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TALL α-RECURSIVE STRUCTURES

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ABSTRACT. The Scott rank of a structure M, sr(M), is a useful measure of its model-theoretic complexity. Another useful invariant is o(M), the ordinal height of the least admissible set above M, defined by Barwise. Nadel showed that sr$(M) \leq$ o(M) and defined M to be tall if equality holds. For any admissible ordinal α there exists a tall structure M such that o$(M) = \alpha$. We show that if $\alpha = \beta^+$, the least admissible ordinal greater than β, then M can be chosen to have a β-recursive presentation. A natural example of such a structure is given when $\beta = \omega_1^L$ and then using similar ideas we compute the supremum of the levels at which $\Pi_1(L_{\omega_1^L})$ singletons appear in L.

The results in this paper concern structures which are complicated model-theoretically, yet recursion-theoretically simple. Fix a structure M for a language $\mathcal{L}$ of finite similarity type. The Scott rank of M is defined as follows: Let $\bar{x}$, $\bar{y}$, $\bar{x}'$, $\bar{y}'$,... range over $|M|^{<\omega}$. By induction define a sequence of relations $\underset{\beta}{\sim}$ on members of $|M|^{<\omega}$ or the same length:

$$\begin{aligned}
\bar{x} \underset{0}{\sim} \bar{y} \quad & \text{iff } \bar{x}, \bar{y} \text{ realize the same atomic type in } M, \\
\bar{x} \underset{\beta+1}{\sim} \bar{y} \quad & \text{iff } \forall \bar{x}' \exists \bar{y}' (\bar{x} * \bar{x}' \underset{\beta}{\sim} \bar{y} * \bar{y}') \text{ and} \\
& \quad \forall \bar{y}' \exists \bar{x}' (\bar{x} * \bar{x}' \underset{\beta}{\sim} \bar{y} * \bar{y}') \\
\bar{x} \underset{\lambda}{\sim} \bar{y} \quad & \text{iff } \bar{x} \underset{\beta}{\sim} \bar{y} \text{ for all } \beta < \lambda, \lambda \text{ limit.}
\end{aligned}$$

In the above, $*$ denotes concatenation of sequences. Finally, Scott rank (M) is the least α such that $\forall \bar{x} \forall \bar{y} (\bar{x} \underset{\alpha}{\sim} \bar{y} \to \bar{x} \underset{\alpha+1}{\sim} \bar{y})$. Scott rank (M) is a useful measure of the model-theoretic complexity of M.

Nadel [**74**] provides a bound on the Scott rank of a structure M in terms of admissible set theory: Scott rank $(M) \leq$ o(M) where o(M) is the ordinal height of the least admissible set above M (see Barwise [**69**]). M is *tall* if equality holds. This bound is best possible in that for any admissible ordinal α there is a tall structure M such that Scott rank $(M) = \alpha$.

Let β be a limit ordinal. M is *β-recursive* if $|M| = \beta$ and all of the relations, functions of M, are β-recursive. (For a definition of β-recursive, see Friedman [**78**]. In this paper we need only consider those β which are either admissible or the limit of admissible ordinals, in which case β-recursive coincides with $\Delta_1 \langle L_\beta, \epsilon \rangle$.) It is

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shown in Nadel [**74**] that there is an ω-recursive (=recursive) structure of Scott rank ω_1^{ck}. (The example is a recursive linear ordering of ordertype $\omega_1^{ck}+\omega_1^{ck}\cdot\eta_1\eta=$ ordertype of the rationals.) §1 of the present paper shows that for every limit ordinal β there is a β-recursive structure of Scott rank β^+, the least admissible ordinal greater than β. Such a structure M_β is tall since it belongs to L_{β^+} and hence $\mathrm{o}(M_\beta)=\beta^+$. Define $L_{\infty\omega_1}$-rank (M) in exactly the same way as Scott rank (M) except where $\bar{x}$, $\bar{y}$, $\bar{x}',\bar{y}',\ldots$ now range over $|M|^{<\omega_1}$. §2 focuses on the special case: $\beta=\omega_1$. Using entirely different methods than in §1 a natural example of an ω_1-recursive structure of $L_{\infty\omega_1}$-rank ω_1^+ is presented (from this an ω_1-recursive structure of Scott rank ω_1^+ is easily obtained). Similar techniques are then used to show that $\Pi_1(L_{\omega_1})$-singletons appear cofinally inside L_σ, where σ is the least stable ordinal greater than ω_1.

1. Game rank versus Scott rank. The goal of this section is to prove

THEOREM 1. *For any limit ordinal β there is a β-recursive structure of Scott rank $\beta^+=$ least admissible ordinal greater than β.*

It clearly suffices to treat the case where β is either admissible or the limit of admissible ordinals. It will also be convenient to assume that β is greater than ω (otherwise the result is known).

The proof of Theorem 1 can be outlined as follows: We first show that there is a β-recursive open game with a winning strategy for the "closed player", but none inside L_{β^+}. This allows one to build a β-recursive tree T of "game rank" β^+. Then a β-recursive structure M of Scott rank β^+ is obtained by building M so that its Scott analysis is very similar to the "game analysis" of T.

We must first describe the "game rank" of a tree. All trees are subtrees of $\beta^{<\omega}=$ all finite sequences of ordinals less than β. Our definition here is rather nonstandard but is designed to allow the transition from game rank to Scott rank to go smoothly.

Let T be a tree. If $\eta=\langle\eta(0),\eta(1),\ldots\rangle\in T$ has even length we let $(\eta)_{\text{even}}=\langle\eta(0),\eta(2),\ldots\rangle$. Let $A_k=\{(\eta)_{\text{even}}\mid\eta\in T,\ l(\eta)=\text{length}(\eta)=2k\}$. For $v\in A_k$ let $B_v=\{\eta\in T\mid(\eta)_{\text{even}}=v\}$. If $\eta\in T$ has even length we define $\mathrm{Rk}(\eta)$ by

$\mathrm{Rk}(\eta)=0\leftrightarrow$ there is $v\supseteq(\eta)_{\text{even}}$ such that η has no extension in B_v, $v\in\bigcup_k A_k$;

$\mathrm{Rk}(\eta)=\alpha>0\leftrightarrow 0\neq\mathrm{Rk}(\eta)\neq\beta$ for all $\beta<\alpha$ and there is $v\supseteq(\eta)_{\text{even}}$ such that

$$\eta'\supseteq\eta,\quad\eta'\in B_v\rightarrow\mathrm{Rk}(\eta')=\beta\text{ for some }\beta<\alpha,$$

$\mathrm{Rk}(\eta)=\infty\leftrightarrow\forall\alpha\,\mathrm{Rk}(\eta)\neq\alpha$, $\mathrm{Rk}(T)=\sup\{\mathrm{Rk}(\eta)\mid\eta\in T\text{ and }\mathrm{Rk}(\eta)\neq\infty\}$.

Thus $\mathrm{Rk}(\eta)$ measures how good a position player I is in after η has been played in the following game: Players I and II alternately choose v_0, η_0, v_1, $\eta_1,\ldots$ with the restrictions that $v_0\subseteq v_1\subseteq\ldots$, $\eta_0\subseteq\eta_1\subseteq\ldots$, $\eta_i\in B_{v_i}$, $v_i\in\bigcup_k A_k$. Player I wins if at some stage player II can make no legal move. Otherwise player II wins.

LEMMA 2. *There is a β-recursive tree T such that* $\mathrm{Rk}(T)=\beta^+$.

PROOF. We use some ideas from β-logic. Enlarge the language of set theory by adjoining (Henkin) constants c_0, $c_1,\ldots$ and a name β' for each ordinal $\beta'\leq\beta$.

Formulas in this language can be easily coded by ordinals less than β. Let S consist of the following sentences in this language:

(a) Axioms for admissibility,

(b) $\underline{\beta'}$ is an ordinal, $\underline{\beta_1} \in \underline{\beta_2}$ (whenever $\beta_1 < \beta_2 \leq \beta$).

Then the tree T consists of all sequences of sentences $\langle \phi_0, \phi_1, \ldots \rangle$ such that

(i) if $\phi_{2n} = \sim\psi$ then $\phi_{2n+1} = \psi$ or $\sim\psi$,

(ii) if $\phi_{2n} = \exists x\, \psi$ then $\phi_{2n+1} = \psi(c_k)$ some k or $\sim\phi_{2n}$,

(iii) if $\phi_{2n} = \psi_1 \vee \psi_2$ then $\phi_{2n+1} = \psi_1$ or ψ_2 or $\sim\phi_{2n}$,

(iv) if $\phi_{2n} =$ "$c_k \in \underline{\beta}$" then $\phi_{2n+1} =$ "$c_k = \underline{\beta'}$" some $\beta' < \beta$ or $\sim\phi_{2n}$,

(v) $\phi_1 \wedge \phi_3 \wedge \phi_5 \wedge \cdots$ is consistent with S.

Since $\beta > \omega$ condition (v) is β-recursive.

CLAIM. $\mathrm{Rk}(T) = \beta^+$.

PROOF OF CLAIM. As the inductive definition of Rk can be carried out in L_{β^+} it is clear that $\mathrm{Rk}(T) \leq \beta^+$. By absoluteness we can assume that β is countable. As S has a model where β is standard, $\mathrm{Rk}(\phi) = \infty$. Now suppose $\mathrm{Rk}(T) = \gamma < \beta^+$. Let ψ_0, $\psi_1, \ldots$ be a listing of the sentences in this language. Define ϕ_0, $\phi_1, \ldots$ by

$$\phi_{2n} = \psi_n,$$

$$\phi_{2n+1} = \text{ least}\, \phi \text{ such that } \langle \phi_0, \ldots, \phi_{2n}, \phi \rangle \text{ has Rk} \geq \gamma.$$

As $\{\eta \in T \mid \mathrm{Rk}(\eta) \geq \gamma\} \in L_{\beta^+}$ the sequence $\langle \phi_0, \phi_1, \ldots \rangle \in L_{\beta^+}$. But $\{\phi_{2n+1} \mid n \in \omega\}$ describes the complete Henkin theory of an end extension of L_{β^+}. This is a contradiction. Q.E.D.

We can now describe the structure M to satisfy Theorem 1. Let T be as in Lemma 2. Define A_k, B_v for $v \in \bigcup_k A_k = A$ as before. Let $P_v =$ all finite subsets of B_v, for $v \in A$. Endow each P_v with a distinct $\varnothing_v$ so that $v_1 \neq v_2 \to P_{v_1} \cap P_{v_2} = \varnothing$. The universe of $M = |M| = \bigcup\{P_v \mid v \in A\}$. Introduce predicates for each P_v.

We now provide P_v with an "affine" group structure; that is, a group structure without a distinguished identity. Note that P_v is a group under the operation Δ of symmetric difference. For $w \in P_v$ let $S_{v,w} = \{(w_1, w_2) \mid w_1 \Delta w_2 = w\}$.

Notice that with these relations, any automorphism of P_v is determined by its action at a single argument.

Finally, we introduce functions connecting the different P_v's. If $v * \langle\alpha\rangle \in A_n$ then $f_{v*\langle\alpha\rangle}$ is defined by: $f_{v*\langle\alpha\rangle}(w) = \{\eta \restriction 2n-2 \mid \eta \in w\}$ for $w \in P_{v*\langle\alpha\rangle}$; $f_{v*\langle\alpha\rangle}(w) = w$ otherwise. Thus any automorphism of $P_{v*\langle\alpha\rangle}$ has a unique extension to P_v preserving the function $f_{v*\langle\alpha\rangle}$.

Thus the desired structure is $M = \langle |M|, P_v, S_{v,w}, f_{v*\langle\alpha\rangle}\rangle$, $v \in A$, $w \in P_v$. It remains to compute the Scott rank of M.

For any collection G of partial functions from M to M define G-$\mathrm{Rk}(g)$ for $g \in G$ by

G-$\mathrm{Rk}(g) \geq 0 \leftrightarrow g \in G$;

G-$\mathrm{Rk}(g) \geq \alpha + 1 \leftrightarrow \forall m \in |M| \exists h \in G(g \subseteq h, m \in \mathrm{Dom}(h)$, G-$\mathrm{Rk}(h) > \alpha)$ and $\forall m \in |M| \exists h \in G(g \subseteq h, m \in \mathrm{Range}(h)$, G-$\mathrm{Rk}(h) \geq \alpha)$;

G-$\mathrm{Rk}(g) \geq \lambda \leftrightarrow \forall \alpha < \lambda$ G-$\mathrm{Rk}(g) \geq \alpha$ for limit λ;

G-$\mathrm{Rk}(g) = \infty \leftrightarrow G$-$\mathrm{Rk}(g) \geq \alpha$ for all α.

Also let $\mathrm{Rk}(G) = \sup\{G\text{-}\mathrm{Rk}(g) \mid g \in G, G\text{-}\mathrm{Rk}(g) < \infty\}$. Thus we are interested in showing that $\mathrm{Rk}(G_0) = \beta^+$ where $G_0 =$ all finite partial isomorphisms of M.

For any $D \subseteq |M|$ let $\overline{D} =$ closure $(D) = \bigcup\{P_v \mid \text{For some}\, v' \supseteq v,\ D \cap P_{v'} \neq \varnothing\}$. As remarked earlier any partial isomorphism of M with domain D has a unique extension to a partial isomorphism with domain (and range) $\overline{D}$. Thus it suffices to show that $\mathrm{Rk}(G_1) = \beta^+$ where $G_1 = \{g \in G_0 \mid \mathrm{Dom}(g) = \overline{\mathrm{Dom}(g)}\}$.

Now if $g \in G_1$ then g is uniquely determined by g^* which is defined by Domain $(g^*) = \{v \mid P_v \subseteq \mathrm{Dom}(g)\}$, $g^*(v) = g(\varnothing_v)$. Moreover, g^* satisfies

$$(*) \qquad f_{v*\langle\alpha\rangle}(g^*(v * \langle\alpha\rangle)) = g^*(v).$$

Conversely, any function h with domain a finite $t \subseteq A$ closed under initial segments, obeying $(*)$ must be of the form g^* for some g. Let $H = \{g^* \mid g \in G_1\}$. Then $\mathrm{Rk}(G_1) = \mathrm{Deg}(H)$ which is defined by

$\mathrm{Deg}(h) \geq 0 \leftrightarrow h \in H$;

$\mathrm{Deg}(h) \geq \alpha + 1 \leftrightarrow \forall v \in A \exists h_1 \supseteq h (v \in \mathrm{Dom}(h_1), \mathrm{Deg}(h_1) \geq \alpha)$;

$\mathrm{Deg}(h) \geq \lambda \leftrightarrow \forall \alpha < \lambda\, \mathrm{Deg}(h) \geq \alpha$ for limit λ;

$\mathrm{Deg}(h) = \infty \leftrightarrow \mathrm{Deg}(h) \geq \alpha$ for all α, $\mathrm{Deg}(H) = \sup\{\mathrm{Deg}(h) \mid \mathrm{Deg}(h) < \infty\}$.

Thus it suffices to show that $\mathrm{Deg}(H) = \beta^+$.

Our final claim establishes the theorem by relating Deg (defined on H) to Rk (defined on $\eta \in T$, length(η) even).

CLAIM. For $h \in H$, $\mathrm{Deg}(H) = \min\{\mathrm{Rk}(\eta) \mid \eta \in h(v) \text{ for some } v\}$.

PROOF. By induction on α we show that $\mathrm{Deg}(h) \geq \alpha$ iff $\mathrm{Rk}(h) \geq \alpha$ iff $\mathrm{Rk}(\eta) \geq \alpha$ for all $\eta \in \bigcup \mathrm{Range}(h)$. This is trivial for $\alpha = 0$ or for limit α (by induction). Let $\alpha = \gamma + 1$. Suppose $\mathrm{Rk}(\eta) \geq \gamma + 1$ for all $\eta \in \bigcup \mathrm{Range}(h)$ and $v \in A$. We show that $\exists h_1 \supseteq h$ ($v \in \mathrm{Dom}(h_1)$ and $\mathrm{Rk}(\eta) \geq \gamma$ for all $\eta \in \bigcup \mathrm{Range}(h_1)$). Let $v_0 \subseteq v$ be maximal, $v_0 \in \mathrm{Dom}(h)$. For each $\eta \in h(v_0)$ choose $\eta' \supseteq \eta$, $\eta' \in B_v$ so that $\mathrm{Rk}(\eta') \geq \gamma$ (this is possible since $\mathrm{Rk}(\eta) \geq \gamma + 1$). Then set $h_1(v') = h(v')$ for $v' \in \mathrm{Dom}(h)$, $h_1(v \restriction k) = \{\eta' \restriction 2k \mid \eta \in h(v_0)\}$ for $k \leq \mathrm{length}(v)$.

Conversely suppose $\mathrm{Deg}(h) \geq \gamma + 1$, $\eta \in \bigcup \mathrm{Range}(h)$. We show that for all $v \supseteq (\eta)_{\mathrm{even}}$ there is $\eta' \supseteq \eta$ such that $\eta' \in B_v$, $\mathrm{Rk}(\eta') \geq \gamma$. For, given $v \supseteq (\eta)_{\mathrm{even}}$ let $h_1 \supseteq h$, $v \in \mathrm{Dom}(h_1)$, $\mathrm{Deg}(h_1) \geq \gamma$. By induction, $\mathrm{Rk}(\eta') \geq \gamma$ for all $\eta' \in h_1(v)$. But η has an extension $\eta' \in h_1(v)$ as $h_1 \in H$. Q.E.D.

Finally as $\mathrm{Rk}(T) = \beta^+$ we conclude $\mathrm{Deg}(H) = \beta^+$ and hence the theorem.

2. ω_1-recursive trees. We use here Gödel condensation methods to build an ω_1-recursive tree T of $L_{\infty\omega_1}$-rank ω_1^+ = least admissible ordinal greater than ω_1. For simplicity assume $\omega_1 = \omega_1^L$. The general case follows from the fact that the proof given below can be easily adapted to any L-cardinal κ such that κ is regular in L_α, α = least admissible greater than κ

Let $S = \{\alpha < \omega_1 \mid \alpha \text{ admissible}, L_\alpha \vDash \omega_1 \text{ exists and is the largest admissible}\}$.
A typical member of S is α where L_α is the transitive collapse of a countable elementary submodel of $L_{\omega_1^+}$.

We first define the tree $T' = \{(\alpha_0, \dots, \alpha_n) \mid \text{For all } i,\ \alpha_i \in S,\ \alpha_i < \alpha_{i+1}$ and there exists $\Pi\colon L_{\alpha_i} \overset{\equiv}{\to} L_{\alpha_{i+1}}\}$. Note that Π as above must be the identity on $\omega_1^{L_{\alpha_i}}$ and every element of L_{α_i} is definable over L_{α_i} from ordinals $\leq \omega_1^{L_{\alpha_i}}$. Thus if Π exists in the definition of T' then Π^{-1} must be the transitive collapse of H = Skolem hull of $\omega_1^{L_{\alpha_i}}$ inside $L_{\alpha_{i+1}}$. This proves that T' is ω_1-recursive.

The desired tree T is obtained via a minor modification of T'. This modification is needed to eliminate certain inhomogeneities on T'. Define $T = \{((\alpha_0, i_0), \dots, (\alpha_n, i_n)) \mid$ For all k, $\alpha_k \in S$, $i_k \in \omega$, $\alpha_k \leq \alpha_{k+1}$ and there exists $\Pi\colon L_{\alpha_k} \overset{\equiv}{\to} L_{\alpha_{k+1}}\}$. (Thus an ordinal $\alpha \in S$ can be "repeated" countably often.) As before T is w_1-recursive. Our goal is to show that T has $L_{\infty\omega_1}$-rank w_1^+. (We shall in fact show that T is isomorphic to the tree $\mathcal{T}$ in §1 of Friedman [**81**].)

We begin by analyzing the structure of T. We show that the structure of T below $((\alpha_0, i_0), \dots, (\alpha_n, i_n))$ is determined by the S-rank (α_n). This is defined by

$S\text{-rk}(\alpha) \geq 0 \leftrightarrow \alpha \in S$;

$S\text{-rk}(\alpha) \geq \gamma + 1 \leftrightarrow$ For uncountably many $\alpha' \exists \Pi : L_\alpha \xrightarrow{\equiv} L_{\alpha'}$, $S\text{-rk}(\alpha') \geq \gamma$;

$S\text{-rk}(\alpha) \geq \lambda \leftrightarrow S\text{-rk}(\alpha) \geq \gamma$ for all $\gamma < \lambda$, for limit λ;

$S\text{-rk}(\alpha) = \infty \leftrightarrow S\text{-rk}(\alpha) \geq \gamma$ for all γ.

Also set $\text{Rank}(S) = \sup\{S\text{-rk}(\alpha) \mid \alpha \in S,\ S\text{-rk}(\alpha) < \infty\}$.

We can also define $\text{rk}((\alpha_0, i_0), \dots, (\alpha_n, i_n)) = S\text{-rk}\,(\alpha_n)$, when $((\alpha_0, i_0), \dots, (\alpha_n, i_n)) \in T$. Then a node on T of rk 0 has exactly ω-many immediate extensions on T. A node on T of rk $\gamma > 0$ has exactly ω-many immediate extensions of rk γ and ω_1-many immediate extensions of rk δ for $\delta < \gamma$. A node on T of rk ∞ has ω_1-many immediate extensions of rk ∞.

Our main goal is to show that for each $\sigma_0 \in T$, rk $\sigma_0 = \infty$ or $\sigma_0 = \emptyset$, $\{\sigma \in T \mid \sigma \supseteq \sigma_0 \text{ and } \text{rk}\,\sigma = \infty\} \notin L_{\omega_1^+}$. From this it follows that $L_{\infty\omega_1}$-rank of $T = \omega_1^+$: Note that the inductive definition of rk as well as the inductive analysis of the $L_{\infty\omega_1}$-rank of T can be carried out in $L_{\omega_1^+}$. If $\sigma_0 \in T$, rk $\sigma_0 = \infty$ then σ_0 must have immediate extensions of rk γ for each $\gamma < \omega_1^+$ as otherwise $\{\sigma \in T \mid \sigma \supseteq \sigma_0 \text{ and } \text{rk}\,\sigma = \infty\} = \{\sigma \in T \mid \sigma \supseteq \sigma_0 \text{ and } \text{rk}\,\sigma \geq \gamma\}$ for some $\gamma < \omega_1^+$ and this latter set is a member of $L_{\omega_1^+}$. Thus we can conclude that if two nodes on T lie on the same level and have the same rk, they can be mapped to each other by an automorphism of T. Thus determining the $L_{\infty\omega_1}$-type of nodes on T is nothing more than determining their rk and the level of T on which they lie. If $L_{\infty\omega_1}$-rank of T is less than ω_1^+ then $\{\sigma \in T \mid \text{rk}\,\sigma = \infty\} = \{\sigma \in T \mid \text{rk}\,\sigma \geq \gamma\}$ for some $\gamma < \omega_1^+$ and this latter set belongs to $L_{\omega_1^+}$. This contradicts our main claim.

CLAIM. $S\text{-rk}(\alpha) = \infty \leftrightarrow \alpha < \omega_1$ and $\exists \Pi : L_\alpha \xrightarrow{\equiv} L_{\omega_1^+}$.

From this claim it is clear that $\{\sigma \in T \mid \sigma \supseteq \sigma_0,\ \text{rk}\ \sigma = \infty\} \notin L_{\omega_1^+}$ when rk $\sigma_0 = \infty$ or $\sigma_0 = \emptyset$, as otherwise $\{\alpha < \omega_1 \mid \exists \Pi : L_\alpha \xrightarrow{\equiv} L_{\omega_1^+}\} \in L_{\omega_1^+}$ which is impossible.

PROOF OF CLAIM. Clearly if $\alpha < \omega_1$ and $\exists \Pi : L_\alpha \xrightarrow{\equiv} L_{\omega_1^+}$ then $S\text{-rk}(\alpha) = \infty$ as if X is the set of all such α's then X is uncountable and each element of X can be elementarily embedded in all larger elements of X. For the converse suppose $\alpha \in S$, $S\text{-rk}(\alpha) = \infty$. Choose $\beta > \alpha$, $\exists \Pi : L_\alpha \xrightarrow{\equiv} L_{\omega_1^+}$. Now inductively define $L_\alpha \xrightarrow{\equiv} L_{\alpha_1} \xrightarrow{\equiv} L_{\alpha_2} \xrightarrow{\equiv} \cdots$ and $L_\beta \xrightarrow{\equiv} L_{\beta_1} \xrightarrow{\equiv} L_{\beta_2} \xrightarrow{\equiv} \cdots$ such that S-rk $\alpha_i = S$-rk $\beta_i = \infty$ for each i and $\beta_i < \alpha_i < \beta_{i+1}$. (This is possible by the definition of S-rk.) If Direct $\text{Lim}\langle L_{\alpha_i} \mid i < \omega\rangle$ is well-founded then it is isomorphic to some $L_{\alpha'}$. If Direct $\text{Lim}\langle L_{\beta_i} \mid i < \omega\rangle$ is well-founded then it is isomorphic to some $L_{\beta'}$. But $\omega_1^{L_{\alpha'}} = \omega_1^{L_{\beta'}}$ so $\alpha' = \beta'$ since $\alpha', \beta' \in S$. We conclude that $\exists \Pi_\alpha : L_\alpha \xrightarrow{\equiv} L_{\alpha'}$, $\Pi_\beta : L_\beta \xrightarrow{\equiv} L_{\alpha'}$, so $\Pi_\beta^{-1} \circ \Pi_\alpha : L_\alpha \xrightarrow{\equiv} L_\beta$ (since Π_α, Π_β is just the inverse of the transitive collapse of the Skolem hull of $\omega_1^{L_\alpha}$, $\omega_1^{L_\beta}$ in $L_{\alpha'}$). So $\exists \Pi : L_\alpha \xrightarrow{\equiv} L_{\omega_1^+}$.

It remains to justify the well-foundedness of the direct limits. This is provided by our final subclaim.

SUBCLAIM. Direct $\text{Lim}\langle L_{\alpha_i} \mid i < \omega\rangle$ is well-founded if $L_{\alpha_1} \xrightarrow{\equiv} L_{\alpha_2} \xrightarrow{\equiv} \cdots$ with $\alpha_1 < \alpha_2 < \cdots$ in S.

PROOF. Let $M = \text{Direct Limit}\langle L_{\alpha_i} \mid i < \omega\rangle$ and we identify $\text{sp}(M)$ = standard part of M with some L_γ. Note that $\omega_1^M = \sup\{\omega_1^{L_{\alpha_i}} \mid i < \omega\} < \gamma$. But γ is admissible as either $L_\gamma = M$ or L_γ is the standard part of a model of KP. As $M \vDash \omega_1$ is the largest admissible, we can conclude that $\gamma = (\omega_1^M)^+$.

Now suppose $L_\gamma \neq M$ and choose i and $\Pi\colon L_{\alpha_i} \overset{\equiv}{\to} M$ so that $\text{Range}(\Pi) \nsubseteq L_\gamma$. Let $\lambda < \alpha_i$ be so that $\Pi(\lambda) \notin L_\gamma$. Then $\omega_1^{L_{\alpha_i}} < \lambda$. $L_{\alpha_i} \vDash \omega_1$ is the largest admissible, we may choose $\eta \in T$ such that $\text{Rk}(\eta) = \lambda$ where T is the $\omega_1^{L_{\alpha_i}}$-recursive tree constructed in Lemma 2 (where $\beta = \omega_1^{L_{\alpha_i}}$). Note that for arbitrary $\eta' \in T$, $\text{Rk}(\eta') < \infty$ if and only if player I has a winning strategy at position η' for the game described immediately before Lemma 2.

If T' = tree obtained from Lemma 2 when $\beta = \omega_1^M$ then $\Pi(T) = T'$ and $\Pi(\eta) = \eta$ has nonstandard Rk' (= Rk for T'). But then player II has a winning strategy in the T'-game. This easily yields a winning strategy for player II in the T-game, contradicting $\text{Rk}(\eta) < \infty$. Q.E.D.

Thus we have established

THEOREM 3. *T is an ω_1-recursive tree of $L_{\infty\omega_1}$-rank ω_1^+.*

An ω_1-recursive structure of Scott rank ω_1^+ can now be obtained by considering T^ω = infinite direct product of ω-many copies of T. For then the analysis of $L_{\infty\omega_1}$-rank for T reduces to the Scott analysis of T^ω.

We end with an observation concerning $\Pi_1(L_{\omega_1})$-singletons. Assume $V = L$. A function $f\colon L_{\omega_1} \to L_{\omega_1}$ is a $\Pi_1(L_{\omega_1})$-singleton if it is the unique solution to a $\Pi_1(L_{\omega_1})$ formula $\phi(f)$ with a single variable for a total function. An ω_1-recursive tree with a unique branch of length ω_1 yields a $\Pi_1(L_{\omega_1})$-singleton. We will show that for any $\beta < \sigma$ = least stable $> \omega_1$ there is an ω_1-recursive tree with a unique branch of length ω_1 which is constructed in L past β. Note that any $\Pi_1(L_{\omega_1})$-singleton must be a member of L_σ.

Note that $L_\sigma = \Sigma_1$ Skolem hull $(L_{\omega_1} \cup \{L_{\omega_1}\})$. Thus we can choose a Σ_1 formula $\phi(x,y,z)$ and $p \in L_{\omega_1}$ such that β is the unique solution to $\phi(x,\omega_1,p)$. Let α be the least admissible such that $\beta < \alpha$, $L_\alpha \vDash \phi(\beta,\omega_1,p)$ and $\alpha^* = \Sigma_1$ projection of $\alpha = \omega_1$.

We describe now an ω_1-recursive tree T whose unique path f consists of an ω_1-sequence of elementary submodels of L_α. This will suffice as clearly $f \notin L_\beta$. S consists of all $\overline{\alpha} < \omega_1$ such that

(a) $L_{\overline{\alpha}} \vDash KP + \omega_1$ exists, $\overline{\alpha}^* = \omega_1^{L_{\overline{\alpha}}}$;

(b) $p \in L_{\overline{\gamma}}$ where $\overline{\gamma} = \omega_1^{L_{\overline{\alpha}}}$, $L_{\overline{\alpha}} \vDash \phi(\overline{\beta},\overline{\gamma},p)$ for some $\overline{\beta} < \overline{\alpha}$;

(c) $L_{\overline{\alpha}} \vDash$ There are no admissible $\delta > \beta$ s.t. $\delta^* = \omega_1$.

Then the tree $T = \{\langle\overline{\alpha}_0,\overline{\alpha}_1,\dots\rangle \in \omega_1^{<\omega_1} \mid \overline{\alpha}_\delta \in S$ for all δ, $\overline{\alpha}_\delta$ = greatest $\overline{\alpha} < \overline{\alpha}_{\delta+1}$ s.t. $\exists\Pi : L_{\overline{\alpha}} \overset{\equiv}{\to} L_{\overline{\alpha}_{\delta+1}}$, $\omega_1^{L_{\overline{\alpha}_\lambda}} = \bigcup\{\omega_1^{L_{\overline{\alpha}_\delta}} \mid \delta < \lambda\}$, λ limit, $\sim\exists\overline{\alpha} < \overline{\alpha}_0 \exists j : L_{\overline{\alpha}} \overset{\equiv}{\to} L_{\overline{\alpha}_0}\}$. It is not hard to check that Π as above is uniquely determined as every element of $L_{\overline{\alpha}}$ is definable over $L_{\overline{\alpha}}$ from $\overline{\beta}$ together with ordinals $\leq \omega_1^{L_{\overline{\alpha}}}$, for $\overline{\alpha} \in S$. So T is ω_1-recursive.

Now define an ω_1-sequence of elementary submodels $M_0 \prec M_1 \prec \cdots$ of L_α by: M_0 = Skolem hull of $\{p,\omega_1,\beta\}$ in L_α, $\gamma_0 = M_0 \cap \omega_1$; $M_{\delta+1}$ = Skolem hull of $\gamma_\delta \cup \{p,\omega_1,\beta\}$ inside L_α, $\gamma_{\delta+1} = M_{\delta+1} \cap \omega_1$; $M_\lambda = \bigcup\{M_\delta \mid \delta < \lambda\}$, $\gamma_\lambda = \bigcup\{\gamma_\delta \mid \delta < \lambda\}$ for limit λ. Then $\langle\overline{\alpha}_0,\overline{\alpha}_1,\dots\rangle$ forms an ω_1-branch through T where $\overline{\alpha}_\delta$ = transitive collapse (M_δ).

If f is an ω_1-branch through T then there are elementary embeddings $L_{f(0)} \xrightarrow{\equiv} L_{f(1)} \xrightarrow{\equiv} \cdots$ and we can form the direct limit $L_{\alpha'}$. Now α' must be the least μ such that μ is admissible, $\mu^* = \gamma'$, $\mu > \beta'$, $L_\mu \models \phi(\beta', \gamma', p)$ for some $\beta' < \alpha'$, $\gamma' = \omega_1^{L_{\alpha'}}$. But $\gamma' = \omega_1$. So $\beta' = \beta$ since β is the unique solution to $\phi(x, \omega_1, p)$. It follows that $\alpha' = \alpha$ and hence $f(\delta) = \overline{\alpha}_\delta$ for all δ. Thus T has a unique ω_1-branch.

We have shown that $\Pi_1(L_{\omega_1})$-singletons are constructed in L cofinally in the least stable ordinal $\sigma > \omega_1$. By way of contrast all $\Pi_1(L_\omega)$-singletons are constructed in L before $\omega^+ = \omega_1^{CK}$. The disparity here is due to the fact that well-foundedness is easily expressible over L_{ω_1}.

FINAL NOTE. The second author has found a way to modify the construction in §1 to produce an ω_1-recursive structure of $L_{\infty\omega_1}$-rank ω_1^+. The key to the argument is in establishing the existence of an ω_1-recursive tree of ω_1-Rk ω_1^+, where ω_1-Rk is defined in analogy to our earlier definition of Rk. Then the appropriate structure is obtained from such a tree much as the structure M was obtained from T in §1.

REFERENCES

[**69**] Jon Barwise, *Admissible sets and structures*, Springer-Verlag, Berlin and New York, 1969.

[**78**] Sy D. Friedman, *An introduction to β-recursion theory*, Generalized Recursive Theory. II (Fenstad, Gandy, Sacks, eds.), North-Holland, Amsterdam, 1978.

[**81**] ——, *Model theory for $L_{\infty\omega_1}$* (Proc. 1980–81 Logic Year, Jerusalem) (to appear).

[**74**] Mark Nadel, *Scott sentences and admissible sets*, Ann. Math. Logic **7** (1974), 267–294.

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