On strong forms of reflection in set theory

SY-DAVID FRIEDMAN and RADEK HONZIK

Kurt Gödel Research Center for Mathematical Logic,
Währinger Strasse 25, 1090 Vienna Austria
sdf@logic.univie.ac.at (S. D. Friedman)
radek.honzik@univie.ac.at (R. Honzik)

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Abstract. In this paper we review the most common forms of reflection and introduce a new form which we call *sharp-generated reflection*. We argue that sharp-generated reflection is the strongest form of reflection which can be regarded as a natural generalization of the Lévy reflection theorem. As an application we formulate the principle *sharp-maximality* with the corresponding hypothesis $\mathrm{IMH}^{\#}$. $\mathrm{IMH}^{\#}$ is an analogue of the IMH (Inner Model Hypothesis, introduced in [4]) which is compatible with the existence of large cardinals.

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Contents

1 Introduction

In this paper, we focus on *vertical reflection*, i.e. reflection of the universe in terms of its height. Vertical reflection for the universe V can be intuitively formulated as the following principle, denoted (Refl):

(Refl) Any property which holds in V already holds in some initial segment of V.

In other words, (Refl) says that V cannot be described as the unique initial segment of the universe satisfying a given property. The strength of such reflection depends on what we consider by "property".[1]

A priori, there is no need to limit ourselves to first-order properties of V. However, there is a problem expressing higher order properties attributable to V in theories such as ZFC or GB.[2] In section 2, we briefly review the standard ways of expressing (Refl) in ZFC and GB. In Section 2.3, we touch on the subject of strong forms of reflection which imply transcendence over L (i.e. imply $V \neq L$). In Section 3 we formulate our own formalization of (Refl) – perhaps the strongest imaginable natural strengthening of Lévy reflection – using indiscernibles.

2 Vertical reflection

2.1 Reflection in ZF

Probably the first major theorem on reflection for ZF is the following theorem by Lévy.

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[1] Properties are often formulated using higher-order quantification. Let M be a set. We say that a variable x is 1-st order (or of order 1) if it ranges over elements of M. In general, we say that a variable R is $n+1$-th order (or of order $n+1$), $0 < n < \omega$, if it ranges over $\mathscr{P}^n(M)$, where $\mathscr{P}^n(M)$ denotes that iteration of the powerset operation n-many times. A formula φ is Π^n_m (or analogously Σ^n_m) if it starts with a block of universal quantification of variables of order $n+1$, followed by existential quantification of variables of order $n+1$, and these blocks alternate at most m-many times; the rest of the formula can contain variables of order at most $n+1$, and quantifications over variables of order at most n. One can define higher-order formulas of transfinite order, see paragraph 6 in [6] (Kanamori gives reference to Jensen's work, reviewed in Drake's book [3], pp 284 ff) and Remark 2.12 in this paper.

[2] If M is a proper class, then we strictly speaking cannot apply the definitions in Footnote 1 to M (because $\mathscr{P}(M)$ is not defined). Depending on the context, we will therefore need to make some modifications to make the discussion concerning higher-order properties of proper classes (such as V) meaningful.

Theorem 2.1 (Lévy) *Let $\varphi(x_1, \dots, x_n)$ be a first-order formula with free variables shown. Then the following is a theorem of* ZF*:*

(2.1)
$$\forall\alpha\, \forall x_1, \dots, x_n \in V_\alpha\, \exists\beta \geq \alpha\, \big(\varphi(x_1, \dots, x_n) \leftrightarrow (V_\beta, \in) \models \varphi(x_1, \dots, x_n)\big).$$

In fact, Lévy showed that the first-order reflection in Theorem 2.1 is actually equivalent over ZF minus Replacement (R) and Infinity (I) to conjunction of R and I. Thus Lévy's reflection principle is an Axiom of Infinity, postulating the existence of cardinals constructed by replacement from ω. Lévy's theorem captures the idea of (Refl) if by property we mean a property expressible by a first-order property with first-order parameters.

As mentioned above, there is no direct way to generalize Lévy's theorem to higher-order formulas because the language of ZF is first-order. Lévy resolved this problem by studying $(V_\kappa, \in)$ instead of V, with higher-order properties of V_κ being expressible by first-order formulas in V (by way of using the powerset operation).

Definition 2.2 *Let $\varphi(R)$ be a Π^n_m-formula which contains only one free variable R which is second-order. Given $R \subseteq V_\kappa$, we say that $\varphi(R)$ reflects in V_κ if there is some $\alpha < \kappa$ such that:*

$$\text{(2.2)} \qquad \textit{If } (V_\kappa, \in, R) \models \varphi(R), \ \textit{then } (V_\alpha, \in, R \cap V_\alpha) \models \varphi(R \cap V_\alpha).$$

Notice that while φ can be of arbitrary finite order, the free variable in φ is not permitted to be of higher order than 2 (the parameter R). The reason is that a correct formulation of reflection for parameters of higher order than 2 requires some care to avoid inconsistency. See Section 2.3.

The extra parameter R allows us to obtain an inaccessible cardinal if we postulate reflection as in Definition 2.2:

Theorem 2.3 (Lévy) *The following are equivalent:*

(i) *κ is inaccessible.*
(ii) *For every $R \subseteq V_\kappa$ and every first-order formula $\varphi(R)$, $\varphi(R)$ reflects in V_κ.*
(iii) *For every $R \subseteq V_\kappa$, the set $\{\alpha < \kappa \,|\, (V_\alpha, \in, R \cap V_\alpha) \prec (V_\kappa, \in, R)\}$ is closed unbounded.*

For a proof, see [6], Section 6.

Remark 2.4 Note that reflection for first-order formulas according to Definition 2.2 needs to include the parameter R to yield inaccessibles as in

Theorem 2.3. Consider the following example: let κ be inaccessible, and let $\alpha < \beta < \kappa$ be two singular cardinals of countable cofinality such as $(V_\alpha, \in) \prec (V_\kappa, \in)$ and $(V_\beta, \in) \prec (V_\kappa, \in)$; such cardinals $\alpha < \beta$ exist by running the usual Löwenheim-Tarski theorem in V_κ. It follows that $(V_\alpha, \in) \prec (V_\beta, \in)$, and β is not even regular.[3] If we relax the relation of elementary substructure to that of elementary embedding, then we can find $\alpha < \beta$ with $\pi : (L_\alpha, \in) \to (L_\beta, \in)$ already in $\mathsf{ZF} + V = L$: in $(L_\beta, \in)$, where β is singular of uncountable cofinality, use the Löwenheim-Skolem argument to find a countable $(M, \in) \prec (L_\beta, \in)$; then the transitive collapse of M is equal to some L_α, $\alpha < \beta$, and hence the inverse of the collapse is an elementary embedding from $(L_\alpha, \in)$ to $(L_\beta, \in)$.

We have seen in Theorem 2.3 that if the reflection provable in V reflects down to some V_κ, then κ must be inaccessible. We might repeat this idea and argue that it should hold for V_κ (in place of V) that there is some $\alpha < \kappa$ inaccessible. Moreover, we might argue that every formula $\varphi(R)$ true in V_κ should reflect at some inaccessible $\alpha < \kappa$. It is easy to see that this implies that κ must be Mahlo. This construction could be repeated over and over again. However, there is a limit to this type of construction (for a proof, see [6], Theorem 6.7):

Fact 2.5 *The reflection – construed as explained in the previous paragraph – with second-order parameters for higher-order formulas (even of transfinite type) does not yield transcendence over L.*

The form of reflection discussed in this section does not seem to be the true embodiment of (Refl), though. The point is that it does not postulate reflection directly for V, or a single structure of the form $(V_\kappa, \in)$, but rather reflects the process of reflection itself.

2.2 Reflection in GB

Since GB, Gödel-Bernays' first-order theory with two sorts of variables (sets and classes), is finitely axiomatizable, there is no analogue of Lévy's theorem 2.1 provable in GB. However if we go above the consistency strength of GB, we can derive the existence of an inaccessible from such reflection (with a second-order parameter).

[3] This example implies that it is consistent from an inaccessible cardinal that there exist $\alpha < \beta$ singular such that $(V_\alpha, \in) \prec (V_\beta, \in)$ and there exists no inaccessible cardinal in any inner model of the universe; this statement holds in L_κ where κ is the least inaccessible in L.

Definition 2.6 *We say that $\varphi(R)$ with a class parameter R reflects if there is α such that*

$$\varphi(R) \to (V_\alpha, V_{\alpha+1}) \models \varphi(R \cap V_\alpha). \tag{2.3}$$

Note that since φ may contain class variables, we need to specify the intended range of class variables in V_α. As in the previous section, where the parameter R ranged over entire $V_{\alpha+1}$, we postulate that the intended range of the class variables in Definition 2.6 is equal to $V_{\alpha+1}$.

Theorem 2.7 *There is a second-order sentence φ which is provable in* GB *such that if φ reflects at α, i.e. if*

$$\varphi \to (V_\alpha, V_{\alpha+1}) \models \varphi, \tag{2.4}$$

then α is an inaccessible cardinal.

Proof. Take φ to say "there is no function from $\gamma \in$ ORD cofinal in ORD and for every $\gamma \in$ ORD, $2^\gamma \in$ ORD". Clearly, if φ reflects at some α, then α is inaccessible (here we use that the second-order variable range over $\mathscr{P}(V_\alpha) = V_{\alpha+1}$). □

As a corollary we obtain:

Corollary 2.8 *Second-order reflection in* GB *implies the existence of an inaccessible cardinal.*

In the context of GB and reflection for ORD it seems unnatural to consider reflection for classes of classes, etc. because the language of GB is not equipped for it. In Section 3 we introduce a more appropriate framework for discussing reflection of formulas of higher order than two.

2.3 Reflection implying transcendence over L

In Definition 2.2, the parameter R is second-order, i.e. R is a subset of V_κ. What about higher order parameters? A natural definition of relativization $\bar{\mathscr{R}}$ of a third-order parameter $\mathscr{R}$ over V_κ to V_α, $\alpha < \kappa$, seems to be the following:

$$\bar{\mathscr{R}} = \{R \cap V_\alpha \mid R \in \mathscr{R}\}. \tag{2.5}$$

However, one can find an easy example which shows that reflection with a third-order parameter – defined according to (2.5) – is inconsistent.[4] We

[4] Consider the following example. For any infinite ordinal κ, let $\mathscr{R}$ be the collection of all $\alpha < \kappa$ (viewed as subsets of κ), and consider $\varphi(\mathscr{R})$ which says that every element of $\mathscr{R}$ is bounded in κ (φ is first-order with a third-order parameter $\mathscr{R}$). Clearly, $\varphi(\mathscr{R})$ is true in V_κ. However, $\varphi(\bar{\mathscr{R}})$ is false in V_α for every $\alpha < \kappa$. See [7] for more discussion of reflection with higher-order parameters.

might attempt to weaken the notion of reflection in the case of third-order, and higher-order, parameters, and demand that instead of reflection with respect to the identity function (which is the motivation behind the flawed definition (2.5)), we ask for an elementary embedding (see Definition 2.9).

Assume GCH. For a regular cardinal $\kappa > \omega$, there is a straightforward analogy between structures $V_\kappa, V_{\kappa+1}, \ldots$ and $H(\kappa), H(\kappa^+), \ldots$, where $H(\lambda)$ is the collection of all sets whose transitive closure has size less than λ. As it turns out, it is more convenient to formulate the higher-order reflection with the H-hierarchy. Given $H(\kappa)$, we will give a definition of reflection for a parameter $\mathscr{R}$ of degree $n+2$ and a formula φ of degree $n+1$, $0 < n < \omega$. The notation in Definition 2.2, though standard, is now less convenient. We write

$$(H(\kappa^{+n}), \in, \mathscr{R}) \models \varphi(\mathscr{R}) \tag{2.6}$$

instead of $(H(\kappa), \in, \mathscr{R}) \models \varphi(\mathscr{R})$ to express that $\varphi(\mathscr{R})$ holds in $H(\kappa)$ with appropriately interpreted higher-order quantifiers. The notation in (2.6) has the advantage that it emphasizes that the properties of order $n+1$ over $H(\kappa)$ actually reduce to first-order properties over $H(\kappa^{+n})$, with $\mathscr{R}$ being second-order over $H(\kappa^{+n})$.

As the first – and inadequate – attempt to define reflection for higher-order parameters, we my define that $\varphi(\mathscr{R})$ reflects at a regular cardinal $\bar{\kappa} < \kappa$ if there is $\bar{\mathscr{R}}$, a parameter of order $n+2$ over $H(\bar{\kappa})$, and

$$\text{If } (H(\kappa^{+n}), \in, \mathscr{R}) \models \varphi(\mathscr{R}), \text{ then } (H(\bar{\kappa}^{+n}), \in, \bar{\mathscr{R}}) \models \varphi(\bar{\mathscr{R}}). \tag{2.7}$$

However, such a definition of reflection seems too arbitrary in that we are allowed to choose $\bar{\mathscr{R}}$ as convenient, depending on φ. In keeping with the reflection using elementary substructures in Theorem 2.3 (iii), we make our definition more uniform:

Definition 2.9 *Let κ be an uncountable regular cardinal. We say that κ* satisfies reflection with parameters of order $n+2$, $0 < n < \omega$, *if for every $\mathscr{R} \subseteq H(\kappa^{+n})$ there are a regular uncountable cardinal $\bar{\kappa} < \kappa$, $\bar{\mathscr{R}} \subseteq H(\bar{\kappa}^{+n})$, and an embedding $\pi : H(\bar{\kappa}^{+n}) \to H(\kappa^{+n})$ with critical point $\bar{\kappa}$, $\pi(\bar{\kappa}) = \kappa$, such that*

$$\pi : (H(\bar{\kappa}^{+n}), \in, \bar{\mathscr{R}}) \to (H(\kappa^{+n}), \in, \mathscr{R}) \tag{2.8}$$

is elementary.

Note that demanding $(H(\bar{\kappa}^{+n}), \in, \bar{\mathscr{R}}) \prec (H(\kappa^{+n}), \in, \mathscr{R})$ is contradictory;[5] thus the requirement that π is not the identity is essential.

[5]Choose $\mathscr{R}$ as in the example in Footnote 4. By elementarity, $\bar{\mathscr{R}}$ is equal to $H(\bar{\kappa}^+) \cap \mathscr{R}$, which leads to contradiction as in Footnote 4.

The definition 2.9 forces no "canonicity" on π; any embedding which satisfies the requirements will do. One might wonder whether more stringent requirements on π, such as demanding constructibility in some sense, might give the definition more structure. Unfortunately, this cannot be done if by canonicity we mean constructibility in L-like models: By an adaptation of Magidor's proof in [8], which shows that supercompactness can be captured by existence of elementary embeddings between initial segments of V, one can show:

Theorem 2.10 *The following hold:*

(i) (GCH) κ satisfies reflection with parameters of order $n+4$, $0 \leq n < \omega$, if and only if κ is κ^{+n}-supercompact. In particular, reflection for parameters of order 4 implies that κ is measurable (κ-supercompact).

(ii) If κ satisfies reflection for parameters of order 3, then $\Box_\kappa$ fails.

Proof. (i) This is a simple corollary of Proposition 7 in [2], which is itself based on [8].

(ii) This follows from a proof of Jensen (circulated notes) that κ^+-subcompactness implies the failure of $\Box_\kappa$. See Remark 2.12. $\Box$

Corollary 2.11 *If there is a cardinal κ which satisfies reflection with third-order parameters (and higher), then $V \neq L$.*

Proof. In L, $\Box_\kappa$ holds for every uncountable cardinal κ. $\Box$

Remark 2.12 Cardinals defined as in Definition 2.9 are called *subcompact cardinals* (κ is κ^{+n}-subcompact for $0 < n < \omega$ iff κ satisfies reflection for parameters of order $n+2$). Subcompact cardinals were defined by Jensen,[6] and apparently for different reasons than the study of reflection (Jensen isolated the concept of subcompact cardinals for his study of the failure of the square). α-subcompact cardinals can be defined for any cardinal $\alpha > \kappa$, not just the κ^{+n}'s for $n < \omega$, and are therefore suitable for expressing reflection with parameters of transfinite order. For more details about subcompact cardinals, see [2].

There are other versions of strong forms of reflection implying transcendence over L; see for instance [9]. However, in our opinion, such strong forms of reflection seem to be too "uncanonical" to count as true formalization of (Refl). See Section 3 for further discussion.

[6]Jensen defined κ to be subcompact if it is κ^+-subcompact according to our definition.

3 Sharp-generated reflection

If we take V to denote not the entire universe of all sets but rather a transitive set which approximates it, then we may consider end-extensions $V \subseteq V^*$ of a larger ordinal length. Now, reflection becomes a more complex, and intriguing, concept. Constructions of this type can be carried out in certain axiomatic theories more complicated than ZF or GB (for example Ackermann's, or theories developed by Reinhardt; see [6], Section 23, for more details). However we think that by treating V as an element of the Hyperuniverse, consisting of all countable transitive models of ZFC (see [5]), we obtain a much stronger (indeed the strongest possible) form of reflection.

Let us extrapolate from the usual reflection and see where it takes us. It is natural to strengthen the reflection of individual first-order properties from V to some V_α to the simultaneous reflection of all first-order properties of V to some V_α, even with parameters from V_α. Thus V_α is an elementary submodel of V. Repeating this process suggests that in fact there should be an increasing, continuous sequence of ordinals $(\kappa_i \mid i < \infty)$ such that the models $(V_{\kappa_i} \mid i < \infty)$ form a continuous chain $V_{\kappa_0} \prec V_{\kappa_1} \prec \cdots$ of elementary submodels of V whose union is all of V (where ∞ denotes the ordinal height of the universe V).

But the fact that for a closed unbounded class of κ's in V, V_κ can be "lengthened" to an elementary extension (namely V) of which it is a rank initial segment suggests via reflection that V itself should also have such a lengthening V^*. But this is clearly not the end of the story, because we can also infer that there should in fact be a continuous increasing sequence of such lengthenings $V = V_{\kappa_\infty} \prec V^*_{\kappa_{\infty+1}} \prec V^*_{\kappa_{\infty+2}} \prec \cdots$ of length the ordinals. For ease of notation, let us drop the *'s and write W_{κ_i} instead of $V^*_{\kappa_i}$ for $\infty < i$ and instead of V_{κ_i} for $i \leq \infty$. Thus V equals W_∞.

But which tower $V = W_{\kappa_\infty} \prec W_{\kappa_{\infty+1}} \prec W_{\kappa_{\infty+2}} \prec \cdots$ of lengthenings of V should we consider? Can we make the choice of this tower "canonical"?

Consider the entire sequence $W_{\kappa_0} \prec W_{\kappa_1} \prec \cdots \prec V = W_{\kappa_\infty} \prec W_{\kappa_{\infty+1}} \prec W_{\kappa_{\infty+2}} \prec \cdots$. The intuition is that all of these models resemble each other in the sense that they share the same first-order properties. Indeed by virtue of the fact that they form an elementary chain, these models all satisfy the same first-order sentences. But again in the spirit of "resemblance", it should be the case that any two pairs $(W_{\kappa_{i_1}}, W_{\kappa_{i_0}})$, $(W_{\kappa_{j_1}}, W_{\kappa_{j_0}})$ (with $i_0 < i_1$ and $j_0 < j_1$) satisfy the same first-order sentences, even allowing parameters which belong to both $W_{\kappa_{i_0}}$ and $W_{\kappa_{j_0}}$. Generalising this to triples, quadruples and n-tuples in general we arrive at the following situation:

$(*)$ Our approximation V to the universe should occur in a continuous elementary chain $W_{\kappa_0} \prec W_{\kappa_1} \prec \cdots \prec V = W_{\kappa_\infty} \prec W_{\kappa_{\infty+1}} \prec W_{\kappa_{\infty+2}} \prec \cdots$

of length the ordinals, where the models W_{κ_i} form a *strongly-indiscernible chain* in the sense that for any n and any two increasing n-tuples $\vec{i} = i_0 < i_1 < \cdots < i_{n-1}$, $\vec{j} = j_0 < j_1 < \cdots < j_{n-1}$, the structures $W_{\vec{i}} = (W_{\kappa_{i_{n-1}}}, W_{\kappa_{i_{n-2}}}, \cdots, W_{\kappa_{i_0}})$ and $W_{\vec{j}}$ (defined analogously) satisfy the same first-order sentences, allowing parameters from $W_{\kappa_{i_0}} \cap W_{\kappa_{j_0}}$.

But this is again not the whole story, as we would want to impose higher-order indiscernibility on our chain of models. For example, consider the pair of models $W_{\kappa_0} = V_{\kappa_0}, W_{\kappa_1} = V_{\kappa_1}$. Surely we would want that these models satisfy the same second-order sentences; equivalently, we would want $H(\kappa_0^+)^V$ and $H(\kappa_1^+)^V$ to satisfy the same first-order sentences. But as with the pair $H(\kappa_0)^V$, $H(\kappa_1)^V$ we would want $H(\kappa_0^+)^V$, $H(\kappa_1^+)^V$ to satisfy the same first-order sentences *with parameters*. How can we formulate this? For example, consider κ_0, a parameter in $H(\kappa_0^+)^V$ that is second-order with respect to $H(\kappa_0)^V$; we cannot simply require $H(\kappa_0^+)^V \vDash \varphi(\kappa_0)$ iff $H(\kappa_1^+)^V \vDash \varphi(\kappa_0)$, as κ_0 is the largest cardinal in $H(\kappa_0^+)^V$ but not in $H(\kappa_1^+)^V$. Instead we need to replace the occurrence of κ_0 on the left side with a "corresponding" parameter on the right side, namely κ_1, resulting in the natural requirement $H(\kappa_0^+)^V \vDash \varphi(\kappa_0)$ iff $H(\kappa_1^+)^V \vDash \varphi(\kappa_1)$. More generally, we should be able to replace each parameter in $H(\kappa_0^+)^V$ by a "corresponding" element of $H(\kappa_1^+)^V$ and conversely, it should be the case that, to the maximum extent possible, all elements of $H(\kappa_1^+)^V$ are the result of such a replacement.This also should be possible for $H(\kappa_0^{++})^V$, $H(\kappa_0^{+++})^V$, ... and with the pair κ_0, κ_1 replaced by any pair κ_i, κ_j with $i < j$.

It is natural to solve this parameter problem using embeddings, as in the last subsection. But the difference here is that there is no assumption that these embeddings are internal to V; they need only exist in the "real universe", outside of V. In this way we will arrive at a principle compatible with $V = L$ in which the choice of embeddings is indeed "canonical".

Thus we are led to the following.

Definition 3.1 *Let V be a transitive set-size model of ZFC of ordinal height ∞. We say that V is* indiscernibly-generated *iff there exists a class-size model W, a continuous sequence $\kappa_0 < \kappa_1 < \ldots$ of the length the ordinals such that $\kappa_\infty = \infty$ and commuting elementary embeddings $\pi_{ij} : W \to W$ where π_{ij} has critical point κ_i and sends κ_i to κ_j. Moreover, for any $i \leq j$, any element of W should be first-order definable in W from elements of the range of π_{ij} together with κ_k's for k in the interval $[i, j)$.*

The last clause in the above definition formulates the idea that to the maximum extent possible, elements of W are in the range of the embedding π_{ij} for each $i \leq j$; notice that the interval $[\kappa_i, \kappa_j)$ is disjoint from this range, but by allowing the κ_k's in this interval as parameters, we can first-order definably recover everything.

Indiscernible-generation as formulated in the above definition does indeed give us our advertised higher-order indiscernibility: For example, in the notation of the definition, if $\vec{i} = i_0 < i_1 < \ldots < i_{n-1}$ and $\vec{j} = j_0 < j_1 < \ldots < j_{n-1}$ with $i_0 \leq j_0$, and $x_k \in H(\kappa_{i_0}^+)^W$ for $k < n$ then the structure $W_{\vec{i}}^+ = (H(\kappa_{i_{n-1}}^+)^W, H(\kappa_{i_{n-2}}^+)^W, \cdots, H(\kappa_{i_0}^+)^W)$ satisfies a sentence with parameters $(\pi_{i_0,i_{n-1}}(x_{n-1}), \ldots, \pi_{i_0,i_0}(x_0))$ iff $W_{\vec{j}}^+$ satisfies the same sentence with corresponding parameters $(\pi_{i_0,j_{n-1}}(x_{n-1}), \ldots, \pi_{i_0,j_0}(x_0))$. There is a similar statement with W^+ replaced by higher-order structures $W^{+\alpha}$ for arbitrary α.

Indiscernible-generation has a clearer formulation in terms of $\#$*-generation*, which we explain next.

Definition 3.2 *A structure $N = (N, U)$ is called a* sharp with critical point *κ, or just a $\#$, if the following hold:*

(i) N is a model of ZFC$^-$ *(*ZFC *minus powerset) in which κ is the largest cardinal and κ is strongly inaccessible.*
(ii) (N, U) is amenable (i.e. $x \cap U \in N$ for any $x \in N$).
(iii) U is a normal measure on κ in (N, U).
(iv) N is iterable, i.e., all of the successive iterated ultrapowers starting with (N, U) are well-founded, yielding iterates (N_i, U_i) and Σ_1 elementary iteration maps $\pi_{ij} : N_i \to N_j$ where $(N, U) = (N_0, U_0)$.

We will use the convention that κ_i denotes the the largest cardinal of the i-th iterate N_i.

If N is a $\#$ and λ is a limit ordinal then $LP(N_\lambda)$ denotes the union of the $(V_{\kappa_i})^{N_i}$'s for $i < \lambda$. (LP stands for "lower part".) $LP(N_\infty)$ is amodel of ZFC.

Definition 3.3 *We say that a transitive model V of* ZFC *is* $\#$-generated *iff for some sharp $N = (N, U)$ with iteration $N = N_0 \to N_1 \to \cdots$, V equals $LP(N_\infty)$ where ∞ denotes the ordinal height of V.*

Fact 3.4 *The following are equivalent for transitive set-size models V of* ZFC*:*

(i) V is indiscernibly-generated.
(ii) V is $\#$-generated.

Proof. The last clause in the definition of indiscernible-generation ensures that the embeddings π_{ij} in that definition in fact arise from iterated ultrapowers of the embedding π_{01}, itself an ultrapower by the measure U_0

on κ_0 given by $X \in U_0$ iff $\pi_{01}(X)$ contains κ_0 as an element. Conversely, if (N, U) generates V, then the chain of embeddings given by iteration of (N, U) witnesses that V is indiscernibly-generated. $\square$

#-generation fulfils our requirements for vertical maximality, with powerful consequences for reflection. L is #-generated iff $0^{\#}$ exists, so this principle is compatible with $V = L$. If V is #-generated via (N, U) then there are embeddings witnessing indiscernible-generation for V which are canonically-definable through iteration of (N, U). Although the choice of # that generates V is not in general unique, it can be taken as a fixed parameter in the canonical definition of these embeddings. Moreover, #-generation evidently provides the maximum amount of vertical reflection: If V is generated by (N, U) as $LP(N_\infty)$ where ∞ is the ordinal height of V, and x is any parameter in a further iterate $V^* = N_{\infty^*}$ of (N, U), then any first-order property $\varphi(V, x)$ that holds in V^* reflects to $\varphi(V_{\kappa_i}, \bar{x})$ in N_j for all sufficiently large $i < j < \infty$, where $\pi_{j,\infty^*}(\bar{x}) = x$. This implies any known form of vertical reflection and summarizes the amount of reflection one has in L under the assumption that $0^{\#}$ exists, the maximum amount of reflection in L.

Thus #-generation tells us what lengthenings of V to look at, namely the initial segments of V^* where V^* is obtained by further iteration of a # that generates V. And it fully realises the idea that V should look exactly like closed unboundedly many of its rank initial segments as well as its "canonical" lengthenings of arbitrary ordinal height.

Therefore we believe that #-generated models are the strongest formalization of the principle of reflection (Refl) – we call this form of reflection *sharp-generated reflection*, and we shall call these models as *vertically maximal*.

Remark 3.5 Notice that a sharp-generated model can satisfy $V = L$, and hence our reflection principle is compatible with L. The reason is that the non-trivial embeddings obtained from the sharp-iteration are external to the model in question. This contrasts with the use of nontrivial embeddings in the discussion ending with Corollary 2.11 which are internal to the relevant models (and thus imply $V \neq L$). Compatibility with L agrees with our intuition that a natural formulation of vertical reflection (Refl) should be determined by the *height* of the universe, and not its *width* (and L has the same height as V).

4 An application

We now apply sharp-generated reflection to formulate an analogue of the IMH principle in [4].

4.1 Vertically maximal models and IMH

The Hyperuniverse is the collection of all countable transitive models of ZFC. We view members of the Hyperuniverse as possible pictures of V which mirror all possible first-order properties of V. The Hyperuniverse Programme, which originated in [4], is concerned with the formulation of natural criteria for the selection of preferred members of the Hyperuniverse. First-order sentences holding in the preferred universes can be taken to be true in the "real V"; in other words, preferred universes may lead to adoption of new axioms. Models satisfying IMH, and IMH$^{\#}$ introduced below, are examples of such preferred universes.

Definition 4.1 *We say that a $\#$-generated model M is $\#$-maximal if and only if the following hold. Whenever M is a definable inner model of M' and M' is $\#$-generated, then every sentence φ, i.e. without parameters, which holds in a definable inner model of M' already holds in some definable inner model of M.*

We say that a $\#$-generated model M satisfies IMH$^{\#}$ if it is $\#$-maximal.

Note that IMH$^{\#}$ differs from IMH by demanding that both M and M', the outer model, are of a specific kind, i.e. should be $\#$-generated (while the outer models considered in IMH are arbitrary). The motivation behind this requirement is that not all outer models count as "maximal"; if our main motivation is formulated in terms of maximality, consideration of non-maximal models as the outer models seems counterintuitive. Indeed, inclusion of such non-maximal models leads to incompatibility of maximal universes satisfying IMH with inaccessible cardinals (see [4]).

Theorem 4.2 *Assume there is a Woodin cardinal with an inaccessible above. Then there is a model satisfying IMH$^{\#}$.*

Proof. For each real R let $M^{\#}(R)$ be $L_\alpha[R]$ where α is least so that $L_\alpha[R]$ is $\#$-generated. Note that $R^{\#}$ exists for each $R \subseteq \omega$ by our large cardinal assumption. The Woodin cardinal with an inaccessible above implies enough projective determinacy to enable us to use Martin's theorem, see [6] Proposition 28.4, to find $R \subseteq \omega$ such that the theory of $(M^{\#}(S), \in)$ for $R \leq_T S$ stabilizes. By this we mean that for $R \leq_T S$, where $\leq_T$ denotes the Turing reducibility relation, the theories of $(M^{\#}(R), \in)$ and $(M^{\#}(S), \in)$ are the same.[7]

[7] In more detail, given a sentence σ in the language with $\{\in\}$ consider the set of Turing degrees $X_\sigma = \{S \mid (M^{\#}(S), \in) \models \sigma\}$. X_σ has a projective definition ($\boldsymbol{\Delta^1_2}$). By Martin's theorem, X_σ or $X_{\neg\sigma}$ contains a cone of degrees. Denote Y_{σ^*} the unique set of the two X_σ and $X_{\neg\sigma}$ which contains the cone. Then $\bigcap_\sigma Y_{\sigma^*}$ contains a cone. Take R to be the base of this cone.

We claim that $M^\#(R)$ satisfies IMH$^\#$: Indeed, let M be a #-generated outer model of $M^\#(R)$ with a definable inner model satisfying some sentence φ. Let α be the ordinal height of $M^\#(R)$ (= the ordinal height of M). By Theorem 9.1 in [1], M has a #-generated outer model W of the form $L_\alpha[S]$ for some real S with $R \leq_T S$. Of course α is least so that $L_\alpha[S]$ is #-generated as it is least so that $L_\alpha[R]$ is #-generated.So W equals $M^\#(S)$. By the choice of R, $M^\#(R)$ also has a definable inner model satisfying φ. So $M^\#(R)$ is #-maximal. □

4.2 IMH$^\#$ is compatible with large cardinals

Finally, we show that – unlike IMH – IMH$^\#$ is compatible with large cardinals.

Theorem 4.3 *Assume there is a Woodin cardinal with an inaccessible above. Then for some real R, any #-generated transitive model M containing R also models IMH$^\#$.*

Proof. Let R be as in the proof of Theorem 4.2. Thus $M^\#(R) = L_\alpha[R]$ is a #-generated model of IMH$^\#$. Now suppose that $M^* = L_{\alpha^*}[R]$ is obtained by iterating $L_\alpha[R]$ past α; we claim that M^* is also a model of IMH$^\#$: Indeed, suppose that W is a #-generated outer model of M^* which has a definable inner model satisfying some sentence φ. Again by Jensen's Theorem 9.1 in [1], we can choose W to be of the form $L_{\alpha^*}[S]$ for some real $S \geq_T R$. But then $L_{\alpha^*}[S]$ is an iterate of $M^\#(S)$ (via the iteration given by $S^\#$) and therefore $M^\#(S)$ also has a definable inner model of φ. By the choice of R, $M^\#(R)$, and therefore by iteration also $L_{\alpha^*}[R]$, has a definable inner model of φ. This verifies the IMH$^\#$ for M^*.

Now any #-generated transitive model M containing R is an outer model of such a model of the form $L_{\alpha^*}[R]$ as above and therefore is also a model of IMH$^\#$. □

Corollary 4.4 *Assume the existence of a Woodin cardinal with an inaccessible above and suppose that φ is a sentence that holds in some V_κ with κ measurable. Then there is a transitive model which satisfies both the IMH$^\#$ and the sentence φ.*

Proof. Let R be as in Theorem 4.3 and let U be a normal measure on κ. The structure $N = (H(\kappa^+), U)$ is a #; iterate N through a large enough ordinal ∞ so that $M = LP(N_\infty)$, the lower part of the model generated by N, has ordinal height ∞. Then M is #-generated and contains the real R. It follows that M is a model of the IMH$^\#$. Moreover, as M is the union of an elementary chain $V_\kappa = V_\kappa^N \prec V_{\kappa_1}^{N_1} \prec \cdots$ where φ is true in V_κ, it follows that φ is also true in M. $\square$

Note that in Corollary 4.4, if we take φ to be any large cardinal property which holds in some V_κ with κ measurable, then we obtain models of the IMH$^\#$ which also satisfy this large cardinal property. This implies the compatibility of the IMH$^\#$ with arbitrarily strong large cardinal properties.

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