(f) \land f(\alpha) \neq \gamma \}$ is smaller than $\gamma$. By induction we have the desired $g'$ for the following modification of $f$:

$$f'(\alpha) = f(\alpha) \text{ (if } f(\alpha) < \gamma),$$
$$f'(\alpha) = \kappa \text{ (if } f(y) = \gamma).$$

From lemma 32 we have an $L$-good bijection $b_\gamma : \gamma \to \kappa$. So define:

$$g(\alpha, y) = g'(\alpha, y) \text{ (if } f(\alpha) < \gamma),$$
$$g(\alpha, y) = (\alpha, b_\gamma(y)) \text{ (if } f(\alpha) = \gamma).$$
If \( x \) is a subset of \( \text{Dom}(g) \) in \( L \) of \( L \)-cardinality \( \kappa \) then \( g \restriction x \) is the union of two constructible functions, and is therefore in \( L \). □

**Corollary 36 (Uniformly \( M \)-good bijections)** Let \( M \) be an inner model with \( \text{Card}^L = \text{Card}^M \). Let \( f \in M \) be from \( \kappa \) to \( [\kappa, \kappa^+)_V \). Then there exists a uniformly \( M \)-good bijection \( g : \bigcup_{\alpha \in \kappa} (\{\alpha\} \times f(\alpha)) \to \kappa \times \kappa \), i.e., for each \( \alpha \in \kappa \), \( g \restriction \{\alpha\} \times f(\alpha) \) is a bijection between \( \{\alpha\} \times f(\alpha) \) and \( \{\alpha\} \times \kappa \), and for \( x \subseteq \text{Dom}(g) \), \( \|x\|^M = \kappa \), we have \( g \restriction x \in M \).

**Proof.** As in Corollary 33. □

**Lemma 37 (Stretching below \( \omega^V \))** Let \( f \) be a constructible function from \( \text{Reg}^L \cap \omega^V \) to \( \text{Card}^L \cap \omega^V \) (obeying the Easton conditions). Define \( f' \) by \( f'(\kappa) = \min\{f(\kappa), \omega^V\} \) for each \( \kappa \in \text{Reg}^L \cap \omega^V \). Suppose that \( M \) is an inner model with the same cofinalities as \( L \) such that \( \alpha < \omega^V \to (2^{\alpha})^M \leq \omega^V \), and in \( V \) there is a generic over \( M \) for \( P' = \prod_{\kappa \in \omega^V \cap \text{Reg}^L} \text{Add}(\kappa, f'(\kappa)) \) of \( M \).

Then in \( V \) there is also a generic over \( M \) for \( P = \prod_{\kappa \in \omega^V \cap \text{Reg}^L} \text{Add}(\kappa, f(\kappa)) \) of \( M \).

**Proof.** By Corollary 36 we have a uniformly \( M \)-good bijection \( g : \bigcup_{\alpha < \omega^V} (\{\alpha\} \times f''(\alpha)) \to \omega^V \times \omega^V \), where:

\[
    f''(\alpha) = \max\{f(\alpha), \omega^V\} \text{ if } \alpha \in \text{Reg}^L \cap \omega^V, \\
    f''(\alpha) = \omega^V \text{ otherwise.}
\]

For \( \alpha < \omega^V \) let \( g_\alpha : f''(\alpha) \to \omega^V \) be defined by \( g_\alpha(\beta) = g(\alpha, \beta) \).

Now for \( \kappa \in \text{Reg}^L \cap \omega^V \) define \( \pi_\kappa : \text{Add}(\kappa, f(\kappa))^M \to \text{Add}(\kappa, f'(\kappa))^M \) by:

\[
    \pi_\kappa(p)(a, b) = p(a, b) \text{ if } f(\kappa) = f'(\kappa), \\
    \pi_\kappa(p)(a, b) = p(a, g_\kappa^{-1}(b)) \text{ if } f(\kappa) > f'(\kappa).
\]

And define \( \pi \) from \( P \) to \( P' \) by: \( \pi(p)(\kappa) = \pi_\kappa(p(\kappa)) \) for each \( \kappa \in \text{Reg}^L \cap \omega^V \).

Choose \( G' \) in \( V \) to be \( P' \)-generic over \( M \). Our candidate for a \( P \)-generic over \( M \) is \( G = \{p \in P \mid \pi(p) \in G' \} \). This is a filter, so we need only show that it intersects maximal antichains on \( P \) which belong to \( M \).
Suppose that $A$ is a maximal antichain in $P$ and define $A' = \{ \pi(a) \mid a \in A \}$. $A'$ is an antichain on $P'$; we will show that $A'$ is in fact a maximal antichain on $P'$. Note that $A'$ belongs to $M$ because (by the hypothesis $\alpha < \omega_1^P \to (2^\alpha)^M \leq \omega_1^V$) $P$ is $(\omega_1^V)^+\text{-}\text{c.c.}$ in $M$ and $g$ is a good bijection.

Let $D(A') = \bigcup \{ \text{Dom}(a') \mid a' \in A' \}$. Then for any $p' \in P'$ with $\text{Dom}(p') \subseteq D(A')$ there is $p \in P$ such that $\pi(p) = p'$. This is again because $D(A')$ has $M$-cardinality at most $\omega_1^P$ and $g$ is good. Now let $p'$ belong to $P'$. We want to find some element in $A'$ which is compatible with $p'$. Set $p_1' = p' \upharpoonright D(A')$. Then there is $p$ such that $\pi(p) = p_1'$, $p$ is compatible with some $a \in A$ (because $A$ is a maximal antichain in $P$), and therefore $p_1' = \pi(p)$ is compatible with $\pi(a) \in A'$. As $p', p_1'$ agree on $\text{Dom}(\pi(a)) \subseteq D(A')$, it follows that $p'$ is also compatible with $\pi(a) \in A'$. So $A'$ is a maximal antichain.

Now as $A'$ is a maximal antichain on $P'$ we may choose some $g' \in A' \cap G'$. Then $g' = \pi(p)$ for some $p$ and $p$ belongs to both $A$ and $G$, so we have shown that $G$ is $P$-generic over $M$, as desired. \( \square \)

**Lemma 38** Let $P = \text{\textit{Easton}}_{\kappa \in \text{Reg}} \text{Add}(\kappa, \kappa)$ in $L$. Then $P$ preserves cofinalities and the GCH, and there exists a $P$-generic over $L$ in $L[0^\#]$.

**Proof.** Preservation of cofinalities and of the GCH are straightforward, using the factoring of $P$ as $P(< \kappa) * P(\kappa) * P(> \kappa)$ for $\kappa \in \text{Reg}^L$: “Cofinality greater than $\kappa$” is preserved as $P(\leq \kappa)$ has a dense subset of $L$-cardinality $\kappa$ and $P(> \kappa)$ is $\kappa^+$-closed. The GCH still holds at the infinite cardinal $\lambda$ as $P(\leq \lambda)$ has a dense subset of $L$-cardinality at most $\lambda^+$ and $P(> \lambda)$ is $\lambda^+$-closed.

To build a $P$-generic in $L[0^\#]$ we proceed as in the previous section (although the proof here is much easier). By induction on $i \in I$ we define a generic $G(\leq i)$ for $P(\leq i)$. We inductively ensure the following coherence property: For indiscernibles $i < j$, $G(\leq i)$ is a subset of $G(\leq j)$ and $G(i) \subseteq \text{Add}(i, i)_{L[G(\leq i)]}$ is a subset of $G(j)$. If $i = \min I$ then we choose $G(\leq i)$ to be some $P(\leq i)$ generic, which exists due to the countability of $i^+$. In $L[0^\#]$. Our coherence property ensures that for $i$ a limit indiscernible we can take $G(\leq i)$ to be $\bigcup_{i < \bar{i}} G(\leq \bar{i})$ and $G(i)$ to be $\bigcup_{i < \bar{i}} G(\bar{i})$. The resulting $G(\leq i)$ is $P(\leq i)$-generic as if $D = \tau(\bar{j}, i, \infty)$ is dense in $P(\leq i)$ we choose an
indiscernible $k$ from $(\max(\vec{j}), i)$ and consider $D_k = \tau(\vec{j}, k, \infty)$, a dense subset of $P(\leq k)$. Then by induction there is $\bar{p}$ in $D_k \cap G(\leq k)$ and $\pi_{ki}(\bar{p}) = \bar{p}$ belongs to $D$. By coherence, $\bar{p}$ belongs to $G(\leq i)$, so $G(\leq i)$ meets $D$, as desired.

Finally suppose that $G(\leq i)$ has been defined and we wish to define $G(\leq i^*)$. Now $P(\leq i^*)$ factors as $P(\leq i^*) \simeq P(\leq i) * P(i, i^*)$, where $P(i, i^*)$ is $i^+$-closed and of cardinality $i^*$ in $L[G(\leq i)]$. It follows from Lemma 8 that we can choose a $P(i, i^*)$-generic in $L[0\#]$, resulting in a $P(\leq i^*)$-generic $G(\leq i^*)$ including $G(\leq i)$. Similarly, we can choose a $P(i^*)$-generic in $L[0\#]$ below the condition $G(i)$ in $P(i^*)$. The result is a $P(\leq i^*)$-generic $G(\leq i^*)$ obeying our coherence property.

Theorem 39 Let $f : \text{Reg}^L \to \text{Card}^L$ be an $L$-definable Easton function with parameters $\leq \omega_1^V$. Then there is a cofinality preserving generic extension $M \subseteq V$ of $L$ such that $M \models 2^\kappa = f(\kappa)$ for all $\kappa \in \text{Reg}^L$.

Proof. Let $\gamma = \min\{\alpha \mid f(\alpha) \geq \omega_1^V\}$. We first use the following forcings:

1. $P_1 = \star_{\alpha < \gamma} \prod_{\alpha < \gamma} \text{Add}(\kappa, f(\kappa))$ in $L$.
2. $P_2 = \star_{\kappa < \omega_1^V} \text{Add}(\kappa, \kappa)$ in $L^{P_1}$.
3. $P_3 = \text{Add}(\omega_1^V, \omega_1^V)$ in $L^{P_1 * P_2}$.
4. $P_4 = \prod_{\gamma \leq \kappa < \omega_1^V} \text{Add}(\kappa, \omega_1^V)$ in $L^{P_1 * P_2 * P_3}$.
5. $P_5 = \prod_{\kappa \in \gamma \cap \text{Reg}_L} \text{Add}(\kappa, f(\kappa))$ in $L^{P_1 * P_2 * P_3 * P_4}$.

Here, $\star$ denotes reverse Easton iteration with Easton supports. By Lemma 28, the iteration $P_1 * P_2 * P_3 * P_4 * P_5$ preserve cofinalities over $L$ and forces the generic extension to realise the following Easton function $f'$:

$f'(\kappa) = f(\kappa)$, if $\kappa < \omega_1^V$ and $f(\kappa) < \omega_1^V$,
$f'(\kappa) = \omega_1^V$, if $\kappa < \omega_1^V$ and $f(\kappa) \geq \omega_1^V$. 

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\[ f'(\kappa) = (\omega_1^V)^+ \text{, if } \kappa = \omega_1^V \text{ and } \]
\[ f'(\kappa) = f(\kappa), \text{ if } \kappa > \omega_1^V. \]

We next find generics for the \( P_i \)'s:

**P₁:** has a generic \( G_1 \) by Corollary 29.

**P₂:** By Lemma 38, there exists a \( P_2 \)-generic \( G_2 \) over \( L \). But as \( P(\omega_1^V)^L = P(\omega_1^V)^{L[G_1]} \), it follows that \( G_2 \) is also \( P_2 \)-generic over \( L[G_1] \).

**P₃:** By Lemma 38, there is a \( P_3 \)-generic \( G_3 \) over \( L[G_2] \); again as \( P(\omega_1^V)^L = P(\omega_1^V)^{L[G_1]} \), it follows that \( G_2 \ast G_3 \) is \( P_2 \ast P_3 \)-generic over \( L[G_2] \).

**P₄:** By Lemma 9, \( P_2 \ast P_4 \) is \( \omega_1^V \)-cc in \( V \) and therefore has a generic \( G_2 \ast G_4 \) over \( L \). We can assume that \( G_2 = G_2 \). Then \( G_4 \) is also generic over \( L[G_1 \ast G_2 \ast G_3] \), as all bounded subsets of \( \omega_1^V \) of the latter model belong to \( L[G_2] \).

**P₅:** If \( \gamma \) is less than \( \omega_1^V \) then \( P_5 \) has only countably many subsets in \( L[G_1 \ast G_2 \ast G_3 \ast G_4] \) and therefore there is a \( P_5 \)-generic \( G_5 \) over that model. If \( \gamma \) equals \( \omega_1^V \), then we can apply Lemma 9 to \( P_2 \ast P_4 \ast P_5 \).

Now let \( M \) be the model \( L[G_1 \ast G_2 \ast G_3 \ast G_4 \ast G_5] \) and define \( P_6 = \text{Add}(\omega_1^V, f(\omega_1^V)) \) in \( M \) and \( P_7 = \prod_{\kappa \in \text{Reg} \cap \omega_1^V} \text{Add}(\kappa, f(\kappa)) \) in \( M^{P_6} \). By Lemma 28 it suffices to find generics for \( P_6 \) and \( P_7 \). A \( P_6 \)-generic \( G_6 \) over \( M \) exists by Lemma 34 (Stretching at \( \omega_1^V \)). And a \( P_7 \)-generic over \( M[G_6] \) exists by Lemma 37 (Stretching below \( \omega_1^V \)). This completes the proof. □

### 3 The parameter \( \omega_2 \)

As mentioned in the introduction, we cannot expect every Easton function which is \( L \)-definable with parameter \( \omega_2 \) to be realisable in an inner model, as CH implies that \( 2^{\omega_1} < \omega_2 \) holds in all inner models. A reasonable conjecture would be that any \( L \)-definable Easton function \( f \) with parameter \( \omega_2^V \) satisfying \( f(\alpha) < \omega_2^V \) for countable \( \alpha \in \text{Reg}^L \) can be realised in an inner model of \( L[0^\#] \) with same cofinalities as \( L \). The following result is a step in that direction.

**Theorem 40** There is an inner model of \( L[0^\#] \) with the same cofinalities as \( L \) in which \( \omega_1^V \) is a strong limit cardinal and \( 2^{\omega_1} = \omega_2^V \).
Proof. Assume $V = L[0^\#]$. We shall use the gap 1 morass at $\omega_1$ whose construction is based on [1]. In particular, a morass point is an ordinal $\sigma$ (with sufficient closure) such that $\sigma < \omega_2$ and $L_\sigma[0^\#] \models \omega_1$ is the largest cardinal. The level of the morass point $\sigma$, denoted $\alpha(\sigma)$, is the $\omega_1$ of $L_\sigma[0^\#]$. A morass level is an ordinal of the form $\alpha(\sigma)$ for some morass point $\sigma$. If $\alpha$ is a countable morass level then $\sigma(\alpha)$ denotes the largest $\sigma$ such that $\alpha(\sigma) = \alpha$. If $\alpha$ is countable then $\sigma(\alpha)$ is also countable. To certain pairs $(\sigma, \tau)$ of morass points with $\alpha(\sigma) < \alpha(\tau)$ is associated a $\Sigma_1$ elementary map $\pi_{\sigma\tau}$ from $L_\sigma[0^\#]$ to $L_\tau[0^\#]$ which is the identity on $\alpha(\sigma)$ and sends $\alpha(\sigma)$ to $\alpha(\tau)$. We write $\sigma <_1 \tau$ when $\pi_{\sigma\tau}$ is defined. Also write $\sigma <_0 \tau$ when $\alpha(\sigma) = \alpha(\tau)$ and $\sigma$ is less than $\tau$. All morass points, and all morass levels, are limit points of $I$.

The desired inner model $M$ is a generic extension of $L$ via the forcings described next.

First we add a function $f : \omega_1^V \rightarrow \omega_1^V$ using a reverse Easton iteration of length $\omega_1$. At $L$-regular stage $\alpha \leq \omega_1^V$, force a function from $\alpha$ to $\alpha$ with initial segments of size less than $\alpha$. A generic for this forcing $P$ can be built using $0^\#$: By induction on $i \in I$ we define a generic for $P(\leq i)$. This is easy when $i = \min I$ by the countability of $i^+$ and also when $i = j^*$ is a successor indiscernible, using the $i$-closure of the forcing together with a decomposition of the collection of dense subsets of $P(j,i]$ into the union of $\omega$-many subcollections in $L[G(\leq j)]$ of size $j$. For limit $i$, we take $G(< i)$ to be the union of the $G(< j)$, $j \in I \cap i$. To obtain $G(i)$ we need to know inductively that $j < k < i$ in $I \rightarrow G(j) \subseteq G(k)$. This we can easily arrange at the successor steps of the construction. The desired generic function is $f = G(\omega_1^V) : \omega_1^V \rightarrow \omega_1^V$.

Now notice that in the above construction we had complete freedom about how to define $f$ at indiscernibles. We choose our $f$ so that for a morass level $i$, $f(i) = \sigma(i)$, the largest morass point on level $i$, and for an indiscernible $i$ which is not a morass level, $f(i) = 0$.

The desired inner model $M$ is a generic extension of $L[f]$ via the reverse Easton iteration $Q$ of $\text{Add}(\alpha, f(\alpha))$ for $L$-regular $\alpha < \omega_1^V$ followed by $\text{Add}(\omega_1^V, \omega_2)$. ($\text{Add}(\alpha, 0)$ is the trivial forcing.) To obtain a generic for this iteration, we inductively build generics $G(\leq i)$ for $P(\leq i)$, $i \in I$, $i \leq \omega_1^V$, which obey the following condition:
(*) For a morass point $\sigma$ let $G(\sigma)$ denote the restriction of $G(\alpha(\sigma)) \subseteq \text{Add}(\alpha(\sigma), \sigma(\alpha(\sigma)))$ to $\text{Add}(\alpha(\sigma), \sigma)$. Then if $\sigma < _1 \tau$, we have $\pi_{\sigma \tau}[G(\sigma)] \subseteq G(\tau)$.

Now we describe the inductive construction of the $G(\leq i)$, $i \in I$, $i \leq \omega^Y_1$. If $i$ is not a limit indiscernible then we take $G(\leq i)$ to be any $P(\leq i)$-generic extending the $G(\leq j)$, $j \in I \cap i$; otherwise, $G(\leq i)$ is the union of the $G(\leq j)$, $j \in I \cap i$. If $i$ is not a morass level then we take $G(i)$ to be trivial and if $i$ equals $\omega^Y_1$ then we take $G(i)$ to be the union of the $\pi_{\sigma \alpha}[G(\sigma)]$ for $\sigma < _1 \alpha$, $\alpha(\sigma) = \omega_1$. So assume now that $i$ is a countable morass level and recall that $\sigma(i)$ denotes the largest morass point $\sigma$ such that $\alpha(\sigma) = i$.

Case 1: $\sigma(i)$ is $<_1$ minimal. For $\sigma < _0 \sigma(i)$ define $G(\sigma)$ to be the union of $\pi_{\sigma \alpha}[G(\sigma)]$ for $\sigma < _1 \alpha$. By an inductive use of (*) and morass properties, $G(\sigma)$ is generic for $\text{Add}(i, \sigma)$ for $\sigma < _0 \sigma(i)$ and $G(\sigma) \subseteq G(\sigma')$ for $\sigma < _0 \sigma' < _0 \sigma(i)$. If $\sigma(i)$ is a $<_0$ limit, then take $G(i) = G(\sigma(i))$ to be the union of the $G(\sigma)$, $\sigma < _0 \sigma(i)$ (this is $\text{Add}(i, \sigma(i))$-generic), and otherwise take $G(i) = G(\sigma(i))$ to be any $\text{Add}(i, \sigma(i))$-generic containing the $G(\sigma)$, $\sigma < _0 \sigma(i)$.

Case 2: $\sigma(i)$ is the $<_1$-successor to some $\sigma$. First suppose that $\sigma(i)$ is $<_0$ minimal. We must choose $G(i) = G(\sigma(i))$ to be $\text{Add}(i, \sigma(i))$-generic and to contain $\pi_{\sigma \sigma(i)}[G(\sigma)]$. This brings us to the main step in the proof, based on the following generalisation of Lemma 27.

**Lemma 41** Suppose that $i$ is an indiscernible and $X$ is a set of indiscernibles greater than $i$ of limit ordertype. Let $j$ be the minimum of $X$ and let $H$ denote the Skolem hull of $X \cup i$ in $L$. Then if $x$ is a constructible set of $L$-cardinality at most $j$, the intersection of $x$ with $H$ is a constructible set of $L$-cardinality at most $i$.

**Proof.** We may assume that $x$ is a set of ordinals. Let $k$ denote the least indiscernible such that $x$ is a subset of $k$. We may assume that $x$ is a subset of $\text{sup} X$ and therefore $k$ is at most $\text{sup} X$. In fact, $k$ is less than $\text{sup} X$ as the latter is regular in $L$. Now we prove the lemma by induction on $k$. If $k$ is at most $j$, then the desired conclusion is immediate, as in this case $x \cap H = x \cap (H \cap j) = x \cap i$. If $k$ is greater than $j$ then it cannot be a limit indiscernible, as indiscernibles are $L$-regular. So assume that $k$ is the least indiscernible greater than the indiscernible $l$, where $l$ is at least $j$.  

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If $l$ does not belong to $H$ then $x \cap H = (x \cap l) \cap H$ so the desired conclusion follows by induction. If $l$ does belong to $H$ then as $x$ has $L$-cardinality at most $l$, there is some finite $n$ such that $x$ is a subset of $H_n = \text{the Skolem hull in } L$ of $l \cup \{l\} \cup \infty_n$, where $\infty_n$ consists of $n$ indiscernibles greater than $l$ in $H$ (recall that $X$ has limit ordertype). But now let $\pi$ be a bijection in $H$ between $l$ and $H_n$. We have $x \cap H = \pi[y \cap H]$, where $y = \pi^{-1}[x]$. By induction, $y \cap H$ is constructible of $L$-cardinality at most $i$ and therefore so is $x \cap H$. □

Using this lemma we proceed with the construction of $G(\sigma(i))$ in Case 2 as follows. First select $G'(\sigma(i))$ to be any Add($i, \sigma(i)$)-generic. We must modify $G'(\sigma(i))$ to the desired $G(\sigma(i))$ containing $\pi_{\sigma(i)}[G(\bar{\sigma})]$. By Lemma 41, we obtain a well-defined condition $p^*$ if we modify a condition $p$ in Add($i, \sigma(i)$) so as to agree with $\pi_{\sigma(i)}[G(\bar{\sigma})]$ on the range of $\pi_{\sigma(i)}$. Let $G(\sigma(i))$ consist of all modification $p^*$ of conditions in $p \in G'(\sigma(i))$. Then as in the construction of the generic in the first section of this paper (see the reference to “small modifications”), this modified $G(\sigma(i))$ is also Add($i, \sigma(i)$)-generic. This completes the construction of $G(\leq i)$ in Case 2 when $\sigma(i)$ is $<\_0$ minimal. When $\sigma(i)$ is the $<\_0$-successor to $\sigma_0$, then we choose $G'(\sigma(i))$ to be any Add($i, \sigma(i)$)-generic extending $G(\sigma_0) = \bigcup_{\sigma_0 < \sigma_1} \pi_{\sigma_0}[G(\sigma_0)]$ and then modify it as in the $<\_0$-minimal case to the desired $G(\sigma(i))$ which agrees with $\pi_{\sigma(i)}[G(\bar{\sigma})]$ on the range of $\pi_{\sigma(i)}$. If $\sigma(i)$ is a $<\_0$-limit, then we set $G(\sigma(i))$ to be the union of the $\pi_{\sigma(i)}[G(\bar{\sigma})]$ for $\sigma_0 < \sigma_0 < \sigma(i)$. By an inductive use of (**) and morass properties, it follows that the resulting $G(\sigma(i))$ is a well-defined Add($i, \sigma(i)$)-generic which agrees with $\pi_{\sigma(i)}[G(\bar{\sigma})]$ on the range of $\pi_{\sigma(i)}$.

Case 3: $\sigma(i)$ is a $<\_1$ limit. In this case we take $G(i)$ to be the union of the $\pi_{\sigma(i)}[G(\bar{\sigma})]$. By an inductive use of (**) together with morass properties, this yields a well-defined Add($i, \sigma(i)$)-generic, which by definition contains $\pi_{\sigma(i)}[G(\bar{\sigma})]$ for $\sigma <\_1 \sigma$.

This completes the construction of the $G(\leq i)$ for indiscernibles $i \leq \omega_1^V$. The model $M = L[f][G(\leq \omega_1^V)]$ is the desired inner model of $V = L[0^\#]$ with same cofinalities as $L$ in which $\omega_1^V$ is a strong limit cardinal and $2^{\omega_1^V} = \omega_2^V$. □
Question. Which \( L \)-definable Easton functions with parameters are realisable in an inner model of \( L[0^\#] \) with the same cofinalities as \( L \)?

References


