On Borel Reducibility in Generalised Baire Space

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Abstract

In this paper we study the Borel reducibility of Borel equivalence relations on the generalised Baire space κ^κ for an uncountable κ with the property $\kappa^{<\kappa} = \kappa$. The theory looks quite different from its classical counterpart where $\kappa = \omega$, although some basic theorems do generalise.

We study the generalisations of classical descriptive set theory of Polish spaces to the setting where instead of the Baire space ω^ω we look at the generalised Baire space κ^κ of all functions from κ to κ where κ is an uncountable cardinal which satisfies $\kappa^{<\kappa} = \kappa$. The topology on this space is generated by the basic open sets

$$[p] = \{\eta \in \kappa^\kappa \mid \eta \supset p\}$$

where $p \in \kappa^{<\kappa}$. The resulting collection of open sets is closed under intersections of length $< \kappa$. The class of κ-Borel sets in this space is the smallest class containing the basic open sets and which is closed under taking unions and intersections of length κ.

In this paper we often work with spaces of the form $(2^\alpha)^\beta$ for some ordinals $\alpha, \beta \leqslant \kappa$. If $x \in (2^\alpha)^\beta$, then technically x is a function $\beta \to 2^\alpha$ and we denote by $x_\gamma = x(\gamma)$ the value at $\gamma < \beta$. Thus x_γ is a function $\alpha \to 2$ for each γ and we denote the value at $\delta < \alpha$ by $x_\gamma(\delta)$. The lengthier notation for $x \in (2^\alpha)^\beta$ is $(x_\gamma)_{\gamma<\beta}$ as a β-sequence of functions $\alpha \to 2$.

We say that a topological space is *κ-Baire*, if the intersection of κ many dense open sets is never empty. The generalised Baire space is κ-Baire [MV93]. If X is a topological space, we say that $A \subseteq X$ is *κ-meager* if its complement contains an intersection of κ many dense open sets. Thus, X is κ-Baire if and

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only if X is itself not meager (we always drop the prefix "κ-"). The complement of a meager set is called *co-meager*. A set $A \subseteq X$ has the *Baire property* if there exists an open set U such that the symmetric difference $U \triangle A$ is meager. When we write $\forall^* x \in A(P(x))$ we mean that there is a co-meager set such that every element of that set which belongs to A satisfies P. We write $\exists^* x \in A(P(x))$ to mean that there exists a non-meager set such that every element of that set which belongs to A satisfies P.

1 Equivalence Relations Induced by a Group Action

Suppose G is a topological group. Let X be a Borel subset of κ^κ. An action $\rho\colon G \times X \to X$ is *Borel* if it is Borel as a function i.e. inverse images of open sets are Borel. This action induces an equivalence relation on X in which two elements x and y are equivalent if there exists $g \in G$ such that $\rho(g,x) = y$. For example, if the action is Borel, then it is easy to see that this equivalence relation is Σ^1_1. This equivalence relation is denoted by $E^X_{G,\rho}$, or just E^X_G if the action is clear from the context.

Here are some examples of equivalence relations induced by a Borel action:

- id the identity relation.
- id^+ the jump of identity. This is an equivalence relation on $(2^\kappa)^\kappa$ where $(x_\alpha)_{\alpha<\kappa}$ and $(y_\alpha)_{\alpha<\kappa}$ are equivalent if the sets $\{x_\alpha \mid \alpha < \kappa\}$ and $\{y_\alpha \mid \alpha < \kappa\}$ are equal (see Definition 5). This is not defined as an equivalence relation induced by a Borel action, but is easily seen to be bireducible with id^+_* which is an equivalence relation on $(2^\kappa)^\kappa$ where $(x_\alpha)_{\alpha<\kappa}$ and $(y_\alpha)_{\alpha<\kappa}$ are equivalent if there exists a permutation $s \in S_\kappa$ (S_κ is the group of all permutations of κ) such that $x_\alpha = y_{s(\alpha)}$ for all α. The latter is induced by a Borel action of S_κ.
- E_0 an equivalence relation on 2^κ where $(\eta,\xi) \in E_0$ if there exists $\alpha < \kappa$ such that for all $\beta > \alpha$ we have $\eta(\beta) = \xi(\beta)$.
- E_1 an equivalence relation on $(2^\kappa)^\kappa$ where $(x_\alpha)_{\alpha<\kappa}$ and $(y_\alpha)_{\alpha<\kappa}$ are equivalent if there exists $\alpha < \kappa$ such that for all $\beta > \alpha$ we have $x_\beta = y_\beta$.

Since all the topologies in this paper are closed under intersections of length $< \kappa$, we replace "finite" by "less than κ" when referring to product topologies below.

1 Theorem. *Let G be a discrete group of cardinality $\leqslant \kappa$ and let it act in a Borel way on a Borel subset $X \subseteq 2^\kappa$. Let E^X_G be the (Borel) equivalence relation induced by this action. Then $E^X_G \leqslant_B E_0$.*

Proof. The group G acts on $\mathcal{P}(G)^\kappa$ coordinatewise by multiplication on the right $g \cdot (X_i)_{i<\kappa} = (X_i g)_{i<\kappa}$. This gives rise to the equivalence relation $E_G^{P(G)^\kappa}$.

1.1 Claim. $E_G^X \leqslant_B E_G^{\mathcal{P}(G)^\kappa}$.

Proof. Let $\pi\colon \kappa \to 2^{<\kappa}$ be a bijection. Let $x \in X$ and for each $\alpha < \kappa$ let

$$Z_\alpha(x) = \{g \in G \mid gx \in [\pi(\alpha)]\}.$$

This defines a reduction: an element $x \in X$ is mapped to $(Z_\alpha(x))_{\alpha<\kappa}$. Suppose there is $g_0 \in G$ such that $y = g_0 x$ for some $x, y \in X$. Then

$$\begin{aligned} Z_\alpha(x) &= \{g \in G \mid gx \in [\pi(\alpha)]\} \\ &= \{gg_0 \in G \mid gy \in [\pi(\alpha)]\} \\ &= Z_\alpha(y)g_0. \end{aligned}$$

On the other hand suppose that there exists $g \in G$ such that $Z_\alpha(x) = Z_\alpha(y)g$ for all $\alpha < \kappa$. It is enough to show that $g^{-1}y \in [p]$ for all basic open neighbourhoods $[p]$ of x. So suppose $U = [p]$ is a basic neighbourhood containing x and let $\alpha = \pi^{-1}(p)$. Now obviously $1_G \in Z_\alpha(x)$, so $1_G \in Z_\alpha(y)g$ and thus $g^{-1} \in Z_\alpha(y)$, i.e. $g^{-1}y \in [p]$. $\square_{\text{Claim 1.1}}$

For a set S, F_S is the free group generated by elements of S. $F_\varnothing = F_0$ is the trivial group.

1.2 Claim. $E_G^{\mathcal{P}(G)^\kappa} \leqslant_B E_{F_\kappa}^{\mathcal{P}(F_\kappa)^\kappa}$.

Proof. Since G has size $\leqslant \kappa$ and F_κ is a free group on κ generators, there is a normal subgroup $N \subseteq F_\kappa$ such that $G \cong F_\kappa/N$. Assume without loss of generality that $G = F_\kappa/N$. Let pr be the canonical projection map $F_\kappa \to F_\kappa/N$. For $(A_\alpha)_{\alpha<\kappa} \in \mathcal{P}(G)^\kappa$, let

$$F((A_\alpha)_{\alpha<\kappa}) = (\mathrm{pr}^{-1} A_\alpha)_{\alpha<\kappa}.$$

This is clearly a continuous reduction. $\square_{\text{Claim 1.2}}$

1.3 Claim. $E_{F_\kappa}^{\mathcal{P}(F_\kappa)^\kappa} \leqslant_B E_0$.

Proof. The space $\mathcal{P}(F_\kappa)^\kappa$ can be canonically thought to be the same as $(2^\kappa)^{F_\kappa}$ the bijection being defined by $(A_\alpha)_{\alpha<\kappa} \mapsto x$ where $x(g)(\alpha) = 1$ if and only if $g \in A_\alpha$. This space is equipped with the product topology (recall that our definition of product topology is non-standard, see page 2): a basic open set is given by

$$\{x \mid x(g) \restriction \alpha = p\}$$

for some ordinal $\alpha < \kappa$, some $g \in F_\kappa$ and some $p \in 2^{<\kappa}$ and the resulting collection of open sets is closed under intersections of length $< \kappa$. The action of F_κ on $(2^\kappa)^{F_\kappa}$ is then defined by $g * x = y$ where $y(f)(\alpha) = 1$ if and only if $x(fg^{-1})(\alpha) = 1$ for all $f \in F_\kappa$.

This space is more convenient for us to work with. Additionally we identify 2^α with $\mathcal{P}(\alpha)$ for all α and write $2^\alpha \subseteq 2^\beta$, meaning that an element p of 2^α is identified with q in 2^β where q is just p with "$\beta - \alpha$" zeros at the end.

Let us look at the sets $X_\alpha = (2^\alpha)^{F_\alpha}$ for $\alpha \leqslant \kappa$.

If $w \in X_\beta$ and $\alpha < \beta$, denote by $w \restriction \alpha$ the element v of X_α such that $v(g) = w(g) \restriction \alpha$ for all $g \in F_\alpha$. Thus, by the identifications we made, $X_\alpha \subseteq X_\beta \subseteq X_\kappa$ for all $\alpha < \beta < \kappa$. For every $\alpha < \kappa$ fix a well-ordering $<_\alpha$ of X_α. Given $g \in F_\alpha$ and $w \in X_\alpha$, let $g * w \in X_\alpha$ be the element such that $(g * w)(f) = w(fg^{-1})$ for all $f \in F_\alpha$. This is an action of F_α on X_α and for $\alpha = \kappa$ it coincides with the original action of F_κ on $\mathcal{P}(F_\kappa)^\kappa$ under the mentioned identifications.

Fix $x \in X_\kappa$. For each α, let $x(\alpha)$ be the $<_\alpha$-least element of

$$\{g * (x \restriction \alpha) \mid g \in F_\alpha\}$$

and let $H(x) = (x(\alpha))_{\alpha<\kappa}$. We claim that for all $x, y \in X_\kappa$, $y = g * x$ for some $g \in F_\kappa$ if and only if there exists $\beta < \kappa$ such that for all $\alpha > \beta$, $x(\alpha) = y(\alpha)$.

Assume first that such $g \in F_\kappa$ exists. Then $g \in F_\beta$ for some $\beta < \kappa$ and for all $\alpha > \beta$, $g \in F_\alpha$. Thus, it is obvious that $x(\alpha) = y(\alpha)$ for $\alpha > \beta$, because for these α

$$\{g * (x \restriction \alpha) \mid g \in F_\alpha\} = \{g * (y \restriction \alpha) \mid g \in F_\alpha\}.$$

Assume now that there exists $\beta < \kappa$ such that for all $\alpha > \beta$, $x(\alpha) = y(\alpha)$. Then for each $\alpha > \beta$ there exists $g_\alpha \in F_\alpha$ such that $x \restriction \alpha = g_\alpha * (y \restriction \alpha)$. For each $\alpha > \beta$, let $\gamma(\alpha)$ be the least ordinal such that $g_\alpha \in F_{\gamma(\alpha)}$. If α is a limit ordinal, then $\gamma(\alpha) < \alpha$ and so there is γ_0 and a stationary $S_0 \subseteq \lim \kappa$ such that for all $\alpha \in S_0$ we have $g_\alpha \in F_{\gamma_0}$. Since $|F_{\gamma_0}| < \kappa$, there is a stationary $S \subseteq S_0$ and $g_* \in F_{\gamma_0}$ such that for all $\alpha \in S$ we have $g_\alpha = g_*$. Since S is unbounded, this obviously implies that $y = g_* * x$.

Fix bijections $f_\alpha \colon X_\alpha \to \kappa$ and map each $x \in X_\kappa$ to the sequence $(f_\alpha(x(\alpha)))_{\alpha<\kappa}$ and denote this mapping by G. By the above we have $x = g * y$ for some $g \in F_\kappa$ if and only if $(G(x), G(y)) \in E_0$. It remains to show that G is continuous.

Suppose $x \in X_\kappa$ and take an open neighbourhood U of $G(x)$. Then there is β such that

$$\{\eta \in \kappa^\kappa \mid \forall \alpha < \beta(\eta(\alpha) = f_\alpha(x(\alpha)))\} \subseteq U.$$

Now, the set $\{y \in F_\kappa \mid y \restriction \beta = x \restriction \beta\}$ is mapped inside U and contains x, so it remains to show that this set is open, but this follows from the definition of the topology on $(2^\kappa)^{F_\kappa}$ in particular that the collection of open sets is closed under intersections of length $< \kappa$. $\square_{\text{Claim 1.3}}$

$\square_{\text{Theorem 1}}$

2 Theorem ($V = L$)**.** *There is a Borel equivalence relation E whose classes have size 2, which is smooth (i.e. Borel reducible to* id*) yet which is not induced by a Borel action of a group of size $\leqslant \kappa$.*

Proof.

2.1 Claim. *There is an open dense set $O \subseteq 2^\kappa$ and a bijection $f\colon O \to 2^\kappa \setminus O$ such that the graph of f is Borel, but f is not Borel as a function on any non-meager Borel set. However the inverse of f is Borel.*

Proof. We let O be the complement of a certain closed set of "master codes" for size κ initial segments of L. This is defined as follows. Let $\mathcal{L}$ be the language of set theory augmented by constant symbols $\bar{\alpha}$ for each ordinal $\alpha < \kappa$. Also let T_0 denote the theory ZFC$^-$ (ZFC minus the power set axiom) plus $V = L$ plus the statement "there are only boundedly many ordinals β such that L_β satisfies ZFC$^-$". We consider complete, consistent theories T which extend T_0 and which in addition satisfy the following:

1. There is no ω-sequence of formulas $\varphi_n(x)$ (mentioning constants $\bar{\alpha}$ for $\alpha < \kappa$) such that for each n both the sentence "$\exists! x\, \varphi_n(x)$" and the sentence "$\exists x, y(\varphi_n(x) \wedge \varphi_{n+1}(y) \wedge y \in x)$" belong to T.
2. For each $\beta < \kappa$ and formula $\varphi(x)$ (mentioning constants $\bar{\alpha}$ for $\alpha < \kappa$) if the sentences "$\exists! x \varphi(x)$" and "$\exists x(\varphi(x) \wedge x < \bar{\beta})$" both belong to T then so does the sentence "$\exists x(\varphi(x) \wedge x = \bar{\gamma})$" for some $\gamma < \beta$.

By identifying sentences of $\mathcal{L}$ with ordinals less than κ we can regard theories in $\mathcal{L}$ as subsets of κ. Now let $C \subseteq 2^\kappa$ be the set of theories T as above. Then C is a closed set. And C is nowhere dense as any set of $\mathcal{L}$-sentences of size less than κ is included in an inconsistent such set.

The theories in C are exactly the first-order theories of structures of the form L_β in which the constant symbol $\bar{\alpha}$ is interpreted as the ordinal α for each $\alpha < \kappa$ and in which the axioms of T_0 hold. And for each T in C there is a unique such model $M(T)$ in which every element is definable from parameters less than κ.

We are ready to define the function $f : O \to C$ where O is the complement of C in 2^κ. List the elements of O in $<_L$-increasing order as $x_0, x_1, \ldots$ and list the elements of C in $<_L$-increasing order as $y_0, y_1, \ldots$; then we set $f(x_i) = y_i$

for each $i < \kappa^+$. The inverse of f is Borel because given $y \in C$ we can identify $f^{-1}(y)$ (viewed as a subset of κ) as the set of $\gamma < \kappa$ such that the sentence "γ belongs to the i-th element in the $<_L$-increasing enumeration of 2^κ where i is the order type of the set of β such that L_β models T_0" belongs to (the theory associated to) y. Thus the graph of f is Borel. If B is a non-meager Borel set and g is a Borel function then we claim that g cannot agree with f on B: Indeed, let β_0 be so that L_{β_0} models ZFC$^-$ and contains Borel codes for both B and g and let $x \in B \cap O$ be κ-Cohen generic over L_{β_0}. Then $f(x) = T$ is the theory of a model $M(T) = L_\beta$ where β is greater than β_0. But $g(x)$ belongs to $L_{\beta_0}[x]$, which by the genericity of x is a model of ZFC$^-$ while $L_{\beta_0}[f(x)]$ does not satisfy ZFC$^-$ as $f(x) = T$ codes the model L_β. □

Define xEy if and only if $x = y$, $y = f(x)$ or $x = f(y)$. Now E has a Borel transversal, i.e., there is a Borel function t such that $xEt(x)$ for all x and xEy if and only if $t(x) = t(y)$ for all x, y: Given $x \in 2^\kappa$, first decide in a Borel way if x is in O or not. If yes, then let $t(x) = x$, otherwise, find $f^{-1}(x)$ in a Borel way (since f^{-1} is Borel) and let $t(x) = f^{-1}(x)$. This $t(x)$ is a Borel transversal. It follows that E is smooth.

Finally, suppose E is given by a Borel action of some group G of size at most κ. Then for each $x \in O$ choose $g_x \in G$ such that $f(x) = g_x \cdot x$; then for some fixed $g \in G$, $f(x) = g \cdot x$ for non-meager many $x \in O$, contradicting the fact that f is not Borel on any non-meager Borel set. □

3 Question. Is there a Borel equivalence relation with classes of size κ which is not reducible to E_0?

2 E_1 and E_{club}

Let E_1 be the equivalence relation on $(2^\kappa)^\kappa$ where $(x_\alpha)_{\alpha<\kappa}$ and $(y_\alpha)_{\alpha<\kappa}$ are equivalent if there exists $\beta < \kappa$ such that for all $\gamma > \beta$, $x_\gamma = y_\gamma$.

4 Theorem. *E_1 and E_0 are bireducible.*

Proof. It is obvious that $E_0 \leqslant_B E_1$, so let us look at the other direction.

To simplify notation, we think of E_0 on κ^κ: two functions η and ξ are E_0-equivalent if the set $\{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\}$ is bounded. It is easy to see that E_0 on 2^κ is bireducible with this equivalence relation.

For all limit $\alpha < \kappa$, define E_1^α to be the equivalence relation on $(2^\alpha)^\alpha$ approximating E_1 i.e. $(x_i)_{i<\alpha} E_1^\alpha (y_i)_{i<\alpha}$ if for some $\beta < \alpha$, $x_i = y_i$ for all $i > \beta$. Now define the reduction $F\colon (2^\kappa)^\kappa \to \kappa^\kappa$ so that for all $(x_i)_{i<\kappa} \in (2^\kappa)^\kappa$,

$F((x_i)_{i<\kappa})(\alpha) = 0$ if α is not a limit and otherwise it is a code for the E_1^α-equivalence class of $(x_i \restriction \alpha)_{i<\alpha}$.

Clearly F is continuous and if $(x_i)_{i<\kappa} E_1 (y_i)_{i<\kappa}$, then also $F((x_i)_{i<\kappa})$ and $F((y_i)_{i<\kappa})$ are E_0-equivalent (if $\beta < \kappa$ witnesses the first equivalence, it witnesses also the second).

Also if $(x_i)_{i<\kappa}$ and $(y_i)_{i<\kappa}$ are not E_1 equivalent, then for all $\alpha < \kappa$ there are $\gamma, \beta < \kappa$ such that $\beta > \alpha$ and $x_\beta(\gamma) \neq y_\beta(\gamma)$. Let $f(\alpha)$ be $max\{\beta, \gamma\}$. Now if $\alpha^* < \kappa$ is such that for all $\alpha < \alpha^*$, $f(\alpha) < \alpha^*$, then clearly $(x_i \restriction \alpha^*)_{i<\alpha^*}$ and $(y_i \restriction \alpha^*)_{i<\alpha^*}$ are not $E_1^{\alpha^*}$-equivalent, and thus $F((x_i)_{i<\kappa})(\alpha^*) \neq F((y_i)_{i<\kappa})(\alpha^*)$. Since the set of such α^* is unbounded, $F((x_i)_{i<\kappa})$ and $F((y_i)_{i<\kappa})$ are not E_0-equivalent. □

5 Definition. If E is an equivalence relation on 2^κ, its *jump* is the equivalence relation denoted by E^+ on $(2^\kappa)^\kappa$ defined as follows. Two sequences $(x_\alpha)_{\alpha<\kappa}$ and $(y_\alpha)_{\alpha<\kappa}$ are E^+-equivalent, if

$$\{[x_\alpha]_E \mid \alpha < \kappa\} = \{[y_\alpha]_E \mid \alpha < \kappa\}$$

where $[x]_E$ is the equivalence class of x in E. Since $(2^\kappa)^\kappa$ is homeomorphic to 2^κ we can assume without loss of generality that E^+ is also defined on 2^κ.

For an ordinal $\alpha < \kappa^+$ define $E^{\alpha+}$ by transfinite induction. To begin, define $E^{0+} = E$. If $E^{\alpha+}$ is defined, then $E^{(\alpha+1)+} = (E^{\alpha+})^+$.

Suppose α is a limit and $E^{\beta+}$ is defined to be an equivalence relation on 2^κ for $\beta < \alpha$. Let X be the disjoint union of α many copies of 2^κ. Denote the β:th copy by X_β, thus $X = \bigcup_{\beta<\alpha} X_\beta$. Let h be a homeomorphism $X \to 2^\kappa$. Two functions η and ξ are defined to be $E^{\alpha+}$-equivalent, if $h^{-1}(\eta)$ and $h^{-1}(\xi)$ belong both to the same X_β and are $E^{\beta+}$-equivalent. This is called the *join* of the equivalence relations $\{E^{\beta+} \mid \beta < \alpha\}$ and is denoted $\bigoplus_{\beta<\alpha} E^{\beta+}$.

6 Theorem. $E_0 <_B \mathrm{id}^+$

Proof. The reduction is defined by

$$E_0 \leqslant_B \mathrm{id}^+ \colon\ \eta \mapsto (p + \eta)_{p \in 2^{<\kappa}}.$$

Suppose $f\colon 2^\kappa \to (2^\kappa)^\kappa$ is a Borel reduction from id^+ to E_0. There is a co-meager set D on which f is continuous. Without loss of generality assume that this D is the intersection $\bigcap_{i<\kappa} D_i$ where D_i are dense open.

For every $i < \kappa$ we will define ordinals γ_i together with sequences $x^i = (x^i_\alpha)_{\alpha<\gamma_i}$ and $y^i = (y^i_\alpha)_{\alpha<\gamma_i}$ where each $x^i_\alpha, y^i_\alpha \in 2^{\gamma_i}$ and permutations $\pi_i \in S_{\gamma_i}$. These will satisfy the following requirements for every $i < j < \kappa$:

1. $\pi_i \subseteq \pi_j$.

2. $\gamma_i \leqslant \gamma_j$,

3. For all $\alpha < \gamma_i$ we have $x^i_\alpha \subseteq x^j_\alpha$ and $y^i_\alpha \subseteq y^j_\alpha$.

4. For all $\alpha < \gamma_i$ we have $x^i_\alpha = y^i_{\pi_i(\alpha)}$.

5. Let $[(x^i_\alpha)_{\alpha<\gamma_i}]$ be the set of all $x = (x_\alpha)_{\alpha<\kappa} \in (2^\kappa)^\kappa$ such that $x^i_\alpha \subseteq x_\alpha$ for all α. There exist $\beta > i$, $\delta < \kappa$ and $p, q \in (2^\delta)^{\beta+1}$ such that

$$f[[(x^i_\alpha)_{\alpha<\gamma_i}] \cap D] \subseteq [p],$$

$$f[[(y^i_\alpha)_{\alpha<\gamma_i}] \cap D] \subseteq [q]$$

and $p(\beta) \neq q(\beta)$.

6. $[(x^{i+1}_\alpha)_{\alpha<\gamma_{i+1}}] \subseteq D_i$ and $[(y^{i+1}_\alpha)_{\alpha<\gamma_{i+1}}] \subseteq D_i$

This will lead to a contradiction as follows. Let $\tilde{x} = (\tilde{x}_\alpha)_{\alpha<\kappa}$ be such that for every α we have $\tilde{x}_\alpha \restriction \gamma_i = x^i_\alpha$ if $\gamma_i > \alpha$. This is possible by (2) and (3). Analogously define $\tilde{y}$. Now by (1) we can define $\pi = \bigcup_{i<\kappa} \pi_i$ which by (4) witnesses that $\tilde{x}$ and $\tilde{y}$ are id^+-equivalent. By (6) they are in D and by continuity in D and by (5) the images $f(\tilde{x})$ and $f(\tilde{y})$ cannot be E_0-equivalent.

Let $x^* = (x^*_\alpha)_{\alpha<\kappa}$ and $y^* = (y^*_\alpha)_{\alpha<\kappa}$ be any sequences in D such that x^* is not id^+-equivalent to y^*. Find these for example as follows: We will define sequences $(\xi_k)_{k<\kappa}$ and $(\eta_k)_{k<\kappa}$ and ordinals ε_k such that for all $k < \kappa$ we have $\xi_k, \eta_k \in (2^{\varepsilon_k})^{\varepsilon_k}$, for $k_1 < k_2$ we have $\varepsilon_{k_1} < \varepsilon_{k_2}$, $\xi_{k_1} \subseteq \xi_{k_2}$ and $\eta_{k_1} \subseteq \eta_{k_2}$, and the unions $\bigcup_{k<\kappa} \xi_k$ and $\bigcup_{k<\kappa} \eta_k$ are in D and not id^+-equivalent. This is easy: Let $\varepsilon_0 = 0$, $\xi_0 = \varnothing$ and $\eta_0 = \varnothing$. If ξ_k and η_k are defined, first extend ξ_k to an element $\xi'_{k+1} \in (2^{\varepsilon'_{k+1}})^{\varepsilon'_{k+1}}$ (for suitable $\varepsilon'_{k+1} > \varepsilon_k$) such that $[\xi'_{k+1}] \subseteq D_k$. Then extend the first component of η_k so that it differs in a diagonal way from every component of ξ'_{k+1}. After that, extend the result into $\eta_{k+1} \in (2^{\varepsilon_{k+1}})^{\varepsilon_{k+1}}$ (for suitable $\varepsilon_{k+1} > \varepsilon'_{k+1}$) so that $[\eta_{k+1}] \subseteq D_k$ and $\varepsilon_{k+1} > \varepsilon'_{k+1}$. Finally extend ξ'_{k+1} to an element of $(2^{\varepsilon_{k+1}})^{\varepsilon_{k+1}}$ so that the first component of η_{k+1} is still diagonally different from every component of ξ_{k+1}; technically this means that $\eta_{k+1}(0)(\alpha) \neq \xi_{k+1}(\alpha)(\alpha)$. At limit k just take the natural limits of the sequences. In this way at the κ:th limit we obtain ξ_κ and η_κ are as required, so we can define $x^* = \xi_k$ and $y^* = \eta_\kappa$.

Let β and δ be such that $f(x^*)(\beta)(\delta) \neq f(y^*)(\beta)(\delta)$ which exist because f is assumed to be a reduction and $f(x^*)$ and $f(y^*)$ are not E_0-equivalent. Now by continuity in D there is $\gamma^*_0 > 0$ such that

$$f[[(x^*_\alpha \restriction \gamma^*_0)_{\alpha<\gamma^*_0}] \cap D] \subseteq [(f(x^*)\restriction(\beta+1)]$$

and

$$f[[(y^*_\alpha \restriction \gamma^*_0)_{\alpha<\gamma^*_0}] \cap D] \subseteq [(f(y^*) \restriction (\beta+1)]$$

Then we glue $(x^*_\alpha \restriction \gamma^*_0)_{\alpha<\gamma^*_0}$ to the end of $(y^*_\alpha \restriction \gamma^*_0)_{\alpha<\gamma^*_0}$ and vice versa: Let $\gamma_0 = \gamma^*_0 + \gamma^*_0$ and for all $\alpha < \gamma^*_0$ define

$$\begin{aligned} x^0_\alpha &= x^*_\alpha \restriction \gamma_0 \\ y^0_\alpha &= y^*_\alpha \restriction \gamma_0 \\ x^0_{\gamma^*_0+\alpha} &= y^*_\alpha \restriction \gamma_0 \\ y^0_{\gamma^*_0+\alpha} &= x^*_\alpha \restriction \gamma_0 \end{aligned}$$

Let π_0 be the permutation which takes α to $\gamma_0+\alpha$ when $\alpha < \gamma_0$ and if $\alpha = \gamma_0+\varepsilon$, then $\pi(\alpha) = \varepsilon$. So we have defined π_0, γ_0, $(x^0_\alpha)_{\alpha<\gamma_0}$ and $(y^0_\alpha)_{\alpha<\gamma_0}$ such that all the conditions (1)–(6) are satisfied so far.

Suppose that π_i, γ_i, $(x^i_\alpha)_{\alpha<\gamma_i}$ and $(y^i_\alpha)_{\alpha<\gamma_i}$ are defined for $i < j$ such that the conditions (1)–(6) are satisfied. If j is a limit, then just define $\pi_j = \bigcup_{i<j} \pi_i$, $x^j_\alpha = \bigcup_{i'<i<j} x^i_\alpha$ and $y^j_\alpha = \bigcup_{i'<i<j} y^i_\alpha$ for some i' such that $\gamma_{i'} > \alpha$ and $\gamma_j = \sup_{i<j} \gamma_i$ and $\gamma_j = \sup_{i<j} \gamma_i$.

Suppose j is a successor, in fact w.l.o.g denote the predecessor by i, i.e. $j = i+1$. Next we want to build elements $x^* = (x^*_\alpha)_{\alpha<\kappa}$ in $[(x_i)_{i<\gamma_i}] \cap D$ and $y^* = (y^*_\alpha)_{\alpha<\kappa}$ in $[(y_i)_{i<\gamma_i}] \cap D$ such that $x^*_\alpha = y^*_{\pi(\alpha)}$ for all $\alpha < \gamma_i$ and which are not id^+-equivalent. To do that, define $\varepsilon_0 = \gamma_i$, $\xi_0 = (x^i_\alpha)_{\alpha<\gamma_i}$ and $\eta_0 = (y^i_\alpha)_{\alpha<\gamma_i}$. Suppose we have defined ξ_k and η_k for some $k < \kappa$.

First we extend ξ_k to $\xi'_{k+1} \in (2^{\varepsilon'_{k+1}})^{\varepsilon'_{k+1}}$ for some suitable $\varepsilon'_{k+1} < \kappa$, $\varepsilon'_{k+1} > \varepsilon_k$ so that $[\xi'_{k+1}] \subseteq D_k$. Then we extend η_k first to a $\eta'_{k+1} \in (2^{\varepsilon'_{k+1}})^{\varepsilon'_{k+1}}$ such that $\eta'_{k+1} \restriction \gamma_i$ equals to the action of π_i applied to $\xi'_{k+1} \restriction \gamma_i$ and $\eta'_{k+1} \restriction \{\gamma_i\}$ diagonally differs from every component of ξ'_{k+1}. Then extend η'_{k+1} to $\eta_{k+1} \in (2^{\varepsilon_{k+1}})^{\varepsilon_{k+1}}$ (for a suitable $\varepsilon_{k+1} > \varepsilon'_{k+1}$) so that $[\eta_{k+1}] \subseteq D_k$. Finally extend ξ'_{k+1} to $\xi_{k+1} \in (2^{\varepsilon_{k+1}})^{\varepsilon_{k+1}}$ in any such way that $\eta_{k+1} \restriction \{\gamma_i\}$ differs from every component of ξ_{k+1} in a diagonal way.

At limit k just take the natural limits of the sequences. In this way at the κ:th limit we obtain ξ_κ and η_κ which are as required, so we can define $x^* = \xi_k$ and $y^* = \eta_\kappa$.

Now by continuity and by the fact that x^* and y^* are not id^+-equivalent, find γ^*_{i+1} and $\beta > i+1$ so that $f(x^*)(\beta) \neq f(y^*)(\beta)$ and

$$f[[(x^*_\alpha \restriction \gamma^*_{i+1})_{\alpha<\gamma^*_{i+1}}] \cap D] \subseteq [(f(x^*) \restriction (\beta+1)]$$

and

$$f[[(y^*_\alpha \restriction \gamma^*_{i+1})_{\alpha<\gamma^*_{i+1}}] \cap D] \subseteq [(f(y^*) \restriction (\beta+1)].$$

Also we make sure that γ^*_{i+1} is big enough so that (6) is satisfied. Now we want to glue a part of $(x^*_\alpha\restriction\gamma^*_{i+1})_{\alpha<\gamma^*_{i+1}}$ to the end of $(y^*_\alpha\restriction\gamma^*_{i+1})_{\alpha<\gamma^*_{i+1}}$ and vice versa: Let $\varepsilon=\gamma_{i+1}-\gamma_i$, i.e. the order type of $\gamma^*_{i+1}\setminus\gamma_i$ and let $\gamma_{i+1}=\gamma^*_{i+1}+\varepsilon$. Define x^{i+1}_α and y^{i+1}_α for all $\alpha<\gamma_{i+1}$ depending on α as follows. If $\alpha<\gamma^*_{i+1}$, let x^{i+1}_α to be $x^*_\alpha\restriction\gamma_{i+1}$ and y^{i+1}_α to be $y^*_\alpha\restriction\gamma_{i+1}$. If $\alpha=\gamma^*_{i+1}+\delta$ for some $\delta<\varepsilon$, then let x^{i+1}_α to be $y^*_{\gamma_i+\delta}$ and y^{i+1}_α to be $x^*_{\gamma_i+\delta}$. This gives us also π_{i+1} and we are done. □

7 Definition. For a regular cardinal $\mu<\kappa$ and $\lambda\in\{2,\kappa\}$ let $E^\lambda_{\mu\text{-cub}}$ be the equivalence relation on λ^κ such that η and ξ are $E^\lambda_{\mu\text{-cub}}$-equivalent if the set $\{\alpha\mid\eta(\alpha)=\xi(\alpha)\}$ contains a μ-cub, i.e. an unbounded set which is closed under μ-cofinal limits. If T is a countable complete first-order theory, denote by $\cong^T_\kappa$ the isomorphism relation on the models of T.

In the following we show that

1. The α:th jump of identity for $\alpha<\kappa^+$ is reducible to $E^\kappa_{\mu\text{-cub}}$ for every regular $\mu<\kappa$,
2. Every Borel isomorphism relation is reducible to $E^\kappa_{\mu\text{-cub}}$ for every regular $\mu<\kappa$,
3. If T a countable complete first-order classifiable (superstable with NDOP and NOTOP) and shallow theory, then $\cong^\kappa_T\leqslant_B E^\kappa_{\mu\text{-cub}}$.

8 Definition. Fix a limit ordinal $\alpha\leqslant\kappa$ and let t be a subtree of $\alpha^{<\omega}$ with no infinite branches. Let h be a function from the leaves of t to $2^{<\alpha}$. Then (t,h) determines the set $B_{(t,h)}$ as follows: $p\in 2^\alpha$ belongs to $B_{(t,h)}$ if player **II** has a winning strategy in the game $G(p,t,h)$: The players start at the root and then one after another choose a successor of the node they are in and then move to that successor. Player **I** starts. Eventually they reach a leaf l and player **II** wins if $h(l)\subset p$. We say that (t,h) is a *Borel code for* α.

If $\alpha=\kappa$, it is easy to see by induction on the rank of the tree that $B_{(t,h)}$ is a usual Borel set and conversely, if $B\subset 2^\kappa$ is any Borel set, then there is a Borel code (t,h) for κ such that $B=B_{(t,h)}$.

If t is replaced by a more general $\kappa^+\kappa$-tree (subtree of $\kappa^{<\kappa}$ without branches of length κ), then the sets that are obtained in this way are the so called Borel* sets, see [Bla81, MV93, Hal96, FHK13].

Suppose (t,h) is a Borel code for κ and $\alpha<\kappa$. Say that α is *good* for (t,h), if for all leaves $l\in t$ with $\mathrm{ht}(l)<\alpha$ we have $h(l)\in 2^{<\alpha}$. Clearly the set of good α for a fixed (t,h) is a cub set.

Define the α:th *approximation* of (t,h), denoted $(t,h)\restriction\alpha$ to be the pair $(t\restriction\alpha, h\restriction\alpha)$ where $t\restriction\alpha = t\cap\alpha^{<\omega}$ and for all leaves l of $t\restriction\alpha$, $(h\restriction\alpha) = h\restriction(t\restriction\alpha)$. It is obvious that if (t,h) is a Borel code for κ and $\alpha<\kappa$ is good for (t,h), then $(t,h)\restriction\alpha$ is a Borel code for α.

By replacing $2^{<\alpha}$ by $(2^{<\alpha})^2$ for the range of h and making necessary changes we can define Borel codes for subsets of $(2^\alpha)^2$.

Note that the game $G(p,t,h)$ is determined for all $p\in 2^\alpha$ (this is not the case for general Borel*-sets).

Make a similar definition for codes of Borel subsets of $2^\kappa\times 2^\kappa$.

9 Lemma. *Suppose that $B = B_{(t,h)}$ is a Borel subset of $2^\kappa\times 2^\kappa$. Then*

$$(\eta,\xi)\in B \iff (\eta\restriction\alpha,\xi\restriction\alpha)\in B_{(t,h)\restriction\alpha}$$

for cub-many α and $(\eta,\xi)\notin B \iff (\eta\restriction\alpha,\xi\restriction\alpha)\notin B_{(t,h)\restriction\alpha}$ for cub-many α.

Proof. Suppose $(\eta,\xi)\in B$ and let σ be a winning strategy of player **II** in $G((\eta,\xi),t,h)$. Let C be the set of those limit α which are good for (t,h) and that $t\restriction\alpha$ is closed under σ. Clearly $(\eta\restriction\alpha,\xi\restriction\alpha)\in B_{(t,h)\restriction\alpha}$ for all $\alpha\in C$ and C is cub.

Conversely, if $(\eta,\xi)\notin B$, then player **I** has a winning strategy τ in $G((\eta,\xi),t,h)$ and by closing under τ we obtain the needed cub set again. $\square$

10 Lemma. *Let S be the set of Borel equivalence relations E such that for some Borel code (t,h), $E = B_{(t,h)}$ and $B_{(t,h)\restriction\alpha}$ is an equivalence relation for cub-many $\alpha<\kappa$. Then S contains* id *and is closed under jump and the join operation $\bigoplus$ as in the definition of iterated jump, Definition 5.*

Proof. Enumerate $2^{<\kappa} = \{p_\alpha \mid \alpha<\kappa\}$. Let $t=\kappa^1$ and $h(\alpha) = (p_\alpha,p_\alpha)$. Clearly $\mathrm{id} = B_{(t,h)}$ and for those α for which $\{p_i \mid i<\alpha\} = 2^{<\alpha}$, $B_{(t,h)\restriction\alpha}$ is the identity on 2^α and this is clearly a cub set.

Suppose E is in S and (t,h) is a code for E witnessing that and that C is a cub set on which $E_{(t,h)}$ is an equivalence relation. It is not difficult to design a Borel code (t^+,h^+) for the jump E^+ and check that for cub many $\alpha\in C$, $B_{(t^+,h^+)\restriction\alpha}$ is the jump of $B_{(t,h)|\alpha}$.

Similarly suppose that $E_i\in S$ are equivalence relations for $i<\kappa$ and witnessing codes (t_i,h_i) are given with C_i cub sets such that $B_{(t_i,h_i)|\alpha}$ is an equivalence relation for each $\alpha\in C_i$. Then it is not difficult to design a code (t,h) so that $B_{(t,h)}$ is $\bigoplus_{i<\kappa}E_i$ and for cub many $\alpha\in\nabla_{i<\kappa}C_i$, $B_{(t,h)\restriction\alpha}$ is $\bigoplus_{i<\alpha}B_{(t_i,h_i)\restriction\alpha}$ $\square$

It follows that S contains all iterates of the jump $id^{+\beta}$, $\beta<\kappa^+$.

11 Theorem. *Let E be an equivalence relation in S. Then E is reducible to $E^{\kappa}_{\mu\text{-}cub}$ for any regular $\mu < \kappa$ (see Definition 7).*

Proof. Let E be $B_{(t,h)}$ where (t,h) witnesses that E belongs to S. To each η assign the function f_η where $f_\eta(\alpha)$ is a code for the $B_{(t,h)\restriction\alpha}$ equivalence class of $\eta\restriction\alpha$ (if $\text{cf}(\alpha) = \mu$ and $B_{(t,h)\restriction\alpha}$ is an equivalence relation, 0 otherwise). By Lemma 9, if $\eta E\xi$ then $f_\eta(\alpha) = f_\xi(\alpha)$ for μ-cub-many α and if $\neg\eta E\xi$ then $f_\eta(\alpha) \neq f_\xi(\alpha)$ for μ-cub-many α. □

12 Corollary. *The iterated jumps $id^{\alpha+}$ of the identity are reducible to $E^{\kappa}_{\mu\text{-}cub}$ for each regular $\mu < \kappa$.* □

13 Corollary. *If $\mathcal{M}$ is a Borel class of models such that $\cong_{\mathcal{M}}$, the isomorphism relation on $\mathcal{M}$ is Borel, then $\cong_{\mathcal{M}}$ is Borel reducible to $E^{\kappa}_{\mu\text{-}cub}$ for all regular $\mu < \kappa$.*

Proof. Using similar techniques as in classical descriptive set theory (see e.g. [Gao09, Lemma 12.2.7]) one can show that a Borel isomorphism can be reduced to an iterated jump of identity. □

14 Corollary. *Suppose T a countable complete first-order classifiable (superstable with NDOP and NOTOP) and shallow theory, then $\cong^{\kappa}_{T} \leqslant_B E^{\kappa}_{\mu\text{-}cub}$.*

Proof. By [FHK13, Theorem 68] the isomorphism relation of a classifiable shallow theory is Borel, so we apply Corollary 13. □

We have shown in [FHK13, Theorem 75] that under certain cardinality assumptions on κ, a complete countable first-order theory T is classifiable if and only if for all regular $\mu < \kappa$, $E^2_{\mu\text{-cub}} \not\leqslant_B \cong_T$, see Definition 7, where $\cong_T$ is the isomorphism on $\text{Mod}(T)$. Clearly $E^2_{\mu\text{-cub}} \leqslant_B E^{\kappa}_{\mu\text{-cub}}$.

15 Question. Is $E^{\kappa}_{\mu\text{-cub}}$ reducible to $E^2_{\mu\text{-cub}}$?

If the answer to Question 15 is "yes" then using [FHK13, Theorem 75] we obtain: Suppose T_1 and T_2 are complete first-order theories with T_1 classifiable and shallow and T_2 non-classifiable. Also suppose that $\kappa = \lambda^+ = 2^\lambda > 2^\omega$ where $\lambda^{<\lambda} = \lambda$. Then $\cong_{T_1}$ is Borel reducible to $\cong_{T_2}$.

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