MUTUAL DIAMOND

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ABSTRACT: We formulate a diamond-like principle for singular cardinals based on the notion of mutual stationarity due to Magidor and prove that it holds in L.

In a joint work with Foreman [1], Magidor used the notion of mutual stationarity to show that $\mathcal{P}_{\kappa\lambda}$ is not λ^+-saturated unless $\kappa = \lambda = \omega_1$. We use it here to formulate a diamond-like principle for singular cardinals.

Definition 1. For $1 < n < \omega$, let $S_n \subseteq \aleph_n$ be such that $\forall\alpha \in S_n(\mathrm{cf}(\alpha) = \omega_1)$. Then $\langle S_n : 1 < n < \omega\rangle$ is mutually stationary if for each structure $\mathfrak{A} = (A, ...)$ for a countable language such that $\aleph_\omega \subseteq A$ there is a $\mathfrak{B} \prec \mathfrak{A}$, with $\mathfrak{B} = (B, ...)$ such that

$$\forall n(1 < n < \omega \wedge \aleph_n \in B \rightarrow \sup(B \cap \aleph_n) \in S_n).$$

Definition 2. A sequence $\langle X_\alpha : \alpha < \aleph_\omega \wedge \mathrm{cf}(\alpha) = \omega_1 \wedge X_\alpha \subseteq \alpha\rangle$ is a mutual diamond sequence ($\diamondsuit$-sequence) if for each $X \subseteq \aleph_\omega$, the sequence $\langle S_n : 1 < n < \omega\rangle$ is mutually stationary, where $S_n = \{\alpha : \alpha < \aleph_n \wedge \mathrm{cf}(\alpha) = \omega_1 \wedge X \cap \alpha = X_\alpha\}$. $\diamondsuit$ holds if there is a mutual diamond sequence.

Theorem 3. $V = L \rightarrow \diamondsuit$

Proof: We construct by induction a sequence $\langle X_\alpha : \alpha < \aleph_\omega \wedge \mathrm{cf}(\alpha) = \omega_1\rangle$ witnessing $\diamondsuit$. An ordinal β is good if

$$L_\beta \models \text{“ } ZF^- + V = L + \aleph_\omega \text{ is largest cardinal ”}.$$

If α is not a cardinal let $\beta(\alpha)$ be the largest limit ordinal β such that $L_\beta \models$ “ α is a cardinal ”, if such β exists, and $\beta(\alpha) = \alpha$ otherwise. For each good $\beta \leq \aleph_{\omega+1}$ we construct a sequence

$$\langle X^\beta_\alpha : L_\beta \models \text{“ } \alpha < \aleph_\omega \wedge \mathrm{cf}(\alpha) = \omega_1 \text{ ”}\rangle$$

which will be a potential witness for $\diamondsuit$ in L_β. So fix a good β and suppose that for all good $\bar\beta < \beta$

$$\langle X^{\bar\beta}_\alpha : L_{\bar\beta} \models \text{“ } \alpha < \aleph_\omega \wedge \mathrm{cf}(\alpha) = \omega_1 \text{ ”}\rangle$$

has been constructed. If $L_\beta \models$ “ $\alpha = \omega_1$ ”, set $X^\beta_\alpha = \emptyset$. Otherwise fix $\alpha < \beta$ such that

$$L_\beta \models \text{“ } \alpha < \aleph_\omega \wedge \mathrm{cf}(\alpha) = \omega_1 \text{ ”}$$

and $\beta(\alpha)$ is defined and less than β. If $\beta(\alpha)$ is not good, let $X^\beta_\alpha = \emptyset$. If $\beta(\alpha)$ is good and

$$\vec{X}^{\beta(\alpha)} = \langle X^{\beta(\alpha)}_{\bar\alpha} : L_{\beta(\alpha)} \models \text{“ } \bar\alpha < \aleph_\omega \wedge \mathrm{cf}(\bar\alpha) = \omega_1 \text{ ”}\rangle$$

is a $\diamondsuit$-sequence in $L_{\beta(\alpha)}$ let $X^\beta_\alpha = \emptyset$. If $\beta(\alpha)$ is good but $\vec{X}^{\beta(\alpha)}$ is not a $\diamondsuit$-sequence, let $\langle\mathfrak{A}, X\rangle$ with $X \subseteq \aleph_\omega^{L_{\beta(\alpha)}}$, be $<_L$-least such that, in $L_{\beta(\alpha)}$, $\mathfrak{A}$ is a witness that $\langle S^{\beta(\alpha)}_n(X) : 1 < n < \omega\rangle$ is not mutually stationary where

$$S^{\beta(\alpha)}_n(X) = \{\bar\alpha : \bar\alpha < \aleph_n \wedge \mathrm{cf}(\bar\alpha) = \omega_1 \wedge X \cap \bar\alpha = X^{\beta(\alpha)}_{\bar\alpha}\}.$$

In this case let $X^\beta_\alpha = X \cap \alpha$. This finishes the construction.

Now

$$L_{\aleph_{\omega+1}} \models \text{" } ZF^- + V = L + \aleph_\omega \text{ is largest cardinal "}.$$

So $\aleph_{\omega+1}$ is good and, by the construction, we have the sequence $\vec{X}^{\aleph_{\omega+1}}$. Let

$$\vec{X} = \vec{X}^{\aleph_{\omega+1}} = \langle X_\alpha : \alpha < \aleph_\omega \wedge \mathrm{cf}(\alpha) = \omega_1 \rangle.$$

We show that $\vec{X}$ is a $\diamondsuit$-sequence. By way of contradiction, suppose not and let $\langle \mathfrak{A}, X \rangle$ be the $<_L$-least witness for this; then $\mathfrak{A} \in L_{\aleph_{\omega+1}}$. Let N_0 be the least $Y \prec L_{\aleph_{\omega+1}}$ such that $\mathfrak{A}, X, \vec{X} \in Y$. Let $N_{\alpha+1}$ be the least $Y \prec L_{\aleph_{\omega+1}}$ such that $N_\alpha \cup \{N_\alpha\} \subseteq Y$. For limit λ, let $N_\lambda = \bigcup_{\alpha<\lambda} N_\alpha$. Let $N = \bigcup_{\alpha<\omega_1} N_\alpha$ and for $1 < n < \omega$, let $\alpha_n = \sup(N \cap \aleph_n)$ and note that $\mathrm{cf}(\alpha_n) = \omega_1$. Let $\mathfrak{B} = N \cap \mathfrak{A}$ and note that $\mathfrak{B}$ is an elementary submodel of $\mathfrak{A}$ and $\alpha_n = \sup(B \cap \aleph_n)$. Then there is an n, with $1 < n < \omega$, such that $\aleph_n \in \mathfrak{B}$ but $\alpha_n \notin S_n(X)$. We work toward a contradiction. Let

$$N[\aleph_{n-1}] = \mathrm{Hull}^{L_{\aleph_{\omega+1}}}(N \cup \aleph_{n-1}).$$

Claim 4. $\sup(N[\aleph_{n-1}] \cap \aleph_n) = \sup(N \cap \aleph_n)$

Proof: Clearly

$$N \cap \aleph_n \subseteq N[\aleph_{n-1}] \cap \aleph_n,$$

so

$$\sup(N \cap \aleph_n) \leq \sup(N[\aleph_{n-1}] \cap \aleph_n).$$

For the reverse direction let $\beta < \sup(N[\aleph_{n-1}] \cap \aleph_n)$. For $m < \omega$, $\gamma \in N \cap \aleph_n$, $\vec{x} \in N$ consider

$$f(\gamma, m, \vec{x}) = \sup\{\delta : \delta < \aleph_n \wedge \delta \text{ is } \Sigma_m\text{-definable from } \vec{x} \cup \gamma\}.$$

Then $f(\gamma, m, \vec{x}) \in N$ and $f(\gamma, m, \vec{x}) < \sup(N \cap \aleph_n)$. But now $\beta < f(\gamma, m, \vec{x})$ for sufficiently large $\gamma \in N \cap \aleph_n$, $\vec{x} \in N$, $m < \omega$. So $\beta < \sup(N \cap \aleph_n)$ and the claim is proved.

In fact $\alpha_n = N[\aleph_{n-1}] \cap \aleph_n$. Now let $\pi : N[\aleph_{n-1}] \simeq L_\beta$ be the transitive collapse. Then $\pi(\aleph_n) = \alpha_n$, so α_n is a cardinal in L_β and β is certainly good. To show that $\beta = \beta(\alpha_n)$ we need to show that for some $m < \omega$, α_n fails to be a cardinal in $L_{\beta+m}$. To this end let N'_0 be the least $Y \prec L_\beta$ such that $\pi(\mathfrak{A}), \pi(X), \pi(\vec{X}) \in Y$. Let $N'_{\alpha+1}$ be the least $Y \prec L_\beta$ such that $N'_\alpha \cup \{N'_\alpha\} \subseteq Y$. Let $N'_\lambda = \bigcup_{\alpha<\lambda} N'_\alpha$ for limit λ and $N' = \bigcup_{\alpha<\omega_1} N'_\alpha$. Now by induction we get $\forall \alpha < \omega_1 (\pi \upharpoonright N_\alpha : N_\alpha \simeq N'_\alpha)$ and $N \simeq N'$. Since $N' \subseteq L_\beta$, $\aleph_{n-1} \subseteq L_\beta$, we can define $N'[\aleph_{n-1}] = \mathrm{Hull}^{L_\beta}(N' \cup \aleph_{n-1})$. And we get $N[\aleph_{n-1}] \simeq N'[\aleph_{n-1}]$. So $N'[\aleph_{n-1}] = L_\beta$ and $\aleph_n^{L_\beta} = N[\aleph_{n-1}] \cap \aleph_n = \alpha_n$ and $\langle N'_\alpha : \alpha < \omega_1 \rangle$ is definable in $L_{\beta+2}$. So over $L_{\beta+2}$ we can define a cofinalizing sequence for α_n of length ω_1. This shows that $\beta = \beta(\alpha_n)$. Now by uniform definability of the construction $\pi(\vec{X}) = \vec{X}^{\beta(\alpha_n)}$ and $\pi(\langle \mathfrak{A}, X \rangle) = \langle \pi(\mathfrak{A}), \pi(X) \rangle$ is the $<_L$-least witness that $\vec{X}^{\beta(\alpha_n)}$ is not a $\diamondsuit$-sequence in $L_{\beta(\alpha_n)}$. But by definition $\pi(X) \cap \alpha_n = X_{\alpha_n}$ and since $N[\aleph_{n-1}] \cap \aleph_n = \alpha_n$, $X \cap \alpha_n = \pi(X) \cap \alpha_n = X_{\alpha_n}$. And this is a contradiction since we assumed $X \cap \alpha_n \neq X_{\alpha_n}$. This proves the theorem. $\square$

References

[1] M. Magidor, M. Foreman: Mutual Stationarity. A talk given by M. Magidor at the ASL meeting in the spring of 1997.