The Appeal of $0^\#$
Mittag-Leffler Lecture, 13. September, 2000

Set-theorists have had great success in solving problems under the hypothesis $V = L$. Under this assumption, Gödel proved the Generalised Continuum Hypothesis and also precisely determined the behaviour of projective sets of reals in L, with regard to regularity properties such as Lebesgue measurabilty and the perfect set property. Further important work on L was accomplished by Jensen.

But the fact remains that $V = L$ is not a theorem of ZFC: The forcing method allows us to consistently enlarge L to models $L[G]$ where G is a set or class that is generic over L with respect to some forcing notion P. Thus it is reasonable to suggest that V, rather than equal to L, should be in fact a generic extension $L[G]$ of L. But this then gives rise to the new and difficult question: Which generic extension is it? Usually, a forcing notion P gives rise not to one, but to many different generics G. Some hope is provided by:

Theorem 1 *Assume some weak large cardinal axioms, consistent with $V = L$ (precisely: an n-ineffable cardinal for each n). Then there is an L-definable forcing notion P with a unique generic.*

(Unless otherwise stated, we take "definable" to mean "definable *without* parameters".) But this does not solve our problem, for P is not the only forcing notion satisfying this Theorem and there does not seem to be a canonical choice for P. Moreover, the hypothesis that generics exist for all definable forcing notions is inconsistent:

Theorem 2 *There exist forcing notions P_0, P_1 which are definable over L and which preserve ZFC, such that there cannot be generics for P_0 and P_1 simultaneously.*

So if we want V to not be L we must decide for which forcing notions P to allow generics. The needed criterion arises naturally through the consideration of *CUB-absoluteness*:

Definition. A class C of ordinals is CUB iff it is closed and unbounded. V is *CUB-absolute over L* iff every L-definable class of ordinals which has a CUB subclass definable with parameters in a generic extension of V has one definable with parameters in V.

Theorem 3 *V is CUB-absolute over L iff* $0^\#$ *exists.*

$0^\#$ is a special set of integers discovered by Silver and Solovay, whose existence is a "transcendence principle for L" in the sense that it implies that V is not a generic extension of L. The existence of $0^\#$ is equivalent to the existence of a nontrivial elementary embedding of L into itself. If $0^\#$ exists then there is a smallest inner model which satisfies "$0^\#$ exists", namely the canonical model $L[0^\#]$.

Thus as an alternative to $V = L$ we could propose the hypothesis: $0^\#$ exists and $V = L[0^\#]$. This allows for the existence of generic extensions of L. Moreover we can now provide a generic existence criterion for L-definable forcings, by declaring an L-definable forcing to have a generic iff it has one definable in $L[0^\#]$.

An open problem is to provide a more convincing characterisation of $0^\#$ in terms of forcing. One possibility is suggested by the following

Theorem 4 *Assume a weak large cardinal axiom, consistent with* $V = L$ *(precisely: an* $\omega + \omega$*-Erdős cardinal). If* $0^\#$ *exists and an* L*-definable forcing has a generic, then it has one definable in* $L[0^\#]$*.*

Thus $L[0^\#]$ is "saturated" with respect to L-definable forcings. A nice result would be the converse to this, giving that $0^\#$ exists iff V is saturated with respect to L-definable forcings.

A second possibility would be to define a new concept of forcing and prove that the existence of $0^\#$ is equivalent to the statement that V is not "generic" over L in this new sense. One cannot simply use the usual notion of class forcing for this purpose; indeed there exist reals R in $L[0^\#]$ which are not class-generic over L and from which $0^\#$ is not constructible.

Cardinal-Preserving Extensions

By not assuming $V = L$ we can compare set-theoretic problems according to their degree of nonconstructibility. We shall now examine a wide class of such problems, under the assumption that $0^\#$ exists.

Definition A subset X of L is Σ_1^{CP} iff X can be written in the form

$$a \in X \text{ iff } \varphi(a) \text{ holds in a cardinal-preserving extension of } L$$

for some Σ_1 formula φ. (We intend our cardinal-preserving extensions of L to satisfy AC and to be contained in a set-generic extension of V.)

Example: A classic result of Baumgartner-Harrington-Kleinberg [1] implies that assuming CH a stationary subset of ω_1 has a CUB subset in a cardinal-perserving set-generic extension of V. This implies that the set

$\{X \in L \mid X \subseteq \omega_1^L$ and X has a CUB subset in a cardinal-preserving extension of $L\}$

is constructible, as it equals the set of constructible subsets of ω_1^L which in L are stationary.

Is there a similar such result for subsets of ω_2^L? Building on work of M. Stanley [9], we show that there is not. We shall also consider a number of related problems, examining the extent to which they are "solvable" in the above sense, as well as defining a notion of reduction between them. We assume throughout that $0^\#$ exists.

Theorem 5 *If X is Σ_1^{CP} then X is constructible from $0^\#$.*

Theorem 6 *$0^\#$ is Σ_1^{CP}. And there are Σ_1^{CP} sets of constructibility degree strictly between 0 and $0^\#$.*

Theorem 7 *The following Σ_1^{CP} sets are equiconstructible with $0^\#$:*
(a) $\{T \mid T \in L$ and T is a tree on κ of height κ with a cofinal branch in a cardinal-preserving extension of $L\}$, for κ an uncountable successor cardinal of L.
(b) $\{X \subseteq \kappa \mid X \in L$ and X contains a CUB subset in a cardinal-preserving extension of $L\}$, for κ regular in L, $\kappa > \omega_1^L$.
(c) $\{X \subseteq \kappa \mid X \in L$ and X is the set of ordinals $< \kappa$ which are admissible relative to some real in a cardinal-preserving extension of $L\}$, for κ uncountable in L.
(d) $\{X \subseteq \kappa \mid X \in L$ and X is the intersection with κ of a class which is Δ_1-definable over $L[R]$ without parameters, for some real R in a cardinal-preserving extension of $L\}$, where κ is at least ω_3^L.

Theorem 7 is proved by "reducing" $0^\#$ to the sets mentioned. In fact we shall need the following more general notion of "reduction".

Definition Suppose that (X_0, X_1) and (Y_0, Y_1) are pairs of disjoint subsets of L. Then we write

$(X_0, X_1) \longrightarrow_L (Y_0, Y_1)$

iff there is a function F in L such that

$a \in X_0 \to F(a) \in Y_0$

$a \in X_1 \to F(a) \in Y_1$.

We write X instead of (X_0, X_1) in case $X = X_0$ is the complement (within some constructible set) of X_1, and similarly for the Y's. It is clear that if (X_0, X_1) is nonconstructible and $(X_0, X_1) \longrightarrow_L (Y_0, Y_1)$, then (Y_0, Y_1) is also nonconstructible. In the proof of Theorem 7 we shall obtain reductions in this sense of $0^{\#}$ to the sets mentioned.

Theorem 7 suggests that the Baumgartner-Harrington-Kleinberg result should be viewed as a rare example of a nontrivial "solvable" Σ_1^{CP} problem. However it is not the only such example:

Theorem 8 *If κ is ω_2^L in the set described in Theorem 7 (d), then the resulting set is constructible.*

About the Proofs of Theorems 5-8

To prove Theorem 5, one shows the following: If φ is a Σ_1 formula, a is a constructible set and $\varphi(a)$ is true in a cardinal-preserving extension of L, then this cardinal-preserving extension of L can be chosen as a set-generic extension of $L[0^{\#}]$. The proof is based on ideas used to prove the Martin-Solovay Basis Theorem.

Theorem 6 is proved using the techniques used to prove the Π_2^1-singleton conjecture (see [3]).

The proof of Theorem 7 is based heavily on the notion of reduction introduced above: The first step is to reduce $0^{\#}$ to the tree problem. Let $\mathcal{T}(\kappa)$ denote the set of constructible trees on κ of height κ with a cofinal, cardinal-preserving branch; i.e., a cofinal branch b such that L and $L[b]$ have the same cardinals. To each n we associate a tree T_n on κ (κ an uncountable successor L-cardinal) in such a way that n belongs to $0^{\#}$ iff T_n has a cofinal, cardinal-preserving branch. Moreover, the sequence of trees T_n, $n \in \omega$ is constructible, so this proves $0^\# \longrightarrow \mathcal{T}(\kappa)$ for uncountable successor L-cardinals κ.

To reduce $0^\#$ to the CUB subset problem for L-regular cardinals greater than ω_1^L it would suffice to reduce the tree problem $\mathcal{T}(\kappa)$ to the CUB subset problem $\mathcal{C}(\kappa)$. However we do not know how to do this. Instead we work with a modified version of the tree problem. Define

$\mathcal{T}^*(\kappa^+) = \{T \in \mathcal{T}(\kappa^+) \mid T$ is Δ_1-definable over L_{κ^+} from the parameter κ and T has a cardinal-preserving, stationary, $\mathcal{P}(\kappa)$-preserving cofinal branch$\}$
$\mathcal{T}^{**}(\kappa^+) = \{T \in \mathcal{T}(\kappa^+) \mid T$ is Δ_1-definable over L_{κ^+} from the parameter κ and T has a cardinal-preserving cofinal branch$\}$

where b is *stationary* if in $L[b]$ its intersection with cof κ is stationary, and b is $\mathcal{P}(\kappa)$*-preserving* if L and $L[b]$ have the same subsets of κ. We show: $0^\# \longrightarrow_L (\mathcal{T}^*(\omega_2^L), \sim \mathcal{T}^{**}(\omega_2^L))$, and $(\mathcal{T}^*(\omega_2^L), \sim \mathcal{T}^{**}(\omega_2^L)) \longrightarrow_L \mathcal{C}(\omega_2^L)$. Finally, using the combinatorial principle $\Box$, which is true in L, we show that for any L-regular cardinal greater than ω_2^L, $\mathcal{C}(\omega_2^L) \longrightarrow_L \mathcal{C}(\kappa)$.

A similar modification of the CUB subset problem is then reduced both to the admissibility spectrum problem at uncountable L-cardinals, and to the Δ_1-definability problem at L-cardinals greater than ω_2^L.

Theorem 8 is proved using a special type of coding construction.

Open Questions

Very simple questions concerning the notion of reduction $\longrightarrow_L$ remain unanswered. For example, is the tree problem at ω_2^L reducible to the CUB subset problem at ω_2^L? Explicitly:

Is there a constructible function that associates to each constructible tree T on ω_2^L a subset X of ω_2^L such that T has a cofinal, cardinal-preserving branch iff X has a cardinal-preserving CUB subset?

If so, the indirect arguments sketched could be avoided. Is the CUB subset problem reducible to the problem of finding a cardinal-preserving homogenous set for a given partition? One can easily formulate a host of similar such open problems.

Other open questions concern a weakening of cardinal-preservation. An example concerns the CUB subset problem: Is the following set constructible?

$\mathcal{C}'(\omega_3^L) = \{X \in L \mid X \subseteq \omega_3^L$ and X has a CUB subset in an extension of L which preserves $\omega_1^L, \omega_3^L\}$

It is possible that the solution to this problem will require genuine use of a gap 2 morass.

Literatur

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