GOOD PROJECTIVE WITNESSES

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ABSTRACT. Developing a new forcing notion for adjoining self-coding cofinitary permutations, we show that consistently there is a Π^1_2-definable maximal cofinitary group of cardinality μ, where $\aleph_1 < \mu < \mathfrak{c}$. Here Π^1_2 is optimal and so the result appears a natural counterpart to the co-analytic Cohen indestructible maximal cofinitary group from [14], as well as the Borel maximal cofiniatry of Horowitz and Shelah from [17]. Our theorem has its maximal almost disjoint families analogue, which extends a long line of results regarding the definability properties of mad families in models of large continuum.

1. INTRODUCTION

We will be interested in subgroups of S_∞, the group of all permutations of the natural numbers which have the additional property that all of their non-identity elements have only finitely many fixed points. Such groups are referred to as *cofinitary groups*, while permutations which have only finitely many fixed points are referred to as *cofinitary permutations*. A cofinitary group which is not properly contained in another cofinitary group, is called a *maximal* cofinitary group, abbreviated *MCG*. The existence of maximal cofinitary groups follows from the axiom of choice, which leaves many questions open regarding their possible cardinalities and their descriptive set-theoretic definability.

The study of the *the spectrum* of maximal cofinitary groups, i.e. of the set of different sizes of MCG's,

$$\mathrm{spec}(\mathrm{MCG}) := \{|\mathcal{G}| : \mathcal{G} \text{ is a maximal cofinitary group}\}$$

was of interest since the early development of the subject. Adeleke [1] proved that every maximal cofinitary groups is uncountable, Neumann showed that

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there is always a maximal cofinitary group of size $\mathfrak{c}$, while Zhang [23] showed whenever $\omega < \kappa \leq \mathfrak{c}$, consistently there is a maximal cofinitary group of size κ. A systematic study of spec(MCG) is found in [5], a study which was later generalized to analyze also the spectrum of the κ-maximal cofinitary groups (see [7]), where κ is an arbitrary regular uncountable cardinal. In [15] it was shown that the minimum of spec(MCG), denoted $\mathfrak{a}_g$, can be consistently of countable cofinality.

Definition 1.1. We refer to maximal cofinitary groups of cardinality μ, as *witnesses* to $\mu \in$ spec(MCG) and to values $\mu \in$ spec(MCG) such that $\aleph_1 < \mu < \mathfrak{c}$ as *intermediate cardinalities (or values).*

Note that any two distinct elements of a cofinitary group are eventually different reals and so cofinitary groups can be viewed as particular instances of almost disjoint families. Exactly this similarity was one of the major driving forces in the early studies of the definability properties of maximal cofinitary groups. While there are no analytic maximal almost disjoint families, a well-known result of A. R. D. Mathias, see [20], in the constructible universe L there is a co-analytic maximal almost disjoint family (see [21]). Regarding the definability properties of maximal cofinitary groups, Gao and Zhang (see [16]) constructed in L a maximal cofinitary group with a co-analytic set of generators, a result which was later improved by Kastermans [19], who showed that in L there is a co-analytic maximal cofinitary group. The existence of analytic maximal cofinitary groups was one of the most interesting open questions in the area, a question which was answered in 2016 by Horowitz and Shelah [17], who showed that there is a Borel maximal cofinitary group.[1] Further studies of the definability properties of maximal almost disjoint families can be found in [3, 10, 13, 22]. Note that in all of those instances, the maximal almost disjoint family of interest is always of cardinality $\mathfrak{c}$ (except in [22]).

The situation regarding maximal cofinitary groups is similar. Even though, there is a large volume of literature concerning the definability properties of witnesses to either $\aleph_1$ or $\mathfrak{c}$ in spec(MCG), there is very little known about the definability properties of witnesses of intermediate size. The present paper is motivated by the question: *What can we say about the definability properties of maximal cofinitary groups $\mathcal{G}$ such that $\aleph_1 < |\mathcal{G}| < \mathfrak{c}$?* Clearly a Borel maximal cofinitary group must be of size continuum and a $\mathbf{\Sigma}^1_2$ maximal cofinitary group must be either of size $\aleph_1$ or continuum, since a $\mathbf{\Sigma}^1_2$ set is the union of $\aleph_1$ many Borel sets. This observation states in particular, that the lowest projective complexity of witnesses to intermediate values in spec(MCG) is Π^1_2. This leads us to the notion of *good projective witness* (see Definition1.5), which will allow us to summarize many of the results

[1]Another interesting dissimilarity between MAD families and MCGs is the fact that consistently $\mathfrak{d} = \omega_1 < \mathfrak{a}_g = \omega_2$ (see [18]), while the consistency of $\mathfrak{d} = \omega_1 < \mathfrak{a} = \omega_2$ is a well-known open problem.

regarding the definability properties of various combinatorial sets of reals in models of $\aleph_1 < \mathfrak{c}$. Our main theorem states:

Theorem 1.2. *Let $2 \leq M < N < \aleph_0$ be given. There is a cardinal preserving generic extension of the constructible universe L in which*

$$\mathfrak{a}_g = \mathfrak{b} = \mathfrak{d} = \aleph_M < \mathfrak{c} = \aleph_N$$

and there is a Π^1_2 definable maximal cofinitary group of size $\aleph_M$.

Remark 1.3. Providing a model in which there is a maximal cofinitary group of cardinality μ where $\aleph_1 < \mu < \mathfrak{c}$ and either $\aleph_\omega < \mathfrak{c}$, or even $\aleph_\omega \leq \mu$ are possible using Jensen coding, however for the sake of clarity we have chosen in this paper to work with values of the continuum below $\aleph_\omega$.

The cardinal characteristics $\mathfrak{b}$ and $\mathfrak{d}$ referred to in the above theorem are the *bounding number* and the *dominating number*. For readers unfamiliar with them, we review definitions of all cardinals characteristics mentioned in this paper in the next section. Our techniques allow us to have also $M = 1$, i.e., to construct a model in which $\mathfrak{a}_g = \mathfrak{d} = \aleph_1 < \mathfrak{c} = \omega_N$ and in which $\mathfrak{a}_g$ is witnessed by a Π^1_2-maximal cofinitary group. The projective definition to the witness of $\mathfrak{a}_g$ though in this model is perhaps not optimal. The consistency of $\mathfrak{a}_g = \mathfrak{d} = \aleph_1 < \mathfrak{c}$ with a Π^1_1 witness to $\mathfrak{a}_g$ is work in progress of the first and third authors (see [12]).

The main result of the paper, should also be compared to [14], where the authors construct a co-analytic, Cohen indestructible maximal cofinitary group in L. Thus, consistently $\mathfrak{a}_g = \omega_1 < \mathfrak{d} = \mathfrak{c}$ with a Π^1_1-witness to $\mathfrak{a}_g$. The methods of [14] and the current paper differ significantly. While the result of [14] is rooted in the preservation properties of a specially constructed cofinitary group in L, and so necessarily of cardinality $\aleph_1$, the techniques of the current paper allow us to control the value of $\mathfrak{a}_g$ beyond $\aleph_1$.

There are two further challenges, which we needed to overcome in obtaining the above theorem:

(1) adjoining a new generator to an uncountable group, while requiring that all new permutations satisfy a self referential recurrence leading eventually to the Π^1_2-definition of the final generic group;
(2) providing enough eventually different reals at initial stages of the forcing construction, which allow (1) and which are not excluded by the generic hitting property of the cofinitary group iterands;

Resolving the first problem, resulted in a carefully designed forcing notion which we present in Section 3 of the paper. Even though this new poset can be compared to earlier forcing notions adjoining generic generators to a given cofinitary group, it is far more intricate and allows for a much finer control over the group members. The second problem was eventually resolved by a very careful arrangement of the entire forcing iteration (see items 4.a and 4.b of the road-map given in the beginning of Section 4) and gives a very necessary flexibility of the entire construction, without which our final goal could not be achieved. These new technical developments not only suggest more elegant proofs to already existing theorems, but also present promising and robust techniques to address existing open problems. Some of the many naturally occurring remaining open questions are discussed in our final section.

Our techniques easily modify to the study of maximal almost disjoint families and provide the following result:

Theorem 1.4. *Let $2 \leq M < N < \aleph_0$ be given. There is a cardinal preserving generic extension of the constructible universe L in which*

$$\mathfrak{a} = \mathfrak{b} = \mathfrak{d} = \aleph_M < \mathfrak{c} = \aleph_N$$

and there is a Π^1_2 definable maximal almost disjoint family of size $\aleph_M$.

The results discussed in this section lead to the following notion:

Definition 1.5. *A good projective witness* to $\mu \in \operatorname{spec}(\mathrm{MCG})$ (resp. $\mu \in \operatorname{spec}(\mathrm{MAD})$[2]) is a mcg (resp. mad family) of cardinality μ in a model of $\aleph_1 < \mathfrak{c}$ which is also of lowest projective complexity.

While earlier results show that good projective witnesses to $\aleph_1$ and $\mathfrak{c}$ being members of spec(MCG) (resp. spec(MAD)) exist, our main theorem states that good projective witnesses for intermediate values can exist. For example, the co-analytic Cohen indestructible maximal cofinitary group from [14] is a good projective witness to $\aleph_1 \in \operatorname{spec}(\mathrm{MCG})$, while the Borel maximal cofinitary group of Horowitz-Shelah is a good projective witness to $\mathfrak{c} \in \operatorname{spec}(\mathrm{MCG})$. A good witness to $\mathfrak{c} \in \operatorname{spec}(\mathrm{MAD})$ is constructed by Brendle and Khomskii in [3], while a Cohen indestructible co-analyitc maximal almost disjoint family in L is a good witness to $\aleph_1 \in \operatorname{spec}(\mathrm{MAD})$. The study of projective witnesses does not limit to mcgs and mad families. Let spec(IND) denote the set of possible cardinalities of maximal independent families. One of the main results of [4] shows that $\aleph_1 \in \operatorname{spec}(\mathrm{IND})$ has a good projective witness, while the existence of a good projective witness to $\mathfrak{c} \in \operatorname{spec}(\mathrm{IND})$ is still open.

Structure of the paper: In section 2 we introduce relevant notation and terminology used throughout the paper. Section 3 presents a new poset, which adjoins self-coding permutations to a given cofinitary group. Section 4 presents the entire forcing construction leading to our main result. Our main result is established in Section 5. List of open problems is given in Section 6.

2. Some Notation and Terminology

Given an index set A, we will call a mapping $\rho : A \to S_\infty$ such that $\operatorname{im}(\rho)$ generates a cofinitary group, a *cofinitary representation.* In particular, given a freely generated cofinitary group with generating set $\{g_a : a \in A\}$, the

[2]Here spec(MAD) denotes the set of cardinalities of maximal almost disjoint families.

mapping $\rho : A \to S_\infty$ sending each a to g_a is a cofinitary representation. Given such a cofinitary representation ρ and an index a which does not occur in $\mathrm{dom}(\rho)$, we denote by $W_{\rho,\{a\}}$ the set of all words w of the form $w = a_n^{j_n} \cdots a_1^{j_1}$ where for each l such that $1 \le l \le n$ we have $a_l \in \mathrm{dom}(\rho) \cup \{a\}$, $j_l \in \{1, -1\}$ and no cancellations are allowed; or $n = 0$ and $w = \emptyset$.[3] An injective partial function $s : \mathbb{N} \rightharpoonup \mathbb{N}$ will be referred to as a partial permutation. Given a word $w \in W_{\rho,\{a\}}$ and a (possibly partial) injective mapping s, we denote by $w[s]$ the (possibly partial) injective mapping $w[s]$ obtained by substituting each occurrence of b^j where $b \in \mathrm{dom}(\rho)$ and $j \in \{-1, 1\}$ with $\rho(b)^j$ and a^j where $j \in \{-1, 1\}$ with s^j. Now, given a word $w \in W_{\rho,\{a\}}$, $w = a_n^{j_n} \cdots a_1^{j_1}$, where $j_l \in \{-1, 1\}$ and a (possibly partial) injective mapping s, *the evaluation path* of a given integer m under $w[s]$ is the sequence $\langle m_k : k \in \omega' \rangle$, where $m_0 = m$, for each k if $k = nl + i$, then

$$m_k = (a_i^{j_i}[s] \circ \cdots \circ a_1^{j_1}[s] \circ w^{nl}[s])(m),$$

where ω' is either ω, or denotes the least natural number for which $m_{\omega'}$ is not defined.

Following the notation of [14], we denote by $\mathrm{use}(w, s, m)$ the set of natural numbers appearing in the evaluation path of m under $w[s]$.

Another notion naturally appearing in the analysis of the fixed points and evaluation paths associated to a given word w, is the notion of a *circular shift* of a word (see [14]). More precisely, given a word $w = w_n \cdots w_1$, where $w_i = a_i^{j_i}$, $j_i \in \{-1, 1\}$ for each i, and a permutation $\sigma : \{1, \cdots, n\} \to \{1, \cdots, n\}$ such that $\sigma(i) = i + k \mod n$ for some $k \in \mathbb{N}$, we will refer to $w_{\sigma(n)} \cdots w_{\sigma(1)}$ as a circular shift of w. Thus, in particular, for each n there are only finitely many circular shifts of a given word.

Finally, for $w_0, w_1 \in W_{\rho,\{a\}}$ we say w_1 is a *proper conjugate subword of* w_0 if $w_0 = w^{-1} w_1 w$ for some word $w \in W_{\rho,\{a\}} \setminus \{\emptyset\}$ and $w_1 \neq \emptyset$.

We review definitions of the well-known cardinal characteristics $\mathfrak{a}$, $\mathfrak{a}_g$, $\mathfrak{b}$, and $\mathfrak{d}$ (for an introduction to cardinal characteristics, see [2]). An *almost disjoint family* is a collection of infinite subsets of ω any two of which have finite intersection. A *maximal almost disjoint* (short MAD) *family* is an almost disjoint family which is not a proper subset of an almost disjoint family.

Write ω^ω for the set of functions from ω to ω. Given $f, g \in \omega^\omega$ write $f \le^* g$ to mean that $\{n : f(n) > g(n)\}$ is finite. Now

$$\begin{aligned}
\mathfrak{a} &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{P}(\omega),\ \mathcal{A} \text{ is an infinite MAD family}\}, \\
\mathfrak{a}_g &= \min\{|\mathcal{G}| : \mathcal{G} \subseteq \omega^\omega,\ \mathcal{G} \text{ is a MCG}\}, \\
\mathfrak{b} &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega, (\forall g \in \omega^\omega)(\exists f \in \mathcal{F})\ f \not\le^* g\}, \\
\mathfrak{d} &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega, (\forall g \in \omega^\omega)(\exists f \in \mathcal{F})\ g \le^* f\}
\end{aligned}$$

where of course $|x|$ denotes the cardinality of x.

[3] Such words are referred to as *reduced words*.

3. Adding cofinitary groups of coding permutations

Fix a recursive bijection

$$\psi : \omega \times \omega \to \omega.$$

Suppose that $\rho : A \to S_\infty$ is a cofinitary group presentation and let a be an index not included in A (i.e., we ask $a, a^{-1} \notin A$). Write $\mathcal{G}$ for the group generated by $\operatorname{im}(\rho)$, $W = W_{\rho,\{a\}}$ for the set of reduced words in the alphabet $\operatorname{dom}(\rho) \cup \{a, a^{-1}\}$, WD for the set of words from W in which a or a^{-1} occurs at least once, and WS for the set of words $w \in \mathrm{WD}$ without a proper conjugate subword.

Further, suppose that we are given

- $\mathcal{F} = \{f_{m,\xi} : m \in \omega, \xi \in \omega_1\}$, a family of almost disjoint permutations (i.e., the graphs are pairwise almost disjoint subsets of $\omega \times \omega$) so that $f_{m,\xi} \notin \operatorname{im}(\rho)$ and $\langle \operatorname{im}(\rho), f_{m,\xi}\rangle$ is cofinitary for each $m \in \omega, \xi \in \omega_1$.
- For each $w \in \mathrm{WS}$, a family $\mathcal{Y}^w = \{Y^w_m : m \in \omega\}$ of subsets of ω_1,
- For each $w \in \mathrm{WS}$ a subset z^w of ω.

Write $\mathcal{F}$ for $\langle \mathcal{F} : w \in \mathrm{WS}\rangle$, $\mathcal{Y}$ for $\langle \mathcal{Y}^w : w \in \mathrm{WS}\rangle$, and $\bar{z}$ for $\langle z^w : w \in \mathrm{WS}\rangle$.

We will define a σ-centered poset, denoted $\mathbb{Q}^{\mathcal{F},\mathcal{Y},\bar{z}}_{\rho,\{a\}}$, which adjoins a generic permutation g such that the mapping $\hat{\rho} : A \cup \{a\} \to S_\infty$, which extends ρ and sends a to g is a cofinitary representation; moreover, for each $w \in \mathrm{WS}$

- the permutation $w[g]$ codes (in a sense about to be defined) the real z^w,
- for each $m \in \psi[g]$, $w[g]$ almost disjointly via the family $\mathcal{F}^m = \{f_{m,\xi} : \xi \in \omega_1\}$ codes Y^w_m.

In order to define the poset we must discuss how each z^w will be coded and introduce some related terminology. To this end, let S_0 be the unique function from WS into the set of words in the alphabet $\{a, a^{-1}, y, y^{-1}\}$ which in each word replaces each letter from A with y (and inverses of letters from A with y^{-1}). Moreover, fix a function $S\colon \mathrm{WS} \to \omega$ such that for all $w, w' \in \mathrm{WS}$:

- $S(w) = S(w') \iff S_0(w) = S_0(w)$,
- $\operatorname{lh}(w) < \operatorname{lh}(w') \Rightarrow S(w) < S(w')$, and
- $S(w) > 1$.

Definition 3.1 (Coding)**.** Let a sequence $\chi \in 2^{\leq\omega}$ be given. Suppose σ is a partial function from ω to ω and $w \in \mathrm{WS}$.

(1) We say (w, σ) *codes* χ *with parameter* m if and only if

$$(\forall k < \operatorname{lh}(\chi))\ \sigma^{S(w)\cdot(k+1)}(m) \equiv \chi(k) \pmod 2. \tag{3.1}$$

(2) Suppose now that $\operatorname{lh}(\chi) < \omega$. Write $w = w_1 w_0$ where w_0 is shortest so that its leftmost letter is a or a^{-1}. We say that (w, σ) *exactly codes* χ *with parameter* m if (w, σ) codes χ and in addition

$$w_0 w^{S(w)\cdot\operatorname{lh}(\chi)}[\sigma](m) \text{ is undefined,}$$

that is, if the path of m under $w[\sigma]$ terminates as soon as possible.

(3) We say that m' is *a critical point in the path of m under* (w, σ) if this path terminates with m' and has length $S(w)(k+1)-1$ for some k.

Note that clearly (w, σ) can only *exactly* code χ if the latter is finite and σ is not a bijection (i.e., σ or σ^{-1} is a partial function).

Finally given $\mathcal{F}, \mathcal{Y}, \bar{z}, \rho, \{a\}$ as above we define $\mathbb{Q} = \mathbb{Q}^{\mathcal{F},\mathcal{Y},\bar{z}}_{\rho,\{a\}}$. First we define an auxiliary forcing $\mathbb{Q}_0$; it consists of all tuples $p = \langle s^p, F^p, \bar{m}^p, s^{p,*}\rangle$ where:

(1) s^p is an injective finite partial function from ω to ω;
(2) F^p is a finite subset of WS which is closed with respect to taking subwords;
(3) $\bar{m}^p = \langle m^p_w : w \in \mathrm{dom}(\bar{m}^p)\rangle$ with $\mathrm{dom}(\bar{m}^p) \subseteq F^p$ and each $m^p_w \in \omega$;
(4) $s^{p,*} = \langle s^{p,*}_w : w \in \mathrm{dom}(s^{p,*})\rangle$ is a finite partial function from F^p to

$$\{f_{m,\xi} : m \in \psi[w[s]], \xi \in Y^w_m\};$$

The extension relation for $\mathbb{Q}_0$ is defined as follows: $q = \langle s^q, F^q, \bar{m}^q, s^{q,*}\rangle \leq_0 p = \langle s^p, F^p, \bar{m}^p, s^{p,*}\rangle$ if and only if

(A) s^q end-extends s^p, $F^q \supseteq F^p$;
(B) for every $w \in F^p$ if $m \in \mathrm{fix}(w[s^q])$, then there is a non-empty subword w' of w such that letting $w = w_1 w' w_0$ and letting $\langle \dots m_1, m_0\rangle$ be the (w, s^q)-path of m, $m_k \in \mathrm{fix}(w'[s^p])$ where k is the length of w_0; i.e., the path has the following form:

$$m \xleftarrow{w_1} m_k \xleftarrow{w'} m_k \xleftarrow{w_0} m$$

(C) $s^{q,*} \supseteq s^{p,*}$ and for all $f \in s^{p,*}$, $s^q \backslash s^p \cap f = \emptyset$.
(D) $\bar{m}^q \restriction (\mathrm{dom}(\bar{m}^p) \cap \mathrm{dom}(\bar{m}^q)) = \bar{m}^p \restriction (\mathrm{dom}(\bar{m}^p) \cap \mathrm{dom}(\bar{m}^q))$

Finally, $\mathbb{Q}$ is defined to be the set of $p \in \mathbb{Q}_0$ which in addition to items (1)–(4) above also satisfy

(5) for each $w \in \mathrm{dom}(\bar{m}^p)$ there exists a (unique) l which we denote by l^p_w such that (w, s) exactly codes $\chi_{z^w} \restriction l$ with parameter m^p_w;

The ordering on $\mathbb{Q}$, which we denote by $\leq$ is just $\leq_0 \cap (\mathbb{Q} \times \mathbb{Q})$.

Proposition 3.2. *Let G be a $\mathbb{Q}$-generic filter and let*

$$\sigma^G = \bigcup \{s : \exists F, \bar{m}, s^* \text{ s.t. } \langle s, F, \bar{m}, s^*\rangle \in G\}.$$

The permutation σ^G has the following properties:

(A) The group $\langle \mathrm{im}(\rho) \cup \{\sigma^G\}\rangle$ is cofinitary.
(B) If f is a ground model permutation, $f \notin \langle \mathrm{im}(\rho)\rangle$, $\langle \{f\} \cup \mathrm{im}(\rho)\rangle$ is cofinitary and f is not covered by finitely many permutation in $\mathcal{F}$, then there are infinitely many n such that $f(n) = \sigma^G(n)$ and so $\langle \mathrm{im}(\rho) \cup \{\sigma^G\} \cup \{f\}\rangle$ is not cofinitary;
(C) For each $w \in$ WS there is $m_w \in \omega$ such that $w[\sigma^G]$ codes the characteristic function of z^w with parameter m_w.

(D) For each $w \in \mathrm{WS}$, for all $m \in \psi[w[\sigma^G]]$, for all $\xi \in \omega_1$

$$|w[\sigma^G] \cap f_{m,\xi}| < \omega \text{ iff } \xi \in Y^w_m.$$

We shall now show these properties to hold, in a series of lemmas. It is most convenient to start with the most involved of the series; it has a precursor in [14, Lemma 3.12] and in conjunction with the following lemmas, it proves Property (C).

Lemma 3.3 (Generic Coding). *For any $w \in \mathrm{WD}$ and any $l \in \mathbb{N}$, let $D^{\mathrm{code}}_{w,l}$ denote the set of $q \in \mathbb{Q}$ such that $w \in \mathrm{dom}(\bar{m}^q)$ and for some $l' \geq l$, q exactly codes $z^w \upharpoonright l'$ with parameter $\bar{m}^q_w$. Then $D^{\mathrm{code}}_{w,l}$ is dense in $\mathbb{Q}$.*

Proof. Suppose $p \in \mathbb{Q}$ and $w \in \mathrm{WS}$ are given. If $w \notin \mathrm{dom}(\bar{m}^p)$ it is clear that we can choose m large enough so that letting

$$q = \langle s^p, F^p, \bar{m}^p \cup \{(w, m)\}, s^{p,*}\rangle$$

we obtain a condition $q \in \mathbb{Q}$ with $l^q_w = 0$ (i.e., that we can chose m so that (w, s^p) codes the trivial string $\emptyset$ with parameter m).

So suppose $w \in \mathrm{dom}(\bar{m}^p)$. Write m for $\bar{m}^p_w$ and l for l^p_w. It suffices to find $s \supseteq s^p$ such that letting

$$q = \langle s, F^p, \bar{m}^p, s^{p,*}\rangle$$

we obtain a condition $q \in \mathbb{Q}$ with $l^q_w = l + 1$.

Let m_0 be the terminating value in the path of m under w and suppose the next letter in w that should be applied is a^i for $i \in \{-1, 1\}$. Let W_0 denote the set of words w' in $\mathrm{dom}(\bar{m}^p)$ whose path from $m^p_{w'}$ also terminates with m_0 and with next letter also a^i (we cannot avoid extending coding paths of words in W_0 and have to ensure exact coding for all of them). Note that this path has length $l^p_{w'} \cdot S(w')$ if the right-most letter of w' is a or a^{-1} and $l^p_{w'} \cdot S(w') + 1$ otherwise.

For each $w' \in W_0$ let $g(w') \in \mathrm{im}(\rho) \cup \{\emptyset\}$ be the rightmost letter if this letter is not a nor a^{-1}, and $g(w') = \emptyset$ otherwise. Then

$$m_0 = g(w') {w'}^{S(w')\cdot l^p_{w'}}[s^p](m).$$

The next point in the path at which we must meet a coding requirement for a word $w' \in W_0$ will be reached after applying $(w')^{S(w')}$ to $g(w')^{-1}(m_0)$. Write $W(w')$ for the set of initial segments of $(w')^{S(w')}$ and consider the tree

$$T = \bigcup_{w' \in W_0} W(w')$$

ordered by end-extension. We make finitely many extensions of s^p, each time extending a coding path starting with m_0 by one step, working along all words in T by induction on their length.

So suppose $w' \in T$ and we have already extended s^p to s' so that

$$w'[s'](m_0) = m',$$

and that for no extension w'' of w' in T is $w''[s'](m_0)$ defined, and fix a word $a^j w' \in T$ where $j \in \{-1, 1\}$. For each $w^* \in W_0$ denote by $l(w^*)$ the length of the path of $m^p_{w^*}$ under (w, s'). We shall now find s'' extending s'.

Let

$$E = \operatorname{dom}(s') \cup \operatorname{ran}(s') \cup \operatorname{ran}(\bar{m}^p)$$

and let F consist of all subwords of circular shifts of words in F^p. Find m'' satisfying the following requirements:

$$m'' \notin \bigcup\{\operatorname{fix}(u[s']) : u \in F \setminus \{\emptyset\}\}, \tag{3.2}$$

$$m'' \notin \bigcup\{\operatorname{fix}(g_0^{-1} g_1[s']) : u_0, u_1 \in F \cap \langle \operatorname{im}(\rho)\rangle \setminus \{\emptyset\}, g_0 \neq g_1\}, \tag{3.3}$$

$$m'' \notin \bigcup\{g^i u^j[s'][E] : i, j \in \{-1, 1\}, u \in F, g \in F \cap \langle \operatorname{im}(\rho)\rangle\}, \tag{3.4}$$

and if m' is a critical point in the path under (w^*, s') of $\bar{m}^p_{w*}$,

$$m'' \equiv z^{w^*}\left(\frac{l(w^*) + 1}{S(w^*)\operatorname{lh}(w^*)}\right) \pmod 2. \tag{3.5}$$

Note that all but the last requirement exclude only finitely many values for m''. To see that m'' as above can be found, we show that m' is a critical point in the path under (w^*, s') of $\bar{m}^p_{w*}$ for at most a single w^*. Therefore we can chose m'' to be any large enough number with the parity prescribed by (3.5).

Claim 3.4. *There is at most one word $w^* \in W_0$ such that the path of $m^p_{w^*}$ under (w^*, s') terminates at m' and $l(w^*) + 1 = (l^p_{w^*} + 1) \cdot S(w^*) \cdot \operatorname{lh}(w^*)$, i.e., so that we must respect the coding requirement* (3.5) *for w^*.*

Proof. Suppose there are $w^*_0 \neq w^*_1$ with the above property. Depending on whether $g(w^*_i) = \emptyset$ or $g(w^*_i) \in \operatorname{im}(\rho)$ we have $l(w^*_i) = k \cdot \operatorname{lh}(w^*_i)$ or $l(w^*_i) = k \cdot \operatorname{lh}(w^*_i) - 1$ for each $i \in \{0, 1\}$. First assume the words are not of equal length, w.l.o.g. $\operatorname{lh}(w^*_0) < \operatorname{lh}(w^*_1)$. But then

$$S(w^*_0) \cdot \operatorname{lh}(w^*_0) < S(w^*_1) \cdot \operatorname{lh}(w^*_1) - 1$$

so for at most one $i \in \{0, 1\}$ can the length of the path from m_0 to m' under (w^*_i, s') be of length $S(w^*_0) \cdot \operatorname{lh}(w^*_0)$ or $S(w^*_0) \cdot \operatorname{lh}(w^*_0) - 1$. If on the other hand $\operatorname{lh}(w^*_0) = \operatorname{lh}(w^*_1)$ then since $w^*_0 \neq w^*_1$ the path of m_0 under (w^*_0, s') must diverge from its path under (w^*_1, s') before reaching m': These paths diverge at some m_k where w^*_0 and w^*_1 disagree at the next letter since by induction, s' was chosen to satisfy Requirements (3.3) and (3.4) each time we made an extension; and these paths are long enough to witness a disagreement between w^*_0 and w^*_1 because $S(w^*_i) > 1$ (this is necessary and sufficient to deal with words where the only difference is in the first letter and this letter is from $\operatorname{im}(\rho)$). Claim 3.4. □

Let $s'' = s' \cup \{(m', m'')\}$; the next two claims shall show that $p' = \langle s'', F^p, \bar{m}^p, s^{p,*}\rangle$ is a condition in $\mathbb{Q}_0$ below p (that is, a condition in $\mathbb{Q}$ except for the requirement of exact coding).

Claim 3.5. *For any $w \in \operatorname{dom}(\bar{m}^p) \setminus W_0$, the path of m^p_w under (w, s^p) is the same as under (w, s'').*

Proof. This is obvious by Requirement (3.4) above. Claim 3.5. □

The next claim shows that $p' \leq_0 p$.

Claim 3.6. *For every $w \in F^p$ and $m \in \operatorname{fix}(w[s''])$ there is a non-empty subword w_0 of w such that letting $w = w' w_0 w''$ and letting $\langle \dots m_1, m_0\rangle$ be the (w, s'')-path of m, $m_k \in \operatorname{fix}(w_0[s'])$ where k is the length of w''; i.e., the path has the following form:*

$$m \xleftarrow{w'} m_k \xleftarrow{w_0} m_k \xleftarrow{w''} m.$$

Proof. Fix $w \in F^p$. Assume that $m_0 \in \operatorname{fix}(w[s'']) \setminus \operatorname{fix}(w[s'])$. As the (w, s')-path of m_0 differs from the (w, s'')-path, the latter must contain an application of a to m' or of a^{-1} to m''. Write this latter path as

$$\dots m_{k(3)} \xleftarrow{w''} m_{k(2)} \xleftarrow{a^j} m_{k(1)} \xleftarrow{w'} m_{k(0)} = m_0 \tag{3.6}$$

where $j \in \{-1, 1\}$ and $m_{k(1)} = n$ when $j = 1$, $m_{k(1)} = n'$ when $j = -1$; moreover we ask that $w', w'' \in W$ are the maximal subwords of w such that from $m_{k(0)}$ to $m_{k(1)}$ and $m_{k(2)}$ to $m_{k(3)}$, the path contains no application of a to m' or of a^{-1} to m'' (allowing either of w', w'' to be empty). Thus, w' and w'' correspond to path segments where s' and s'' agree:

$$w'[s''](m_{k(0)}) = w'[s'](m_{k(0)}) = m_{k(1)},$$
$$w''[s''](m_{k(2)}) = w''[s'](m_{k(2)}) = m_{k(3)}.$$

It is impossible that $w = w'' a^j w'$ and $m_0 = m_{k(3)}$ (for then

$$m'' = (w'w'')^{-j}[s''](m'),$$

again contradicting the choice of m''). Therefore, at step $k(3)$ again a is applied to m' or a^{-1} to m'' by maximality of w''. Write the path as

$$\dots \xleftarrow{a^{j'}} m_{k(3)} \xleftarrow{w''} m_{k(2)} \xleftarrow{a^j} m_{k(1)} \xleftarrow{w'} m_{k(0)} = m_0$$

with $j' \in \{-1, 1\}$ and observe:

1. $m_{k(2)} = m_{k(3)}$; for otherwise, $m'' = (w'')^i[s'](m')$ for some $i \in \{-1, 1\}$, contradicting the choice of m''.
2. Thus, $w'' \neq \emptyset$, since on one side of w'' we have a and on the other a^{-1} and w is in reduced form.
3. As $m'' \notin \operatorname{fix}(w''[s'])$, we have that $m_{k(2)} = m_{k(3)} = m'$.

So $m' \in \operatorname{fix}(w''[s'])$ proving the claim. Claim 3.6. □

Repeating the above argument for each relevant word in T we obtain a condition $q \leq p$ also satisfying the exact coding condition (5) and such that for each $w^* \in W_0$, $l^q_{w*} = l^p_{w*} + 1$ as promised. Lemma 3.3. □

The next lemma shows that g is permutation of ω.

Lemma 3.7. *For each $n \in \omega$ the sets $D_n = \{q \in \mathbb{Q} : n \in \operatorname{dom}(s^q)\}$ and $D^n = \{q \in \mathbb{Q} : n \in \operatorname{ran}(s^q)\}$ are dense in $\mathbb{Q}$.*

Proof. To see D_n is dense, let $p \in \mathbb{Q}$ be given and find $q \in D_n$, $q \leq p$.

If n occurs as the last value in a coding path, the previous lemma applies. Otherwise let W^* be the set of subwords of circular shifts of words in F^p and pick n' arbitrary such that

$$\begin{aligned} &n' \notin \bigcup \{ \operatorname{fix}(w'[s^p]) \colon w' \in W^* \setminus \{\emptyset\}\},\\ &n' \notin \bigcup \{w'[s^p]^i(n) \colon i \in \{-1,1\}, w' \in W^*\}, \text{ and}\\ &n' \notin \operatorname{ran}(s^p). \end{aligned}$$

Let $s' = s \cup \{(n, n')\}$ and $q = \langle s', F^p, \bar{m}^p, s^{p,*}\rangle$. Then $q \in \mathbb{Q}$ and $q \leq p$ by exactly the same argument as in Claims 3.5 and 3.6 above. The case D^n is symmetrical and is left to the reader. □

Property (A) above is established by the previous lemma and the following one.

Lemma 3.8. *For each $w \in W_{\rho,\{a\}}$, the set*

$$D_w = \{q \in \mathbb{Q} : q \Vdash |\operatorname{fix}(w[\sigma_G])| < \infty\}$$

is dense in $\mathbb{Q}$.

Proof. First note that $q \Vdash |\operatorname{fix}(w[\sigma_G])| < \infty$ if $w \in F^q$: This is because such q forces—by the definition of the ordering on $\mathbb{Q}$—that any fixed point of $w[\sigma_G]$ must arise from a fixed point of $w'[s^q]$ where w' is a subword of w and there are only finitely many such points.

Therefore clearly D_w is dense, since we may always add the shortest conjugated subword of any word w to F^q to form a new condition, and of course $w[\sigma^G]$ has the same number of fixed points as its shortest conjugated subword. □

The next lemma shows Property (B) above. Moreover, Property (D) is a direct corollary to this lemma and the almost disjoint requirement in the extension relation of our poset.

Lemma 3.9. *Suppose we are given $m \in \omega$, $w \in \mathrm{WS}$ and $\tau \in S_\infty$.*

(1) If $\tau \notin \langle \operatorname{im}(\rho)\rangle$, $\langle \operatorname{im}(\rho), \tau\rangle$ is cofinitary, and τ is not covered by finitely many elements of $\mathcal{F}$, the set $D^{\text{hit}}_{\tau,m} = \{q \in \mathbb{Q} : (\exists n \geq m)\ w[s^q](n) = \tau(n)\}$ is dense.

(2) If $\tau \in \mathcal{F}$, $\tau = f^w_{n,\xi}$, and $\xi \notin Y^w_m$ then too is the set $D^{\text{hit}}_{\tau,m}$ dense.

(3) If $\tau \in \mathcal{F}$, $\tau = f^w_{n,\xi}$, and $\xi \in Y^w_m$ the set $D^{\text{hit}}_{\tau,m} \cup \{p \in \mathbb{Q} : n \in \psi[w[s^p]]\}$ is dense in $\mathbb{Q}$.

Proof. Let τ and m as in the lemma be given. Note that in all three cases $\tau \notin \langle \operatorname{im}(\rho)\rangle$ and $\langle \operatorname{im}(\rho), \tau\rangle$ is cofinitary and we can assume $\tau \notin s^{*,p}$ (for in the third case, otherwise $n \in \psi[w[s^p]])$ and therefore that

$$|\tau \setminus \bigcup s^{p,*}| = \omega. \tag{3.7}$$

Let $E' = \mathrm{dom}(s^p) \cup \mathrm{ran}(s^p) \cup \mathrm{ran}(\bar{m}^p)$, and find $n \in \omega \setminus m$ such that

$$n \notin \tau^{-1}\Big[\bigcup \big\{ \mathrm{fix}(w[s]) \colon w \in F^* \setminus \{\emptyset\}\big\}\Big],$$

$$n \notin \tau^{-1}\Big[\bigcup \big\{g^{-1}w'[s]^i[E'] \colon i \in \{-1,1\}, w' \in F^*, g \in F^* \cap \langle\, \mathrm{im}(\rho)\rangle\big\}\Big],$$

$$n \notin \bigcup \big\{ \mathrm{fix}(\tau^{-1}g^{-1}w'[s]^i) \colon i \in \{-1,1\}, w' \in F^*, g \in F^* \cap \langle\, \mathrm{im}(\rho)\rangle\big\}, \text{ and}$$

$\tau(n) \neq f(n)$ for each $f \in s^{p,*}$.

The first two requirements obviously exclude only finitely many n; the same holds for the third requirement since $\tau \notin \langle \mathrm{im}(\rho)\rangle$ and $\langle \mathrm{im}(\rho), \tau\rangle$ is cofinitary. Since the last requirement holds for infinitely many n by (3.7), we can pick n satisfying all the requirements.

It follows that letting $n' = \tau(n)$ and $E = \{n\} \cup \mathrm{dom}(s^p) \cup \mathrm{ran}(s^p) \cup \mathrm{ran}(\bar{m}^p)$, n' satisfies the requirements from (3.2)–(3.5). Therefore as in Lemma 3.3 we can let $s = s^p \cup \{(n, n')\}$ and $q = \langle s, F^p, \bar{m}^p, s^{p,*}\rangle$ is a condition below p satisfying $q \in D^{\mathrm{hit}}_{\tau,m}$. □

Finally we show the following.

Lemma 3.10. *The forcing $\mathbb{Q}$ is Knaster.*

Proof. It is straightforward to check that if $p, q \in \mathbb{Q}$ are such that $s^p = s^q$ and $\bar{m}^p$ agrees with $\bar{m}^q$ on $\mathrm{dom}(\bar{m}^p) \cap \mathrm{dom}(\bar{m}^q)$ then

$$r = \langle s^p, F^p \cup F^q, \bar{m}^p \cup \bar{m}^q, s^{p,*} \cup s^{q,*}\rangle$$

is a condition in $\mathbb{Q}$ and $r \leq p, q$. Therefore $\mathbb{Q}$ is Knaster by a standard Δ-systems argument. □

4. The forcing iteration

Since the proof is long and involved, we present a short road-map which may also be used as a reference for notation. We proceed in several steps:

(1) We start with the constructible universe L as the ground model. We chose a sequence $\langle S_\delta : \delta < \omega_M\rangle$ of stationary subsets of ω_{M-1} and force to add a sequence $\langle C_\delta : \delta < \omega_M\rangle$ such that C_δ is a club in ω_{M-1} which is disjoint from S_δ, "killing" the stationarity of S_δ. Then we force to add a sequence $\langle Y_\delta : \delta < \omega_M\rangle$ such that $Y_\delta \subseteq \omega_1$ and Y_δ "locally codes" C_δ. By "locally coding" we mean the property $(***)_{\gamma,m}$ below. For this purpose we also have to add a sequence $\mathcal{W} = \langle W^0_\gamma : \gamma \in \mathrm{Lim}(\omega_M)\rangle$ of auxillary subsets of ω_1 where W^0_γ will serve as a code for the ordinal γ.

The forcing that adds $\langle C_\delta : \delta < \omega_M\rangle$, the auxillary sets $\mathcal{W}$, as well as $\langle Y_\delta : \delta < \omega_M\rangle$ is denoted by $\mathbb{P}^*_0$, and the $(\mathbb{P}^*_0, L)$-generic extension is denoted by V_1. It will be the case that $\mathcal{P}(\omega)^{V_1} = \mathcal{P}(\omega)^L$.

(2) We force over V_1 to add a sequence

$$\mathcal{C} = \langle c^W_\gamma : \gamma \in \mathrm{Lim}(\omega_M)\rangle$$

of reals such that c^W_γ codes W^0_γ. We denote the forcing that adds $\mathcal{C}$ by $\mathbb{P}(\mathcal{C})$ and the $(V_1, \mathbb{P}(\mathcal{C}))$-generic extension by V_2.

(3) We increase 2^ω by adding ω_N-many reals forcing with $\mathrm{Add}(\omega, \omega_N)$. Write V_3 for the $(V_2, \mathrm{Add}(\omega, \omega_N))$-generic extension.

(4) We now force to add the definable MCG. This is done in an iteration $\mathbb{P}(\mathcal{G}) := \langle \mathbb{P}^{\mathcal{G}}_\alpha, \dot{\mathbb{Q}}^{\mathcal{G}}_\alpha : \alpha \in \omega_M\rangle$ of length ω_M over V_3. The final $(V_3, \mathbb{P}(\mathcal{G}))$-generic extension is denoted by $L[G]$.

We denote the $(V_3, \mathbb{P}^{\mathcal{G}}_\alpha)$-generic extension by $V_3[G^{\mathcal{G}}_\alpha]$. At step $\alpha < \omega_M$ in the iteration we force over $V_3[G^{\mathcal{G}}_\alpha]$ with $\mathbb{Q}_\alpha = \mathbb{P}_{\mathcal{F}_\alpha} * \mathbb{P}^{\mathrm{cd}}_{\mathcal{F}_\alpha} * \mathbb{P}^{\mathcal{G}}_\alpha$ where:

(a) The first forcing $\mathbb{P}^{\mathcal{F}}$ adds a family $\mathcal{F}_\alpha$ of size ω_1 consisting of cofinitary permutations of ω. We do this so that in the final model $L[G]$ the graphs of any two elements of $\bigcup_{\alpha<\omega_M} \mathcal{F}_\alpha$ will be almost disjoint.

(b) The next forcing $\mathbb{P}^{\mathrm{cd}}_{\mathcal{F}_\alpha}$ adds a real $c^{\mathcal{F}}_\alpha$ which almost disjointly codes $\mathcal{F}_\alpha$ via a definable almost disjoint family $\mathcal{F}^* \in L$ which remains fixed throughout the iteration.

(c) Finally $\mathbb{P}^{\mathcal{G}}_\alpha$ is the forcing discussed in the previous section adding a single generator of our MCG, using all the machinery added in the previous steps to ensure definability of the resulting group.

Step (1) is described in Section 4.1 below. In this part we do not add countable sequences. Steps (2) and (3) are described in Section 4.2. Finally Steps (4a)–(4c), in which we force to add a MCG of size less than 2^ω, are described in Section 4.3.

4.1. **Preparing the Universe.** We will work over the constructible universe L. Fix $2 \leq M < N < \omega$ arbitrary. We will show that consistently $\mathfrak{a}_g = \omega_M < \mathfrak{c} = \omega_N$ with a Π^1_2 definable witness to $\mathfrak{a}_g$.

Let $\bar{S} = \langle S_\delta : \delta < \omega_M\rangle$ be a sequence of stationary costationary subsets of ω_{M-1} consisting of ordinals of cofinality ω_{M-2} and such that for $\delta \neq \delta'$, $S_\delta \cap S_{\delta'}$ is non-stationary. We also ask that $\bar{S}$ be definable in L_{ω_M}. Every element of the intended Π^1_2-definable maximal cofinitary group will witness itself by encoding a pattern of stationarity, non-stationarity on a segment (a block of the form $[\gamma, \gamma+\omega)$ for $\gamma \in \mathrm{Lim}(\omega_M)$) of $\bar{S}$. To achieve this, the following terminology will be useful.

Definition 4.1. A *suitable model* is a transitive model $\mathcal{M}$ such that $\mathcal{M} \vDash \mathrm{ZF}^-$, $(\omega_M)^{\mathcal{M}}$ exists and $(\omega_M)^{\mathcal{M}} = (\omega_M)^{L^{\mathcal{M}}}$ (by ZF^- we mean an appropriate axiomatization of set theory without the Power Set Axiom).

For each ordinal $\gamma \in \mathrm{Lim}(\omega_M)$ write W_γ for the L-least subset of ω_{M-1} such that

$$\langle \gamma, <\rangle \cong \langle W_\gamma, \in\rangle.$$

For each $m = 1, \cdots, M-2$, let $\bar{S}^m = \langle S^m_\xi : \xi < \omega_{M-m} \rangle$ be a sequence of almost disjoint subsets of ω_{M-m-1} which is definable L_{M-m-1} (without parameters). Successively using almost disjoint coding with respect to the sequences $\bar{S}^m$ (see [11]), we can code each W_γ into a set $W^0_\gamma \subseteq \omega_1$ such that the following holds:

If $\omega_1 < \beta \leq \omega_2$ and $\mathcal{M}$ is a suitable model with $\omega_2^{\mathcal{M}} = \beta$, $\{W^0_\gamma\} \cup \omega_1 \subseteq \mathcal{M}$, then $\mathcal{M} \models$ "Using the sequences $\{\bar{S}^m\}_{m=1}^{m=M-2}$, the set W^0_γ almost disjointly codes a set W such that for some $\gamma < \omega_M$, $\langle \gamma, < \rangle \cong \langle W, \in \rangle$".

Write $\mathbb{P}^{\mathcal{W}}$ for the forcing which adds $\mathcal{W} = \langle W^0_\gamma : \gamma \in \mathrm{Lim}(\omega_M) \rangle$. It is easy to see that this forcing preserved stationarity of each S_δ for $\delta < \omega_M$, preserves cofinalities, and does not add countable sequences (see again [11]).

Fix (until the last paragraph of this section) some $\delta < \omega_M$. Using bounded approximations adjoin a closed unbounded subset C_δ of ω_{M-1} such that $C_\delta \cap S_\delta = \emptyset$. The forcing $\mathbb{P}^{\mathrm{cl}}_\delta$ which adds C_δ preserves stationarity of S_η for each $\eta \in \omega_M \setminus \{\delta\}$, has size ω_{M-1}, preserves cardinals and cofinalities, and doesn't add any countable sequences.

Following the notation of [11], for a set of ordinals X, $\mathrm{Even}(X)$ denotes the subset of all even ordinals in X. Furthermore reproducing the ideas of [11], in $L[C_\delta]$ we can find subsets $Z_\delta \subseteq \omega_{M-1}$ such that

$(*)_\delta$: If $\beta < \omega_{M-1}$ and $\mathcal{M}$ is a suitable model such that $\omega_{M-2} \subseteq \mathcal{M}$, $(\omega_{M-1})^{\mathcal{M}} = \beta$, and $Z_\delta \cap \beta \in \mathcal{M}$, then $\mathcal{M} \models \theta(\omega_{M-1}, Z_\delta \cap \beta)$, where $\theta(\omega_{M-1}, X)$ is the formula "$\mathrm{Even}(X)$ codes a triple $(\bar{C}, \bar{W}, \bar{X})$ where $\bar{W}$, $\bar{X}$ are the L-least codes of ordinals γ, $\delta < \omega_M$ respectively such that γ is the largest limit ordinal not exceeding δ, and $\bar{C}$ is a club in ω_{M-1} disjoint from S_δ".

Using the same sequences $\bar{S}^m$ as when coding W_δ into W^0_δ, we code the sets Z_δ into subsets X_δ of ω_1 with the following property (again using the construction from [11]):

$(**)_\delta$: Suppose that $\omega_1 < \beta \leq \omega_2$, $\mathcal{M}$ is a suitable model with $\omega_2^{\mathcal{M}} = \beta$, and letting γ be the largest limit ordinal below δ, it holds that $\{W^0_\gamma, X_\delta\} \cup \omega_1 \subseteq \mathcal{M}$. Then $\mathcal{M} \models \varphi(W^0_\gamma, X_\delta)$, where $\varphi(W, X, m)$ is the formula: "Using the sequences $\{\bar{S}^m\}_{m=1}^{m=M-2}$, the set W almost disjointly codes $\bar{W}^0 \subseteq \omega_{M-1}$ and X almost disjointly codes a subset Z of ω_{M-1} whose even part codes the triple $(\bar{C}, \bar{W}, \bar{X})$ with $\bar{W} = \bar{W}^0$ and where $\bar{W}$, $\bar{X}$ are the L-least codes of ordinals γ, $\delta < \omega_M$ such that $\delta = \gamma + m$ and $\bar{C}$ is a club in ω_{M-1} disjoint from S_δ".

Note that φ is a statement about $(\omega_{M-1})^{\mathcal{M}}$ and $(\{\bar{S}^m\}_{m=1}^{m=M-2})^{\mathcal{M}}$, i.e., about the interpretation of their *definition* in $\mathcal{M}$ (indeed of course these objects are generally too large to be a parameter in φ).

The forcing $\mathbb{P}^{\mathrm{cd}}_\delta$ over $L[\mathcal{W}][C_\delta]$ described above which codes Z_δ into X_δ preserves stationarity of preserves stationarity of S_η for each $\eta \in \omega_M \setminus \{\delta\}$, has size ω_{M-1}, preserves cardinals and cofinalities, and doesn't add countable sequences.

Next, suppose $\delta = \gamma + m$ for $\gamma \in \mathrm{Lim}(\omega_M)$. We will force over $L[\mathcal{W}][X_\delta]$ (which is the same as $L[\mathcal{W}][C_\delta][X_\delta]$) to achieve localization of the pair of sets W^0_γ, X_δ (see [11, Definition 1]). Let φ be as above.

Definition 4.2. Let W, X be subsets of ω_1 such that $\varphi(W, X, m)$ holds in any suitable model $\mathcal{M}$ with $(\omega_1)^{\mathcal{M}} = (\omega_1)^L$ containing both W and X as elements. Denote by $\mathcal{L}(W, X, m)$ the poset of all functions $r : |r| \to 2$, where the domain $|r|$ of r is a countable limit ordinal such that

1. if $\xi < |r|$ then $\xi \in X$ iff $r(3 \cdot \xi) = 1$,
2. if $\xi < |r|$ then $\xi \in X'$ iff $r(3 \cdot \xi + 1) = 1$,
3. if $\xi \leq |r|$, $\mathcal{M}$ is a countable suitable model containing $r \restriction \xi$ as an element and $\xi = \omega_1^{\mathcal{M}}$, then
$$\mathcal{M} \vDash \varphi(W \cap \xi, X \cap \xi, m).$$

The extension relation is end-extension.

For each $\gamma \in \mathrm{Lim}(\omega_M)$ and $m \in \omega$ we use the poset $\mathcal{L}(W^0_\gamma, X_{\gamma+m}, m)$ to add the characteristic functions of a subset $Y_{\gamma+m}$ of ω_1 such that:

$(* * *)_{\gamma,m}$: If $\beta < \omega_1$, $\mathcal{M}$ is suitable with $\omega_1^{\mathcal{M}} = \beta$, $W^0_\gamma \cap \beta \in \mathcal{M}$, and $Y_{\gamma+m} \cap \beta \in \mathcal{M}$, then $\mathcal{M} \vDash \varphi(W^0_\gamma \cap \beta, X_\gamma \cap \beta, m)$.

With this the preliminary stage of the construction is complete. We let $\mathbb{P}^0$ denote the forcing
$$\mathbb{P}^{\mathcal{W}} * \prod_{\delta \in \omega_M} \mathbb{P}^{\mathrm{cl}}_\delta * \mathbb{P}^{\mathrm{cd}}_\delta * \mathcal{L}\big(W^0_{\gamma(\delta)}, X_\delta, m(\delta)\big).$$
where $\gamma(\delta)$ is the greatest limit ordinal below δ and $m(\delta)$ is the unique m such that $\delta = \gamma(\delta) + m$ and where the product is with the appropriate support as in [11]. Denote by V_0 the resulting model. Note that $V_0 \cap [\omega]^\omega = L \cap [\omega]^\omega$.

4.2. **Adding reals.** Fix (for the rest of the proof) a constructible almost disjoint family
$$\mathcal{F}^* := \{a_{i,j,\xi} : i \in \omega, j \in 2, \xi \in \omega_1 \cdot 2\}$$
which is Σ_1 (without parameters) in L_{ω_2} and such that $a_{i,j,\xi} \in L_\mu$ whenever $L_\mu \vDash |\xi| = \omega$. Next force with the finite support iteration
$$\mathbb{P}(\mathcal{C}) := \langle \mathbb{P}^W_\delta, \dot{\mathbb{Q}}^{\mathcal{C}}_\delta : \delta \in \mathrm{Lim}(\omega_M) \}$$
where for each δ, $\dot{\mathbb{Q}}^2_\delta$ adds the real c^W_δ which almost disjointly via the family $\mathcal{F}^*$ codes W^0_δ. Let V_2 be the $(\dot{\mathbb{P}}(\mathcal{C}), V_0)$-generic extension .

Using the standard forcing $\mathrm{Add}(\omega, \omega_N)$ (finite partial functions from $\omega_N \times \omega$ into 2) adjoin ω_N-many reals to V_2 to increase the size of the continuum to ω_N and denote the resulting model to obtain a model V_3.

4.3. **Adding the MCG.** We shall now define a finitely supported iteration $\mathbb{P}(\mathcal{G}) := \langle \mathbb{P}^{\mathcal{G}}_{\alpha}, \dot{\mathbb{Q}}^{\mathcal{G}}_{\alpha} : \alpha \in \omega_M \rangle$ which adds a self-coding MCG to the model V_3.

Along the iteration, for each $\alpha \in \omega_M$ we will define a $\mathbb{P}^{\mathcal{G}}_{\alpha}$-name $\dot{I}_\alpha \subseteq [\beta_\alpha, \beta_{\alpha+1})$ for a set of ordinals, such that at stage α of the construction we adjoin reals encoding a stationary kill of S_δ (that is, a real locally coding C_δ) for $\delta \in I_\alpha$. We then show that there is "no accidental coding of a stationary kill" in Lemma 5.1.

In order to define $\mathbb{P}(\mathcal{G}) := \langle \mathbb{P}^{\mathcal{G}}_{\alpha}, \dot{\mathbb{Q}}^{\mathcal{G}}_{\alpha} : \alpha \in \omega_M \rangle$, first fix primitive set recursive bijections

$$\psi : \omega \times \omega \to \omega$$

and $\psi' : \omega_1 \times \omega \times \omega \to \omega_1$. The function ψ' will be used to identify the family $\mathcal{F}_\alpha$ which we add at stage α with a subset of ω_1.

Suppose now by induction we are in the $(V_3, \mathbb{P}^{\mathcal{G}}_{\alpha})$-generic extension by $V_3[G^{\mathcal{G}}_{\alpha}]$. We presently define $\mathbb{Q}_\alpha = \mathbb{P}_{\mathcal{F}_\alpha} * \mathbb{P}^{\mathrm{cd}}_{\mathcal{F}_\alpha} * \mathbb{P}^{\mathcal{G}}_{\alpha}$.

For the definition of $\mathbb{P}_{\mathcal{F}_\alpha}$ assume by induction that at previous stages we have added families $\mathcal{F}_\beta$ for $\beta < \alpha$ consisting of cofinitary permutations. We now adjoin a family

$$\mathcal{F}^\alpha = \langle f^\alpha_{m,\xi} : m \in \omega, \xi \in \omega_1 \rangle$$

of permutations such that $|f^\alpha_\xi \cap f^\beta_{\xi'}| < \omega$ when $\beta < \alpha$ or $\xi \neq \xi'$. For this we can use a finite support iteration of the σ-centered posets with finite conditions defined in [15]. Denote this forcing by $\mathbb{P}_{\mathcal{F}_\alpha}$ and by $V_{\alpha,1}$ the resulting model.

Next let $\mathbb{P}^{\mathrm{cd}}_{\mathcal{F}_\alpha}$ be the forcing to add a real c_α which almost disjointly via the family $\mathcal{F}^*$ (see Section 4.2) codes

$$\psi' \left[\bigcup_{\xi < \omega_1} \{\omega \cdot \xi + m\} \times f^\alpha_{m,\xi} \right],$$

a subset of ω_1 which via ψ' codes $\mathcal{F}_\alpha$. Let $V_{\alpha,2}$ be the extension of $V_{\alpha,1}$ which contains $c^{\mathcal{F}}_\alpha$.

Finally, working in $V_{\alpha,2}$ we define $\mathbb{P}^{\mathcal{G}}_{\alpha}$, the forcing which adds a new group generator.

Suppose by induction that $\mathbb{P}^{\mathcal{G}}_{\alpha}$ has added a cofinitary representation ρ_α. Its image generates a cofinitary group $\mathcal{G}_\alpha$. Suppose by induction that $\mathrm{dom}(\rho_\alpha) = \{\beta_\xi\}_{\xi<\alpha}$ and write $\mathrm{CD}_\alpha = \{\beta_\gamma\}_{\gamma<\alpha}$, the set of generators used at a stage before α. Moreover suppose $\rho_\alpha(\beta_\xi) = g_\xi$ for each $\xi < \alpha$. Our next forcing will add the generic permutation g_α thus enlarging our group to $\mathcal{G}_{\alpha+1}$, the group generated by $\mathcal{G}_\alpha \cup \{g_\alpha\}$.

If α is a limit, let

$$\beta_\alpha = \sup\{\beta_\xi : \xi < \alpha\}$$

and otherwise, let

$$\beta_\alpha = \beta_{\alpha-1} + |\alpha \cdot \omega|$$

(we mean ordinal addition of course). This is the ordinal generator to which we associate the generic generator g_α so that

$$\rho_{\alpha+1} = \rho_\alpha \cup \{(\beta_\alpha, g_\alpha)\}$$

is a cofinitary representation.

Every element of the group freely generated by $\mathrm{CD}_\alpha \cup \{a\}$ corresponds to a reduced word in the alphabet $\mathrm{CD}_\alpha \cup \{a\}$, where $a = \beta_\alpha$. Let WD_α be the set of such words in *which a occurs*. Note that the set WD_α corresponds to the new permutations in the group $\mathcal{G}_{\alpha+1}$. More precisely, every permutation in $\mathcal{G}_{\alpha+1} \setminus \mathcal{G}_\alpha$ is of the form $w[g_\alpha]$ (which is the same as $\rho_{\alpha+1}(w)$ by definition) for some $w \in \mathrm{WD}_\alpha$.

As before write WS_α for the set of words in WD_α which do not have a proper conjugated subword. Let $i_\alpha : \mathrm{WS}_\alpha \to \mathrm{Lim}(|\alpha|)$ be a bijection sending a to 0; we shall use i_α to associate the ordinal $\beta_\alpha + i_\alpha(w)$ to each $w \in \mathrm{WS}_\alpha$. We note that those elements of $\mathcal{G}_{\alpha+1} \setminus \mathcal{G}_\alpha$ which correspond via $\rho^{-1}_{\alpha+1}$ to words in WS_α will be associated to ordinals in $[\beta_\alpha, \beta_{\alpha+1})$, and in fact g_α is associated to β_α (elements of $\mathcal{G}_{\alpha+1}$ which are not of the form $\rho_\alpha(w)$ for $w \in \mathrm{WD} \setminus \mathrm{WS}$ we can ignore for now) .

For each $w \in \mathrm{WS}_\alpha$ it is the pattern of stationarity on the block of $\bar{S}$ consisting of the next ω ordinals after $\beta_\alpha + i_\alpha(w)$ that will code w. Let for such $w \in \mathrm{WS}_\alpha$

$$z^w = \{2^m : m \in c^{\mathcal{F}}_\alpha\} \cup \{3^m : m \in c^W_{\beta_\alpha + i_\alpha(w)}\}$$

and define

$$\bar{z} = \langle z^w : w \in \mathrm{WS}_\alpha \rangle.$$

Further, define

$$Y^w_m = Y_{\beta_\alpha + i_\alpha(w) + m}$$

for each $w \in \mathrm{WS}_\alpha$ and let

$$\mathcal{Y} = \langle Y^w_m : w \in \mathrm{WS}_\alpha, m \in \omega \rangle.$$

With the notation from Section 3 we now define

$$\mathbb{Q}^{\mathcal{G}}_\alpha = \mathbb{Q}^{\mathcal{F}_\alpha, \mathcal{Y}, \bar{z}}_{\rho_\alpha, \{\beta_\alpha\}}.$$

In Proposition 3.2 we have seen that $\mathbb{Q}^{\mathcal{G}}_\alpha$ adjoins a new generator g_α such that the following properties hold:

(A_α) The group $\langle \mathrm{im}(\rho_\alpha) \cup \{g_\alpha\} \rangle$ is cofinitary.

(B_α) If $f \in V^{\mathbb{P}_\alpha} \setminus \mathcal{G}_\alpha$ is a permutation which is not covered by finitely many members of $\mathcal{F}_\alpha$ and $\langle \mathcal{G}_\alpha \cup \{f\} \rangle$ is cofinitary, then for infinitely many k, $f(k) = g_\alpha(k)$. This property, will eventually provide maximality of $\mathcal{G}_{\omega_M}$.

(C_α) for each $w \in \mathrm{WS}_\alpha$ there is $m_w \in \omega$ such that for all $k \in \omega$, $w^{2k}[g_\alpha](m_w) = \chi_{z^w}(k) \mod 2$. That is, every new permutation $w[g_\alpha]$ encodes $\mathcal{F}_\alpha$ via the real $c^{\mathcal{F}}_\alpha$ as well as $W^0_{\beta_\alpha + i_\alpha(w)}$ via the real $c^W_{\beta_\alpha + i_\alpha(w)}$.

(D_α) for each $w \in WD_\alpha$, for all $m \in \psi\big[w[g_\alpha]\big]$, for all $\xi \in \omega_1$

$$|w[g_\alpha] \cap f^\alpha_{m,\xi}| < \omega \text{ iff } \xi \in Y^w_m.$$

That is, $w[g_\alpha]$ encodes Y^w_m for each $m \in \psi^{-1}(w[g_\alpha])$.

As we are going to see in the next section, property (D_α) implies that the new permutation $w[g_\alpha]$ encodes itself via a stationary kill on the segment $\langle S_\delta : \beta_\alpha + i_\alpha(w) \leq \delta < \beta_\alpha + i_\alpha(w) + \omega\rangle$. Furthermore, this stationary kill is accessible to countable suitable models containing $w[g_\alpha]$.

Let $\dot{I}_\alpha$ be a $\bar{\mathbb{P}}^{\mathcal{G}}_{\alpha+1}$-name for

$$I_\alpha = \big\{\beta_\alpha + i_\alpha(w) + m : w \in \mathrm{WS}_\alpha, m \in \psi\big[w[g_\alpha]\big]\big\}.$$

Thus I_α denotes the set of indices of the stationary sets for which we explicitly adjoin reals encoding a stationary kill at stage α of the iteration. Note that $\beta_\alpha = \sup I_\alpha$. With this the inductive construction is complete.

5. Definability and maximality of the group

Forcing with $\mathbb{P}(\mathcal{G})$ over V_3 we obtain a generic G over L for the entire forcing

$$\mathbb{P} := \mathbb{P}^*_0 * \mathbb{P}(\mathcal{C}) * \mathrm{Add}(\omega, \omega_N) * \mathbb{P}(\mathcal{G})$$

recalling that $\mathbb{P}^*_0$ was the product which added the sets W^0_α and $Y_{\alpha+m}$, and $\mathbb{P}(\mathcal{C})$ added a real c^W_α "locally coding" the ordinal α for each $\alpha \in \mathrm{Lim}(\omega_M)$; $\mathrm{Add}(\omega, \omega_N)$ made $2^\omega = \omega_N$; and finally $\mathbb{P}(\mathcal{G})$ added a generic self-coding subgroup of S_∞. Also recall that all the forcings after $\mathbb{P}^*_0$ are Knaster, and $\mathbb{P}^*_0$ did not add any countable sequences.

Work in $L[G]$ from now on. We shall now show that in this model there is a MCG of size $\aleph_N$. First we must show that no real codes an "accidental" stationary kill.

Lemma 5.1. *For each δ which is not in*

$$I = \bigcup\{I_\gamma : \gamma \in \mathrm{Lim}(\omega_M)\}$$

there is no real in $L[G]$ coding a stationary kill of S_δ, i.e., there is no $r \in \mathcal{P}(\omega) \cap L[G]$ such that $L[r] \vDash S_\delta \in \mathsf{NS}$.

Proof. The argument closely follows [11, Lemma 3]; for the readers convenience we give a brief sketch. Let $\dot{I}$ be a name for I and suppose that for all $\gamma \in \mathrm{Lim}(\omega_M)$ we have $p \Vdash \check{\delta} \notin \dot{I}$. In the $(L, \mathbb{P}^{\mathcal{W}})$-generic extension, write

$$\mathbb{P}^{\neq\delta}_0 = \prod_{\xi \in \omega_M \setminus \{\delta\}} \mathbb{P}^{\mathrm{cl}}_\xi * \mathbb{P}^{\mathrm{cd}}_\xi * \mathcal{L}(W^0_{\sup \xi \cap \mathrm{lim}}, X_\xi)$$

and

$$\mathbb{P}^\delta_0 = \mathbb{P}^{\mathrm{cl}}_\delta * \mathbb{P}^{\mathrm{cd}}_\delta * \mathcal{L}(W^0_\gamma, X_\gamma, m).$$

where γ is the greatest limit ordinal below δ and m is the unique m such that $\delta = \gamma + m$.

Use that $\mathbb{P}_0^* = \mathbb{P}^{\mathcal{W}} * (\mathbb{P}_0^{\neq\delta} \times \mathbb{P}_0^\delta)$ to decompose the $\mathbb{P}_0$-generic G_0 as follows:

$$G_0 = G^{\mathcal{W}} \; * (G_0^{\neq\delta} \times G_0^\delta).$$

Working in $L[G_0] = L[G^{\mathcal{W}}][G_0^{\neq\delta}][G_0^\delta]$ let

$$\mathbb{P}' = \big(\operatorname{Add}(\omega, \omega_N) * \mathbb{P}(\mathcal{G})\big) \restriction p$$

be the quotient $\mathbb{P}/\mathbb{P}_0^*$ below p, it is easy to verify that $\mathbb{P}' \in L[G^{\mathcal{W}}][G_0^{\neq\delta}]$ since the iteration never uses Y_δ. Thus letting G' be shorthand for the $\mathbb{P}'$ generic, we may decompose $G = (G^{\mathcal{W}} * G^{\neq\delta_0} * G') \times G_0^\delta$.

Let r be any real in $L[G] = L[G^{\mathcal{W}}][G_0^{\neq\delta}][G'][G_0^\delta]$ and write

$$V_* = L[G^{\mathcal{W}}][G_0^{\neq\delta}]$$

Then in fact $r \in V_*[G'] = L[G^{\mathcal{W}}][G_0^{\neq\delta}][G']$ since $\mathbb{P}_0^\delta$ adds no countable sequences over V_* and since $\mathbb{P}'$ is Knaster and so $\mathbb{P}_0^\delta$ also adds no countable sequences over $V_*[G']$. But since $\mathbb{P}^{\mathcal{W}} * \mathbb{P}^{\neq\delta} * \mathbb{P}'$ preserves stationarity of S_δ, the latter is still stationary in $V_*[G'] = L[G^{\mathcal{W}}][G_0^{\neq\delta}][G']$ and hence in $L[r]$. □

Let $\mathcal{G}$ be the group generated by $\{g_\alpha : \alpha \in \omega_M\} = \bigcup_{\alpha<\omega_M} \operatorname{im}(\rho_\alpha)$. Given $w \in \mathrm{WD}_\alpha$, we write w^G for $\rho_\alpha(w)$, i.e., for the interpretation of w that replaces every generator index β_γ by the corresponding generic permutation g_γ.

Lemma 5.2. *The group $\mathcal{G}$ is a maximal cofinitary group.*

Proof. By property (A_α) of the iterands $\dot{\mathbb{Q}}_\alpha$ the group $\mathcal{G}$ is cofinitary. It remains to show maximality. Suppose by contradiction that $\mathcal{G}$ is not maximal. Then there is a cofinitary permutation $h \notin \mathcal{G}$ such that the group generated by $\mathcal{G} \cup \{h\}$ is cofinitary. Find β such that $h \in V_3[G_\beta]$. Then there is $\beta' \in \{\beta, \beta+1\}$ such that h is not a subset of the union of finitely many members of $\mathcal{F}_{\beta'}$: For otherwise by the pigeonhole principle we find $f \in \mathcal{F}_\beta$ and $f' \in \mathcal{F}_{\beta+1}$ such that $|f \cap f'| = \omega$, contradicting the choice of $\mathcal{F}_\beta$ and $\mathcal{F}_{\beta+1}$. Letting $\alpha = \beta'+1$, by property (B_α) of the poset $\mathbb{Q}_\alpha$ in $V_3[G_\alpha]$, the generic permutation g_α infinitely often takes the same value as h, and so $g_\alpha \circ h^{-1}$ is not cofinitary, which is a contradiction. □

It remains to show that $\mathcal{G}$ is Π^1_2.

Lemma 5.3. *Let $g \in S^\infty \cap L[G]$. Then $g = w^G$ for some $w \in \bigcup_{\alpha<\omega_M} \mathrm{WS}_\alpha$ if and only if there is $\gamma \in \operatorname{Lim}(\omega_N)$ and $k \in \omega$ such that*

$$\psi[g] = \{m \in \omega : L[g] \vDash S_{\gamma+m} \in \mathsf{NS}\} = \{m \in \omega : (\exists r \in \mathcal{P}(\omega))\; L[r] \vDash S_{\gamma+m} \in \mathsf{NS}\} \tag{5.1}$$

Proof. Suppose $g = w^G$ for $w \in \mathrm{WS}_\alpha$ and w has no proper conjugated subword. We prove the lemma for $\gamma = \beta_\alpha + i_\alpha(w)$. By property (C_α) of the poset $\mathbb{Q}_\alpha$ the real g codes z^w and therefore

$$\mathcal{F}_{\beta_\alpha + i_\alpha(w)} \in L[g].$$

By property (D_α) of the poset $\mathbb{Q}_\alpha$ the real g codes almost disjointly via the family $\mathcal{F}_\alpha$ codes $Y_{\beta_\alpha+i_\alpha(w)+m}$ for each $m \in \psi[g]$. However $Y_{\beta_\alpha+i_\alpha(w)+m}$ codes $X_{\beta_\alpha+i_\alpha(w)+m}$ which implies that for every $m \in \psi[g]$, the real g codes the closed unbounded subset $C_{\beta_\alpha+i_\alpha(w)+m}$, which is disjoint from $S_{\beta_\alpha+i_\alpha(w)+m}$.

If $m \notin \psi[g]$, then $\beta_\alpha + i_\alpha(w) + m \notin I_\alpha$ and so by Lemma 5.1, there is no real r in $L[G]$ coding the stationary kill of $S_{\beta_\alpha+i_\alpha(w)+m}$ (i.e., such that in $L[r]$, $S_{\beta_\alpha+i_\alpha(w)+m}$ is no longer stationary).

Now, suppose there is $\gamma \in \mathrm{Lim}(\omega_M)$ and $k \in \omega$ such that the following holds for all $n \in \omega$: $L[g] \models S_{\gamma+m} \in \mathsf{NS}$ if and only if $m \in \psi[g]$. Then by Lemma 5.1, $\psi[g] = \{n \in \omega : \gamma + n \notin I_\alpha\} = \psi[w^G]$ where w is such that $\beta_\alpha + i_\alpha(w) = \gamma$ for some $\alpha < \omega_M$. So $g = w[g_\alpha] = w^G$. □

Lemma 5.4. *Let $g = w^G$ for some $w \in \mathrm{WS}_\alpha$ with $\alpha < \omega_M$. Then for every countable suitable model $\mathcal{M}$ such that $g \in \mathcal{M}$ there is a limit ordinal $\gamma < (\omega_M)^{\mathcal{M}}$ such that*

$$(L[w^G])^{\mathcal{M}} \models \psi[g] = \{m \in \omega : L[w^G] \models S_{\gamma+m} \in \mathsf{NS}\}.$$

Proof. Let $\mathcal{M}$ be a countable suitable model and let $g \in \mathcal{M}$. Let $\gamma = i_\alpha(w)$. Since w^G encodes z^w (by property (C_α) of $\mathbb{Q}_\alpha$) we have that

$$\{f^\alpha_{m,\xi} : m \in \omega, \xi < (\omega_1)^{\mathcal{M}}\} \in \mathcal{M}$$

and $W^0_\gamma \cap (\omega_1)^{\mathcal{M}} \in \mathcal{M}$. By property (D_α), w^G almost disjointly codes $Y_{\gamma+m} \cap (\omega_1)^{\mathcal{M}}$ for each $m \in \psi[g_\alpha]$ and hence $Y_{\gamma+m} \cap (\omega_1)^{\mathcal{M}} \in \mathcal{M}$ and also $X_{\gamma+m} \cap (\omega_1)^{\mathcal{M}} \in \mathcal{M}$. These sets belong also to $L[g]^{\mathcal{M}}$. Then for each $m \in \psi[g]$, by $(***)_{\gamma,m}$ we have that $L[g]^{\mathcal{M}} \models \varphi(W^0_\gamma \cap \beta, X_{\gamma+m} \cap \beta)$ where $\beta = (\omega_1)^{\mathcal{M}}$. This means:

Using the sequences $\{\bar{S}^k\}_{k=1}^{k=M-2}$, the set $W^0_\gamma \cap \beta$ almost disjointly codes $\bar{W}^0 \subseteq \omega_{N-1}$ and $X_{\gamma+m} \cap \beta$ almost disjointly codes a subset Z of ω_{M-1} whose even part codes the triple $(\bar{C}, \bar{W}, \bar{X})$ with $\bar{W} = \bar{W}^0$ and where $\bar{W}$, $\bar{X}$ are the L-least codes of ordinals $\bar{\gamma}$, $\bar{\delta} < \omega_M$ such that $\bar{\delta} = \bar{\gamma} + m$ and $\bar{C}$ is a club in ω_{M-1} disjoint from $S_{\bar{\gamma}}$.

In particular, in the above $\bar{\gamma} = \gamma$, $\bar{\delta} = \gamma + m$ and $\bar{C}$ is a club disjoint from $S_{\gamma+m}$. As $m \in \psi^{-1}[g]$ was arbitrary, γ indeed witnesses that the lemma holds. □

Lemma 5.5. *Let g be a real such that for every countable suitable model $\mathcal{M}$ containing g as an element there is $\gamma < (\omega_M)^{\mathcal{M}}$ such that*

$$(L[g])^{\mathcal{M}} \models \psi[g] = \{m \in \omega : L[g] \models S_{\gamma+m} \in \mathsf{NS}\}.$$

Then for some $\alpha < \omega_M$, $g = w^G$ where $w \in \mathrm{WS}_\alpha$.

Proof. By Löwenheim-Skolem take a countable elementary submodel $\mathcal{M}_0$ of $L_{\omega_{n+1}}$ such that $g \in \mathcal{M}_0$ and let $\mathcal{M}$ be the unique transitive model isomorphic to $\mathcal{M}_0$. Then by assumption

$$(L[g])^{\mathcal{M}} \models \big(\exists \gamma \in \mathrm{Lim}(\omega_M)\big)\ \psi[g] = \{m \in \omega : S_{\gamma+m} \text{ is non-stationary}\}$$

so by elementarity the same holds with $(L[g])^{\mathcal{M}}$ replaced by $L_{\omega_{M+1}}[g]$, and hence for some $\gamma \in \mathrm{Lim}(\omega_M)$

$$L[g] \models \psi[g] = \{m \in \omega : L[g] \models S_{\gamma+m} \in \mathsf{NS}\}.$$

But at some stage $\alpha < \omega_M$ we adjoined a generic permutation w^G such that $\beta_\alpha + i_\alpha(w) = \gamma$ and by (5.1) we have

$$\psi[w^G] = \{m \in \omega : (\exists r \in \mathcal{P}(\omega)\, L[r] \models S_{\gamma+m} \in \mathsf{NS}\}.$$

Since there is no accidental coding of a stationary kill (Lemma 5.1) $\psi[g] \subseteq \psi[w^G]$, and so $g = w^G$. □

Lemma 5.6. *The MCG $\mathcal{G}$ is Π^1_2 in $L[G]$.*

Proof. Recall that we denote by g_0 the first generator added by $\mathbb{P}^{\mathcal{G}}_1 = \mathbb{Q}^{\mathcal{G}}_0$ over V_3. Note first that $g \in \mathcal{G}$ if and only if there is $k \in \omega$, $\alpha < \omega_M$, and $w \in \mathrm{WS}_\alpha$ (i.e., w has no proper conjugated subwords) such that $(g_0)^k g = w^G$.

By the previous lemmas, $g \in \mathcal{G}$ if and only if $g \in S_\infty$ and the following statement $\Phi(g)$ holds: For every suitable countable model $\mathcal{M}$ *if* for some $g_* \in \mathcal{M} \cap S_\infty$

$$L[g_*]^{\mathcal{M}} \models \psi[g_*] = \{m \in \omega : S_m \text{ is stationary}\}$$

then for some $k \in \omega$

$$L[(g_*)^k g]^{\mathcal{M}} \models \big(\exists \gamma \in \mathrm{Lim}(\omega_M)\big)\ \psi\big[(g_*)^k g\big] = \big\{m \in \omega : S_{\gamma+m} \text{ is stationary}\big\}.$$

It is standard to see $\Phi(g)$ can be expressed by a Π^1_2 formula. □

Thus we obtain our main result:

Theorem 5.7. *Let $2 \le M < N < \aleph_0$ be given. There is a cardinal preserving generic extension of the constructible universe L in which*

$$\mathfrak{a}_g = \mathfrak{b} = \mathfrak{d} = \aleph_M < \mathfrak{c} = \aleph_N$$

and in which there is a Π^1_2 definable maximal cofinitary group of size $\mathfrak{a}_g$.

Proof. The construction outlined in steps (1) – (4) and developed in detail in Sections 4 and 5, provide a generic extension in which there is a Π^1_2-definable maximal cofinitary group of cardinality $\aleph_M$, while $\mathfrak{c} = \aleph_N$. To guarantee that in the same model there are no maximal cofinitary groups of cardinality strictly smaller than $\aleph_M$, we slightly modify the definition of $\mathbb{Q}_\alpha$ from step (4) to $\mathbb{P}_{\mathcal{F}_\alpha} * \mathbb{P}^{\mathrm{cd}}_{\mathcal{F}_\alpha} * \mathbb{P}^{\mathcal{G}}_\alpha * \dot{\mathbb{D}}$, where $\mathbb{D}$ is Hechler's forcing for adding a dominating real. Thus in the final model, there is a scale of length ω_M and so $\mathfrak{b} = \mathfrak{d} = \aleph_M$. Since $\mathfrak{b} \le \mathfrak{a}_g$ we obtain $\mathfrak{a}_g = \aleph_M$. □

6. Questions

In this section, we state some of the remaining open questions.

(1) Can one construct in *ZFC* a countable cofinitary group which can not be enlarged to a Borel MCG? Note that in L, every countable group can be enlarged to a $\boldsymbol{\Pi}^1_1$ MCG.

(2) Can we add a countable cofinitary group which cannot be enlarged to a $\mathbf{\Pi}^1_1$ MCG using forcing?
(3) Is there a model where $2^\omega > \omega_1$ and every cofinitary group $\mathcal{G}_0$ of size $< 2^\omega$ is a subgroup of a definable MCG *of the same size* as $\mathcal{G}_0$?
(4) Suppose that $\alpha < 2^\omega$ is a cardinal and there is a Σ^1_2 MCG of size α. Is there a Π^1_1 MCG of size α?
(5) Is there a model where there is a projective MCG of size α with $\omega_1 < \alpha < 2^\omega$ but there is no MED family of size α?

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