Rank-into-rank hypotheses and the failure of GCH

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Abstract

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1 Introduction

The behaviour of the power function has been under scrutiny since the birth of set theory. Already in 1878 Cantor proposed the Continuum Hypothesis [1] and years later Hausdorff [8] extended it globally stating for the first time the Generalized Continuum Hypothesis. Both the local and global approaches will be developed: as is well known, Gödel defining the constructible universe provided a model for the Generalized Continuum Hypothesis, while its negation, a tougher problem, was first proved locally by Cohen and then globally by Easton, [3], who managed with a class iteration of Cohen forcing to prove the consistency of the failure of GCH at regulars, leaving open the problem at singulars. This is called *Singular Cardinal Problem*, and is in fact much more difficult.

At the same time, another line of research was thriving. Just before the advent of forcing, Dana Scott [12] connected the power function with large cardinals, proving that if a measurable cardinal violates GCH, then GCH is violated by a set of measure 1 of cardinals below it. After that, Silver

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[13] proved that if the GCH holds below a singular cardinal of uncountable confinality it must hold at that, and Solovay [14] that all the strong limit singular cardinals above a strongly compact cardinal satisfy GCH. Therefore large cardinals have a big impact on the power function, and the investigation on the possible behaviours of the power functions under large cardinal hypotheses is now a fundamental subject in the Singular Cardinal Problem analysis.

This paper is a contribution of such subject in a novel case. After Kunen's proof that a Reinhardt cardinal is inconsistent [9], some very large cardinal hypotheses appeared, at the borders of inconsistency, sometimes called rank-into-rank hypotheses, e.g. I0-I3. The fundamental difference between this case and the precedent ones is that for these axioms the relevant cardinal is *singular*, and not regular. Therefore, we are not looking at the consequences of the existence of a large cardinal of the power functions on singular ones, but at the power function of the large cardinal itself.

In Section 2 all the preliminary facts are collected. In section 3 we present a proof of the consistency of I0 and the failure of GCH at regular cardinals. In Section 4 we use deep facts from the study by Woodin of I0 to prove that, under I0, we can find a model of ZFC where I1 (or larger hypotheses) holds and also the power function at the relevant singular cardinal violates GCH in many possible ways.

2 Preliminaries

To avoid confusion or misunderstandings, all notation and standard basic results are collected here.

The double arrow (e.g. $f : a \twoheadrightarrow b$) denotes a surjection.

If M and N are sets or classes, $j : M \prec N$ denotes that j is an elementary embedding from M to N. We write the case in which the elementary embedding is the identity, i.e., M is an elementary submodel of N, simply as $M \prec N$, while when j is indicated we always suppose that it is not the identity.

If $j : M \prec N$ and either $M \vDash \mathsf{AC}$ or $N \subseteq M$ then it moves at least one ordinal. The *critical point*, $\operatorname{crt}(j)$, is the least ordinal moved by j.

Let j be an elementary embedding and $\kappa = \operatorname{crt}(j)$. Define $\kappa_0 = \kappa$ and $\kappa_{n+1} = j(\kappa_n)$. Then $\langle \kappa_n : n \in \omega \rangle$ is the *critical sequence* of j.

Kunen [9] proved under AC that if $M = N = V_\eta$ for some ordinal η, and λ is the supremum of the critical sequence, then η cannot be bigger than $\lambda + 1$ (and of course cannot be smaller than λ).

Kunen's result leaves room for a new breed of large cardinal hypotheses, sometimes referred to in the literature as rank-into-rank hypotheses:

I3 iff there exists λ s.t. $\exists j : V_\lambda \prec V_\lambda$;

I2 iff there exists λ s.t. $\exists j : V \prec M$, with $V_\lambda \subseteq M$ and λ is the supremum of the critical sequence;

I1 iff there exists λ s.t. $\exists j : V_{\lambda+1} \prec V_{\lambda+1}$.

The consistency order of the above hypotheses is reversed with respect to their numbering: I1 is strictly stronger than I2, which in turn is strictly stronger than I3. All of these hypotheses are strictly stronger than all of the other large cardinal hypotheses ever considered.

Note that if j witnesses a rank-into-rank hypothesis, then λ is uniquely determined by j, so in the following λ always denotes the first nontrivial fixed point of the embedding j under consideration.

An interesting point is that every elementary embedding $j : V_\lambda \prec V_\lambda$ has a unique extension to $V_{\lambda+1}$: let $\langle \kappa_i : i \in \omega \rangle$ be its critical sequence; then for any $X \subseteq V_\lambda$ define $j(X) = \bigcup_{n\in\omega} j(X \cap V_{\kappa_n})$. Then j is a Σ_0-elementary embedding from $V_{\lambda+1}$ to itself. A consequence of this is the possibility to define finite iterates of j, i.e., $j^2 = j(j)$, since $j \in V_{\lambda+1}$, and j^2 is in fact an elementary embedding from V_λ to itself. We define also $j^{\circ n}$ as the composition of n copies of j, i.e., $j^{\circ 0} = j$ and $j^{\circ n+1} = j \circ j^{\circ n}$. Note that these are different from the iterates: for example, $j^{\circ n}$ and j have the same critical point, but $\mathrm{crt}(j^2) = j(\mathrm{crt}(j))$.

If X is a set, then $L(X)$ denotes the smallest inner model that contains X; it is defined like L but starting with the transitive closure of $\{X\}$ as $L_0(X)$.

In 1984 Woodin proposed an axiom even stronger than all the previous ones:

I0 For some λ there exists a $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$, with $\mathrm{crt}(j) < \lambda$.

Note that if λ witnesses I0, then $L(V_{\lambda+1}) \nvDash \mathsf{AC}$, because otherwise by Kunen's result such an elementary embedding could not exist.

The elements of the critical sequence of elementary embeddings that witness rank-into-rank hypotheses are really large cardinals.

Definition 2.1. *Let κ be an uncountable cardinal. We say that κ is:*

- strong limit *if for any $\eta < \kappa$, $2^\eta < \kappa$;*
- measurable *if there is a κ-complete ultrafilter over κ;*

- strongly compact *if for any $\eta \geq \kappa$ there is a fine measure on $\mathcal{P}_\kappa(\eta)$, i.e., a measure U such that for any $\xi < \eta$, $\{A : \xi \in A\} \in U$;*
- n-huge *if there is a sequence $\kappa = \lambda_0 < \lambda_1 < \cdots < \lambda_n = \lambda$ and a κ-complete normal ultrafilter U over $\mathcal{P}(\lambda)$ such that for any $i < n$,*

$$\{x \in \mathcal{P}(\lambda) : \mathrm{ot}(x \cap \lambda_{i+1}) = \lambda_i\} \in U.$$

Let $j : V_\lambda \prec V_\lambda$ and let $\langle \kappa_i : i \in \omega \rangle$ be its critical sequence. Note that for any n, $\kappa_n = \mathrm{crt}(j^{n+1})$, so any property of κ_0 is shared by all of the κ_n's. We have

- "κ_0 is measurable" is witnessed by $U = \{X \subseteq \kappa_0 : \kappa_0 \in j(X)\}$;
- "κ_0 is n-huge" is witnessed by $\kappa_0 < \kappa_1 < \cdots < \kappa_n$ and

$$U = \{X \subseteq \mathcal{P}(\kappa_n) : j''\kappa_n \in j(X)\};$$

- "$V_\lambda \vDash \kappa_0$ is strongly compact": let $\kappa_0 \leq \eta < \lambda$, and suppose $\eta < \kappa_n$; then this is witnessed by

$$U = \{X \subseteq \mathcal{P}_{\kappa_0}(\eta) : j^{\circ n}``\kappa_0 \in j^{\circ n} X\}.$$

The situation is radically different for λ, since it is a singular cardinal, so it cannot have large cardinal properties. However, since it is a limit of strong limit cardinals, λ is a strong limit cardinal. Moreover, we trivally have that $V_{\kappa_0} \prec V_{\kappa_1}$ and for any $i \in \omega$, $V_{\kappa_i} \prec V_{\kappa_{i+1}}$. But then the V_{κ_i}'s form a direct system with limit V_λ and $V_{\kappa_i} \prec V_\lambda$ for any $i \in \omega$. In particular, V_λ is a model of ZFC.

For proving results about rank-into-rank elementary embeddings and the continuum function two forcings will be used: Easton forcing, that will be explained in the following section, and Prikry forcing. For both forcings we use the notation $\check{a}$ to indicate the canonical name of an element in the ground model, and $\dot{G}$ to indicate the canonical name for the generic.

Prikry forcing (a detailed discussion about of it can be found in [6]) is defined as follows: fix an ultrafilter U over κ measurable; $p \in \mathbb{P}$ iff $p = (s, A)$, where $s \in [\kappa]^{<\omega}$, $A \in U$. For $p = (s, A)$, $q = (t, B) \in \mathbb{P}$, we say $q \leq p$ iff $s \subseteq t$, $B \subseteq A$ and $t \setminus s \subseteq A$.

Theorem 2.2. *The Prikry forcing on κ is κ^+-c.c. and doesn't add bounded subsets of κ.*

3 GCH and its negation at regulars

There are various results already published about the interaction of rank-into-rank hypotheses with different behaviours of the continuum function. The following definition captures the concept of "right behaviour" of the continuum function at regulars:

Definition 3.1. *Let* $E : \mathrm{Reg} \to \mathrm{Card}$ *a class function. Then* E *is an* Easton function *iff*

- $\alpha < \beta \to E(\alpha) \leq E(\beta)$*;*
- $\mathrm{cof}(E(\alpha)) > \alpha$ *for all* $\alpha \in \mathrm{Reg}$*.*

Then the current knowledge can be summarized:

Theorem 3.2 ([2], [4], [7])**.** *Let* I^* *be I3, I2, I1 or I0, and let* E *be a definable Easton function. Then*

$$\mathrm{Con}(\mathsf{ZFC} + I^*) \to \mathrm{Con}(\mathsf{ZFC} + \mathsf{GCH} + I^*)$$

and given a model of I^**, there is an inner model of* ZFC *and cofinality preserving forcing extension in which* $I^* + 2^\kappa = E(\kappa)$ *holds for all regular* κ*.*

The proof for I2 is due to Friedman [4], the proof for I1 is due to Hamkins [7] and the proof for I3 is due to Corazza [2]. I0 has never been considered before: although the framework of the proof for I0 is the same as in the cases of I1, I2, I3, there is a new feature, Corollary 3.8, and for this reason we give the proof.

Definition 3.3. *Let* $\mathbb{P}_\lambda$ *be a forcing iteration of length* λ*, where* λ *is either a strong limit or is equal to* ∞*, the class of all ordinals, and we indicate with* $\mathbb{Q}_\delta$ *its* δ*-th stage. Then* $\mathbb{P}_\lambda$ *is*

- reverse Easton *if nontrivial forcing is done only at infinite cardinal stages, direct limits are taken at all inaccessible cardinal limit stages, and inverse limits at all other limit stages; moreover,* $\mathbb{P}_\lambda$ *is the direct limit of the* $\langle \mathbb{P}_\delta, \delta < \lambda \rangle$ *if* λ *is regular or* ∞*, the inverse limit of the* $\langle \mathbb{P}_\delta, \delta < \lambda \rangle$*, otherwise;*
- directed closed *if for all* $\delta < \gamma$*,* $\mathbb{Q}_\delta$ *is* $< \delta$*-directed closed;*
- λ-bounded *if for all* $\delta < \lambda$*,* $\mathbb{Q}_\delta$ *has size* $< \lambda$*. Note that in the case* $\lambda = \infty$*, this just means that each* $\mathbb{Q}_\delta$ *is a set-forcing;*

- above ω *if $\mathbb{Q}_\omega$ is the trivial forcing.*

Moreover, if j is any elementary embedding and $\mathbb{P}_\gamma \subset \mathrm{dom}(j)$, we say that $\mathbb{P}_\gamma$ is j-coherent if for any $\delta < \gamma$, $j(\mathbb{P}_\delta) = \mathbb{P}_{j(\delta)}$.

Theorem 3.4 (Easton, [3])**.** *Let E be a definable Easton function. Then there exist definable, directed closed, reverse Easton iterations $\mathbb{P}$ of length the ordinals such that, if G is generic for $\mathbb{P}$, $V[G] \vDash \mathsf{GCH}$, or $V[G] \vDash \forall\kappa(\kappa \text{ regular } \to 2^\kappa = E(\kappa))$.*

We will prove the following:

Theorem 3.5. *Let $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$ and $\mathbb{P}_\lambda$ a j-coherent, directed closed and λ-bounded reverse Easton iteration. Then for $\mathbb{P}_\lambda$-generic G, j lifts to $j^* : L(V_{\lambda+1})[G] \prec L(V_{\lambda+1})[G]$ and the restriction of such a lifting to $L(V[G]_{\lambda+1})$ witnesses I0 in $V[G]$.*

Lemma 3.6. *Let $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$ and $\mathbb{P}_\lambda$ ve a j-coherent, directed closed, λ-bounded reverse Easton iteration. Then there exists $q \in \mathbb{P}_\lambda$ such that $q \Vdash p \in \dot{G} \to j(p) \in \dot{G}$.*

Proof. We define q piece by piece. Let $q \upharpoonright \kappa_1$ be the trivial condition. Now, for any $n \in \omega$;

$$\Vdash |\{j(p)(\kappa_n) : p \in \dot{G}\}| \leq 2^{\kappa_{n-1}} < \kappa_n,$$

so, since $\mathbb{Q}_{\kappa_n}$ is $< \kappa_n$-directed closed in $V^{\mathbb{P}_{\kappa_n}}$, there exists a name τ such that $\Vdash \forall p \in \dot{G}\ \tau \leq j(p)(\kappa_n)$, and define $q(\kappa_n) = \tau$. As for the definition of q between elements of the critical sequence, we have that $\Vdash |j``\dot{G} \upharpoonright (\kappa_n, \kappa_{n+1})| \leq \kappa_n$, and so again there exists a name τ such that

$$\Vdash \forall p \in \dot{G}\ \forall \beta \in (\kappa_n, \kappa_{n+1})\ \tau \leq j(p)(\beta)$$

and define $q \upharpoonright (\kappa_n, \kappa_{n+1}) = \tau$. Now, suppose that $p, q \in G$. Clearly $q \upharpoonright [\kappa_1, \lambda) \leq j(p) \upharpoonright [\kappa_1, \lambda)$, so $j(p) \upharpoonright [\kappa_1, \lambda) \in G \upharpoonright [\kappa_1, \lambda)$. But $j(p) \upharpoonright \kappa_0 = p$, and $j(p) \upharpoonright [\kappa_0, \kappa_1)$ is trivial, so $j(p) \in G$.

□

To use the typical lifting lemma, we have to prove that in fact the model $L(V_{\lambda+1})$ as constructed in the forcing extension is in the domain of the lifting.

Lemma 3.7 ([7])**.** *If $\mathbb{P}_\lambda$ is a simple, directed closed forcing iteration, then $V[G]_{\lambda+1} = V_{\lambda+1}[G]$.*

Proof. Let p be a condition forcing that σ is a name for a subset of λ. Then there is an extension q of p and a sequence $\langle \sigma_n : n < \omega \rangle$ in V such that for each n, q forces $\sigma \cap \kappa_n = \sigma_n$ and σ_n is a canonical $\mathbb{P}_{\kappa_n}$-name for a subset of κ_n. Then each σ_n belongs to V_λ and therefore q forces $\sigma = \sigma^*$ for some name σ^* in $V_{\lambda+1}$. This proves that $V[G]_{\lambda+1}$ is contained in $V_{\lambda+1}[G \cap V_\lambda] = V_{\lambda+1}[G]$. The converse is clear, as any element of $V_{\lambda+1}[G]$ belongs to $L(X, G \cap V_\lambda)$ for some X in $V_{\lambda+1}$ and therefore belongs to $V[G]_{\lambda+1}$. $\square$

Corollary 3.8. *If $\mathbb{P}_\lambda$ is a λ-bounded, directed closed, reverse Easton iteration, and G is $\mathbb{P}_\lambda$-generic then $L(V[G]_{\lambda+1})$ is contained in $L(V_{\lambda+1})[G]$. If $\mathbb{P}_\lambda$ is above ω then we have equality.*

Proof. The first conclusion follows immediately from Lemma 3.7. For the second conclusion, note that if $\mathbb{P}_\lambda$ is above ω and X is a subset of λ in $V[G]$ then X belongs to V iff $X \cap \kappa_n$ belongs to V for each n, and therefore $V_{\lambda+1}$ belongs to $L(V[G]_{\lambda+1})$; from this it follows that G also belongs to $L(V[G]_{\lambda+1})$, as it belongs to $L(V_{\lambda+1}, G \cap V_\lambda)$. As both $V_{\lambda+1}$ and G belong to $L(V[G]_{\lambda+1})$ we can then conclude that $L(V_{\lambda+1})[G]$ is contained in $L(V[G]_{\lambda+1})$. $\square$

Corollary 3.9. *Suppose I0 and let E be a definable Easton function such that $E \restriction \lambda$ is V_λ-definable. Then* $\mathrm{Con}(\mathsf{ZFC} + I0 + \mathsf{GCH})$ *and there is an inner model of* ZFC *in which I0 + $2^\kappa = E(\kappa)$ holds for all regular κ.*

Proof. Fix a $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$ and let $\mathbb{P}$ be the Easton iteration on all ordinals from Theorem 3.4. First we consider the first λ steps of the iteration, $\mathbb{P}_\lambda$. We can suppose that the condition q in Lemma 3.6 is in the generic G. Then by Corollary 3.8 we can extend j to $j' : L(V_{\lambda+1})[G] \prec L(V_{\lambda+1})[G]$, letting $j(\tau_G) = j(\tau)_G$ for any $\tau \in L(V_{\lambda+1})$. By the usual argument, j' is an elementary embedding. As $j'(V[G]_{\lambda+1}) = V[G]_{\lambda+1}$, the restriction of j' to $L(V[G]_{\lambda+1})$ witnesses I0 in $V[G \cap V_\lambda]$. Now, the rest of the iteration is λ^+-closed, so it doesn't change $L(V[G]_{\lambda+1})$, and j' witnesses I0 in $V[G]$. $\square$

After having proved the consistency of rank-into-rank hypotheses with the failure of GCH at regulars, the next step is to prove it at some singular. However, there are some well-known limitations:

Theorem 3.10 (Solovay)**.** *Let κ be a strongly compact cardinal. Let λ be a singular strong limit cardinal greater than κ. Then $2^\lambda = \lambda^+$.*

Suppose that $j : V_\lambda \prec V_\lambda$. The critical point of j is strongly compact in V_λ, so for any $\mathrm{crt}(j) < \eta < \lambda$ singular strong limit we have $2^\eta = \eta^+$, and this is impossible to kill while preserving the embedding. For the rest of this article we will focus on the failure of GCH at λ, the first point not covered by Solovay's result.

Theorem 3.11. *Suppose there exists $j : V_\lambda \prec V_\lambda$. Then for any $\delta < \aleph_1$ there exists a generic extension $V[G]$ such that j lifts to $V[G]_\lambda$ and $2^\lambda = \lambda^{+\delta+1}$*

This is a simple corollary of Gitik's construction using short extenders for blowing up a singular cardinal [5]:

Theorem 3.12. *Suppose λ is a cardinal of cofinality ω such that $V_\lambda \vDash \mathsf{ZFC}$ and suppose there exists a sequence $\langle \kappa_n : n \in \omega \rangle$ cofinal in λ such that for any $n \in \omega$ there exists $j : V_\lambda \prec M \vDash ZFC$ with $\operatorname{crt}(j) = \kappa_n$ and $V_{\kappa_n+n} \subseteq M$. Then there exist a cardinal preserving extension having the same bounded subsets of λ and satisfying $2^\lambda = \lambda^{+\delta+1}$ for any $\delta < \aleph_1$.*

We use the critical sequence of j as the κ_n-sequence and V_λ as M. Since Gitik's forcing doesn't add bounded subsets of λ, we have trivially $V_\lambda = V[G]_\lambda$, so there is no need to lift j, and the Theorem is proved.

Unfortunately, the same cannot be said for I1. There are many obstacles for lifting an I1 elementary embedding to a forcing that kills GCH at λ, like the fact that the names for elements of $V[G]_{\lambda+1}$ live outside $V_{\lambda+1}$, or the difficulties of finding a master condition for such forcing. This is why we change strategy, and we reflect the embedding instead of lifting it. For this we need more information about elementary embeddings that witness I0.

4 Consistency of I1 and the negation of GCH

First, we want to restrict ourselves to special elementary embeddings, i.e., those which are the result of an ultrapower construction. This is always possible, by Lemma 5 in [15]:

Theorem 4.1. *Let $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ be such that $\operatorname{crt}(j) < \lambda$. Let*

$$U = U_j = \{X \in L(V_{\lambda+1}) \cap V_{\lambda+2} : \ j \upharpoonright V_\lambda \in j(X)\}$$

Then U is a $L(V_{\lambda+1})$-ultrafilter such that $\operatorname{Ult}(L(V_{\lambda+1}), U)$ is well-founded. By condensation the collapse of $\operatorname{Ult}(L(V_{\lambda+1}), U)$ is $L(V_{\lambda+1})$, and $j_U : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$, the inverse of the collapse, is an elementary embedding. Moreover, there is an elementary embedding $k_U : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\operatorname{crt}(k_U) > \Theta^{L(V_{\lambda+1})}$ such that $j = k_U \circ j_U$, where $\Theta^{L(V_{\lambda+1})} = \{\alpha : \exists \pi : V_{\lambda+1} \twoheadrightarrow \alpha, \ \pi \in L(V_{\lambda+1})\}$.

Definition 4.2. *Let $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ be such that $\operatorname{crt}(j) < \lambda$. Then j is* proper *iff $j = j_{U_j}$.*

By Theorem 4.1 any elementary embedding $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ can be substituted with a proper one, that coincides on $L_{\Theta^{L(V_{\lambda+1})}}(V_{\lambda+1})$. This gives us an important property:

Definition 4.3. *Let $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$ be a proper elementary embedding, and let $U = U_j$ be the relevant ultrafilter. Define*

$$j(U) = \bigcup \{j(\mathrm{ran}(\pi)) : \pi \in L(V_{\lambda+1}), \pi : V_{\lambda+1} \to U\}$$

and then define the second iterate of j as the map associated to $j(U)$.

Define the successive iterates in the usual way: let α be an ordinal. Then

- *if $\alpha = \beta + 1$, M_β is well-founded and $j_\beta : M_\beta \prec M_\beta$ is the ultrapower via W, then $M_\alpha = \mathrm{Ult}(M_\beta, j_\beta(W))$ and $j_\alpha = j_\beta(j_\beta)$.*
- *if α is a limit, let (M_α, j_α) be the direct limit of (M_β, j_β) with $\beta < \alpha$.*

We say that j is iterable, if for every $\alpha \in \mathrm{Ord}$, M_α is well-founded and $j_\alpha : M_\alpha \prec M_\alpha$. In this case, we call $j_{\alpha,\beta}$ the natural embeddings between M_α and M_β.

The following is Lemma 21 in [15]:

Theorem 4.4. *Let $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$ be a proper elementary embedding. Then j is iterable.*

Note that for any $n \in \omega$, $M_n = L(V_{\lambda+1})$, but M_ω is definitively different. The key point is that $j_{0,\omega}(\mathrm{crt}(j)) = \lambda$, so in M_ω, λ is a measurable cardinal. Moreover, there is a well-ordering of $M_\omega \cap V_{\lambda+1}$ in M_ω, and in fact $V_{j_{0,\omega}(\lambda)} \cap M_\omega \vDash \mathsf{AC}$.

However, adding again the critical sequence to M_ω makes it a little more similar to the original $L(V_{\lambda+1})$:

Theorem 4.5 (Generic Absoluteness)**.** *Let $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$ be a proper elementary embedding. Let $\langle \kappa_i : i \in \omega \rangle$ be the critical sequence of j and let (M_ω, j_ω) be the ω-th iterate of j. Then for all $\alpha < \lambda$ there exists an elementary embedding*

$$\pi : L_\alpha(M_\omega[\langle \kappa_i : i \in \omega \rangle] \cap V_{\lambda+1}) \prec L_\alpha(V_{\lambda+1})$$

such that $\pi \upharpoonright \lambda$ is the identity.

The previous theorem can be found in [15] labeled as Theorem 135.

Theorem 4.6 (Main)**.** *If I0 is consistent then so is I1 with a failure of* GCH *at λ, i.e., the statement that there is $j : V_{\lambda+1} \prec V_{\lambda+1}$ with $2^\lambda > \lambda^+$.*

Proof. Let $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ witness I0. By Theorem 4.1 we can suppose that j is proper, and by Corollary 3.9 we can suppose that for all regular cardinals κ, $2^\kappa = \kappa^{++}$. Let $\langle \kappa_i : i \in \omega \rangle$ be the critical sequence of j. Let (M_ω, j_ω) be the ω-th iterate of j. Then, by elementarity, $M_\omega \vDash$ there exists a bijection between $\mathcal{P}(\lambda)$ and λ^{++}.

Let $U = \{X \subseteq \kappa_0 : \kappa_0 \in j(X)\}$ be the measure on κ_0 derived from j and let $\mathbb{P}$ be the Prikry forcing on λ with measure $j_{0,\omega}(U)$.

Claim 4.7. *$\langle \kappa_i : i \in \omega \rangle$ is generic for $\mathbb{P}$ over M_ω.*

Proof of Claim. We use the Mathias characterization of genericity for Prikry forcing, i.e., we prove that for any $A \in j_{0,\omega}(U)$, the set $\langle \kappa_i : i \in \omega \rangle \setminus A$ is finite. First, note that if $A \in j_{0,n}(U)$, then $\kappa_n \in j_{n,\omega}(A)$: by definition of U and elementarity $A \in j_{0,n}(U)$ iff $\kappa_n \in j_n(A)$; the critical point of $j_{n+1,\omega}$ is κ_{n+1}, so by elementarity

$$\kappa_n = j_{n+1,\omega}(\kappa_n) \in j_{n+1,\omega}(j_n(A)) = j_{n,\omega}(A).$$

Now, let $A \in j_{0,\omega}(U)$. There exists $n \in \omega$ and $\bar{A} \in L(V_{\lambda+1})$ such that $A = j_{n,\omega}(\bar{A})$, and by elementarity $\bar{A} \in j_{0,n}(U)$ and, more in general, $j_{n,n+1}(\bar{A}) \in j_{0,n+1}(U)$. So for any $i \in \omega$, $\kappa_{n+i} \in j_{n+i,\omega}(j_{n,n+1}(\bar{A})) = A$. $\square$

Because of the claim we can use the usual properties of the Prikry forcing:

Claim 4.8. *$M_\omega[\langle \kappa_i : i \in \omega \rangle] \vDash$ there exists a bijection from 2^λ to λ^{++}.*

Proof of Claim. Since Prikry forcing does not add bounded subsets of λ, $V_\lambda \cap M_\omega = V_\lambda \cap M_\omega[\langle \kappa_i : i \in \omega \rangle]$, therefore for any name τ for a subset of $V_\lambda \cap M_\omega$, we can suppose that $\mathrm{dom}(\tau) \subseteq \{\check{a} : a \in V_\lambda \cap M_\omega\}$. Prikry forcing is λ^+-c.c., so there are only

$$(2^\lambda)^{M_\omega} = (\lambda^{++})^{M_\omega} = (\lambda^{++})^{M_\omega[\langle \kappa_i : i \in \omega \rangle]}$$

possible nice names for subsets of $V_\lambda \cap M_\omega$, and this proves the claim. $\square$

This proves that λ in $M_\omega[\langle \kappa_i : i \in \omega \rangle]$ has the desired properties, and we will use generic absoluteness (Theorem 4.5) to prove the existence of the I1-elementary embedding.

Consider $j \upharpoonright V_{\lambda+1} : V_{\lambda+1} \prec V_{\lambda+1}$. We can define it as

$$(j \upharpoonright V_{\lambda+1})(a) = \bigcup_{n \in \omega} (j \upharpoonright V_\lambda)(a \cap V_{\kappa_n}),$$

using only $j \restriction V_\lambda$ and the critical sequence as parameters, both elements in $V_{\lambda+1}$, so $j \restriction V_{\lambda+1} \in L_1(V_{\lambda+1})$ and $L_1(V_{\lambda+1}) \vDash \exists k : V_{\lambda+1} \prec V_{\lambda+1}$. By generic absoluteness (Theorem 4.5) then

$$L_1(M_\omega[\langle \kappa_i : i \in \omega \rangle] \cap V_{\lambda+1}) \vDash \exists k : V_{\lambda+1} \prec V_{\lambda+1},$$

and this is enough, since $L_1(M_\omega[\langle \kappa_i : i \in \omega \rangle] \cap V_{\lambda+1})$ computes correctly the satisfaction relation of

$$M_\omega[\langle \kappa_i : i \in \omega \rangle] \cap V_{\lambda+1} = (V_{\lambda+1})^{M_\omega[\langle \kappa_i : i \in \omega \rangle]}$$

and

$$L_1(M_\omega[\langle \kappa_i : i \in \omega \rangle] \cap V_{\lambda+1}) \subseteq M_\omega[\langle \kappa_i : i \in \omega \rangle].$$

Therefore

$$M_\omega[\langle \kappa_i : i \in \omega \rangle] \vDash \exists k : V_{\lambda+1} \prec V_{\lambda+1}.$$

For the last step, we need a model of ZFC, since $M_\omega[\langle \kappa_i : i \in \omega \rangle]$ is just a model of ZF. However, V_{κ_1} is a model of ZFC, so by the elementarity of $j_{0,\omega}$, $V_{j_{0,\omega}(\kappa_1)} \cap M_\omega[\langle \kappa_i : i \in \omega \rangle]$ is a model of ZFC such that there exists $k : V_{\lambda+1} \prec V_{\lambda+1}$ and $2^\lambda = \lambda^{++}$. □

The proof can be generalized in two different directions. First, consider that Theorem 3.4 gives many more choices than $2^\lambda = \lambda^{++}$, and this reflects immediately to 4.6. Second, also the Generic Absoluteness gives more choices: we considered just the case $\alpha = 1$, but in fact we could have considered something more than I1, i.e., there exists $j : L_\alpha(V_{\lambda+1}) \prec L_\alpha(V_{\lambda+1})$, if $\alpha < \lambda$. These unnamed hypotheses have been proven by Laver [10], [11] to be strictly stronger than I1 and strictly weaker than I0. Note that if j witnesses I0, then $j \restriction L_\alpha(V_{\lambda+1})$ witnesses the corresponding hypotheses iff α is a fixed point of j, but considering j^n with n big enough we can always assume this. Note also that $j \restriction L_\alpha(V_{\lambda+1}) \in L_{\alpha+1}(V_{\lambda+1})$.

Theorem 4.9. *Suppose there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with* $\mathrm{crt}(j) < \lambda$ *and let $\alpha < \lambda$. Then for every Easton function E such that $E \restriction \lambda$ is V_λ definable, there is an inner model of* ZFC *in which $\exists k : L_\alpha(V_{\lambda+1}) \prec L_\alpha(V_{\lambda+1}) + 2^\lambda = j_{0,\omega}(E)(\lambda)$ holds. In particular, for any $\delta < \lambda$, there is an inner model of* ZFC *in which $\exists k : L_\alpha(V_{\lambda+1}) \prec L_\alpha(V_{\lambda+1}) + 2^\kappa = \kappa^{+\delta+1}$ holds for all regular $\kappa < \lambda$ and $\kappa = \lambda$.*

We make a short comment regarding the meaning of "larger", in the expression "one large cardinal is larger than the second one". During the years the meaning of this sentence grew more and more ambiguous: it is mostly used to indicate the consistency strength of a large cardinal hypotheses, but it sometimes implies that the least cardinal which is large in the first sense is larger than the least cardinal which is large in the second sense with respect to the cardinal order. While at the beginning the two concepts coincided, during the exploration of the upper part of the large cardinals hierarchy the two concepts often differed completely. Theorem 4.6, coupled with Solovay's Theorem 3.10, gives us another such example.

Theorem 4.10. *Suppose there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$. Then it is consistent that there exists $j : V_{\lambda+1} \prec V_{\lambda+1}$ and that every strongly compact cardinal is larger than λ.*

Finally, we cast a look into the future. Generic absoluteness (Theorem 4.5) has a key role in the proof of Theorem 4.6, yet there is no evidence that it is optimal. On the contrary, there are hints that it could be possibly extended, and Woodin outlined such situations (see for example Lemma 130 and Remark 139 in [15]). The study of the structure of the sets $L_\alpha(V_{\lambda+1})$ for $\alpha < \Theta^{L(V_{\lambda+1})}$ made by Laver in [11] gives us the tools to approach Theorem 4.6 under stronger hypotheses.

Definition 4.11. *Suppose there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \kappa$ and let $\alpha < \Theta^{L(V_{\lambda+1})}$. We say that* generic absoluteness holds at α *iff there exists*

$$\pi : L_{\alpha+1}(M_\omega[\langle \kappa_i : i \in \omega \rangle] \cap V_{\lambda+1}) \prec L_{\alpha+1}(V_{\lambda+1})$$

such that $\pi \upharpoonright \lambda$ is the identity.

Definition 4.12 ([11])**.** *Let λ be a cardinal and let $\alpha < \Theta^{L(V_{\lambda+1})}$. Then α is* good *iff every element of $L_\alpha(V_{\lambda+1})$ is definable in $L_\alpha(V_{\lambda+1})$ from an element in $V_{\lambda+1}$.*

Lemma 4.13 ([11])**.** *Let λ be a strong limit cardinal. Then the good ordinals are unbounded in $\Theta^{L(V_{\lambda+1})}$.*

Lemma 4.14 ([11])**.** *Let λ and α be such that α is good and there exists $k : L_\alpha(V_{\lambda+1}) \prec L_\alpha(V_{\lambda+1})$ with $\mathrm{crt}(k) < \lambda$. Then k is induced by $k \upharpoonright V_\lambda$, and therefore $k \in L_{\alpha+1}(V_{\lambda+1})$.*

Lemma 4.15 (Woodin)**.** *Let λ be a cardinal. If there exists $j : L_{\Theta^{L(V_{\lambda+1})}}(V_{\lambda+1}) \prec L_{\Theta^{L(V_{\lambda+1})}}(V_{\lambda+1})$, then for any $\alpha < \Theta^{L(V_{\lambda+1})}$ there exists a $k : L_\alpha(V_{\lambda+1}) \prec L_\alpha(V_{\lambda+1})$.*

Proof. Suppose it is false. Then $L_{\Theta^{L(V_{\lambda+1})}}(V_{\lambda+1}) \vDash \exists\alpha$ (α is a counterexample). Let α_0 be the least counterexample. Then α_0 is definable in $L_{\Theta^{L(V_{\lambda+1})}}(V_{\lambda+1})$ and $j(\alpha_0) = \alpha_0$. Therefore $j \upharpoonright L_{\alpha_0}(V_{\lambda+1})$ is as in the lemma, contradiction. □

Suppose now that there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$ and that the generic absoluteness holds at α, with α good. Then by Lemma 4.15 there exists $k : L_\alpha(V_{\lambda+1}) \prec L_\alpha(V_{\lambda+1})$, and by Lemma 4.14, $k \in L_{\alpha+1}(V_{\lambda+1})$. If π witnesses the generic absoluteness, $\pi^{-1}(k)$ witnesses the following theorem:

Theorem 4.16. *Suppose that there exists $j : L(V_{\lambda+1}) \prec L(V_{\lambda+1})$ with $\mathrm{crt}(j) < \lambda$ and that the generic absoluteness holds at α, with α good. Then for every Easton function E such that $E \restriction \lambda$ is V_λ definable, there exists a model of* ZFC *in which $\exists k : L_\alpha(V_{\lambda+1}) \prec L_\alpha(V_{\lambda+1}) + 2^\lambda = j_{0,\omega}(E)(\lambda)$ holds*

Even if a generalization of generic absoluteness could be proved, there would still be questions to answer. Let I0(λ) and I1(λ) be the the corresponding hypotheses with fixed λ. While we used I0 for the consistency strength of $\exists\lambda$ I1(λ) + the failure of GCH at λ, it is not known whether this is optimal.

Question 1. *Does* $\mathrm{Con}(\mathsf{ZFC} + I1)$ *imply* $\mathrm{Con}(\mathsf{ZFC} + \exists\lambda\ (I1(\lambda) \wedge 2^\lambda > \lambda^+))$*?*

Another way to improve the results in this paper would be to approach the consistency of I0(λ) + the failure of GCH at λ.

Question 2. *Is* $\mathsf{ZFC} + \exists\lambda\ (I0(\lambda) \wedge 2^\lambda = \lambda^+)$ *consistent? If so, what is its consistency strength?*

A generic absoluteness theorem for hypotheses stronger than I0 seems difficult to obtain, therefore the previous question appears to be a compelling challenge.

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