0# and Inner Models

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In this paper we examine the cardinal structure of inner models that satisfy GCH but do not contain \(0^\#\). We show, assuming that \(0^\#\) exists, that such models necessarily contain Mahlo cardinals of high order, but without further assumptions need not contain a cardinal \(\kappa\) which is \(\kappa\)-Mahlo. The principal tools are the Covering Theorem for \(L\) and the technique of reverse Easton iteration.

Let \(I\) denote the class of Silver indiscernibles for \(L\) and \(\langle i_\alpha \mid \alpha \in \text{ORD}\rangle\) its increasing enumeration. Also fix an inner model \(M\) of GCH not containing \(0^\#\) and let \(\omega_\alpha\) denote the \(\omega_\alpha\) of the model \(M[0^\#]\), the least inner model containing \(M\) as a submodel and \(0^\#\) as an element.

**Theorem 1** Suppose that \(\alpha\) is greater than 0. (a) \(i_{\omega_{\omega_1}\cdot \alpha}\) is an \(M\)-cardinal, and unless \(\alpha\) is a limit ordinal of countable \(M[0^\#]\)-cofinality, so is its \(L\)-cardinal successor.

(b) If \(\beta\) is less than \(i_{\omega_{\omega_1\cdot \omega}}\) then there is a proper inner model \(M\) of \(L[0^\#]\) satisfying GCH in which the only ordinals between \(\omega\) and \(\beta\) which are \(M\)-cardinals are those which are required to be by part (a).

It follows from (a) that for finite \(n\), \(\omega_{2n+1}^M\) is at most \(i_{\omega_{\omega_1\cdot (n+1)}}\) and that \(\omega_{2n+2}^M\) is at most the \(L\)-cardinal successor to \(i_{\omega_{\omega_1\cdot (n+1)}}\). It follows from (b) that these bounds are optimal. The restriction in (b) on \(\beta\) cannot be weakened, as otherwise an increasing \(\omega\)-sequence of Silver indiscernibles, and hence \(0^\#\)

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itself, would belong to $M$. In fact the supremum of the $i_{\omega_1 \cdot n}$’s must be large in $M$:

**Theorem 2** (a) $i_{\omega_1 \cdot \alpha}$ is inaccessible in $M$ for limit $\alpha$.

(b) If $\beta$ is less than $i_{\omega_1 \cdot \omega^L \cdot \omega}$, then there is a proper inner model $M$ of $L[0^\#]$ satisfying GCH in which the only ordinals less than $\beta$ which are $M$-inaccessible are those which are required to be by part (a).

It follows from (a) that for finite $n$, the $n$-th $M$-inaccessible is at most $i_{\omega_1 \cdot \omega \cdot n}$. It follows from (b) that these bounds are optimal. As before, the restriction in (b) on $\beta$ cannot be weakened, as otherwise $0^\#$ would belong to $M$.

We can also obtain Mahlo cardinals of high order in $M$. Define: $\kappa$ is 0-Mahlo (or simply Mahlo) iff the set of inaccessible $\bar{\kappa} < \kappa$ is stationary in $\kappa$, $\kappa$ is $\alpha + 1$-Mahlo iff the set of $\alpha$-Mahlo $\bar{\kappa} < \kappa$ is stationary in $\kappa$, and for limit $\lambda$, $\kappa$ is $\lambda$-Mahlo iff $\kappa$ is $\alpha$-Mahlo for every $\alpha < \lambda$.

**Theorem 3** (a) $i_{\omega_1 \cdot \beta}$ for $\beta$ of $M[0^\#]$-cofinality at least $\omega_{\alpha + 1}$ is $\alpha$-Mahlo in $M$.

(b) Suppose that $\alpha$ is not $L[0^\#]$-inaccessible. If $\gamma$ is less than $i_{\omega_1 \cdot \omega^L \cdot \omega}$, then there is a proper inner model of $L[0^\#]$ satisfying GCH in which the only $\alpha$-Mahlo cardinals less than $\gamma$ are those which are required to be by part (a).

It follows from (a) that for finite $n > 0$, the $n$-th $\alpha$-Mahlo cardinal of $M$ is at most $i_{\omega_1 \cdot \omega \cdot \omega_{\alpha + 1}}$. It follows from (b) that these bounds are optimal when $\alpha$ is not $M[0^\#]$-inaccessible. And, as before, the bound on $\gamma$ in (b) cannot be improved. Part (a) of Theorem 3, when $\alpha = 0$, was proved independently by Amir Leshem.

**Theorem 4** (a) If $\alpha$ is inaccessible in $M[0^\#]$ then it is $\alpha$-Mahlo in $M$.

(b) If there are only finitely many $L[0^\#]$-inaccessibles less than $\beta$ then there is a proper inner model of $L[0^\#]$ satisfying GCH in which the only cardinals $\alpha$ less than $\beta$ which are $\alpha$-Mahlo are inaccessible in $L[0^\#]$. If there are no inaccessibles in $L[0^\#]$ then there is a proper inner model $M$ of $L[0^\#]$ satisfying GCH which contains no $\alpha$ which is $\alpha$-Mahlo.
Thus we have reached the limit of large cardinal properties which must hold in $M$, in the theory ZFC+$0^\#$ exists. Of course if $0^{\#\#}$ exists, then much stronger large cardinal properties, such as $n$-subtlety, are witnessed in inner models of $L[0^\#]$, as these properties are witnessed in $L[0^\#]$ and are downward absolute to inner models. The next result implies that the assumption of $0^\#$ plus an $\omega + \omega$-Erdős cardinal maximizes the large cardinal properties that must hold in inner models which are generic extensions of $L$.

**Theorem 5** Suppose that $0^\#$ exists, there is an $\omega + \omega$-Erdős cardinal and $M$ is an inner model generic over $L$. Suppose that $\varphi$ is a sentence true in $M$. Then $\varphi$ is also true in an inner model of $L[R]$ for some real $R$ such that $R^\#$ exists.

To see the implications of Theorem 5, suppose that $P$ is a property downward absolute to inner models, and that $P$ is witnessed in $L[R]$ whenever $R$ is a real and $R^\#$ exists. An example of such a property is the existence of $\alpha$-Erdős cardinals for $\alpha$ countable in $L$. Then by Theorem 5, $P$ is also witnessed in all inner models which are class-generic extensions of $L$. This applies to the strongest large cardinal properties that hold in $L$. It follows from the last statement of Theorem 4 (b) that the hypothesis of an $\omega + \omega$-Erdős cardinal cannot be deleted in Theorem 5.

**Proof of Theorem 1.** (a) We show that if $\kappa$ is an indiscernible of the form $i_{\omega_1 \cdot \alpha}$, $\alpha > 0$ then $\kappa$ is a cardinal of $M$, and if $\kappa$ is an indiscernible of uncountable cofinality in $M[0^\#]$ then its $L$-cardinal successor is a cardinal of $M$.

**Lemma 6** Suppose that $\kappa$ is $L$-regular and has uncountable $M[0^\#]$-cofinality. Then $\kappa$ is a limit of indiscernibles and $\kappa^+$ of $M$ equals $\kappa^+$ of $L$.

This is proved as follows. For each finite $n$ consider $C_n = \{\bar{\kappa} < \kappa \mid$ No ordinal between $\bar{\kappa}$ and $\kappa$ is $L$-definable from ordinals less than $\bar{\kappa}$ together with $n$ indiscernibles $\geq \kappa\}$. For each $n$, $C_n$ is CUB in $\kappa$, by the $L$-regularity of $\kappa$. And the intersection of the $C_n$’s is equal to $I^* = I \cap \kappa$, since every ordinal is $L$-definable from finitely many indiscernibles. As $\kappa$ has uncountable $L[0^\#]$-cofinality, it follows that $I^*$ is unbounded in $\kappa$, and hence $\kappa$ is a limit of indiscernibles. Now if $\kappa^+$ of $L$ were collapsed in $M$, then there would be in $M$ only $\kappa$-many constructible CUB subsets of $\kappa$. By taking diagonal intersection,
there would then be in $M$ a CUB subset $C$ of $\kappa$ which is almost contained in (i.e., contained in except for a bounded subset of $\kappa$) each constructible CUB subset of $\kappa$. But then since $\kappa$ has uncountable cofinality in $M[0^\#]$ and $I^*$ is the countable intersection of constructible CUB subsets of $\kappa$, $C$ is almost contained in $I^*$. It follows that $M$ contains an infinite set of indiscernibles, and therefore $0^\#$ belongs to $M$.

**Lemma 7** If $\kappa$ is both $L$-regular and at least $\omega^M_2$, then the cofinality of $\kappa$ in $M$ equals its cardinality in $M$.

This is proved as follows. By the Covering Theorem ([2]), there is a constructible cofinal subset of $\kappa$ of $M$-cardinality at most the maximum of $\omega_1^M$ and the $M$-cofinality of $\kappa$. By hypothesis, the $M$-cardinality of $\kappa$ is greater than $\omega^M_1$. Therefore if the $M$-cofinality of $\kappa$ were less than the $M$-cardinality of $\kappa$, it would follow that $\kappa$ would be singular in $L$, against our hypothesis.

Now we show by induction on $\kappa$, that if $\kappa$ is an indiscernible of the form $i_{\omega_1 \cdot \alpha}$, $\alpha > 0$ then $\kappa$ is a cardinal of $M$. The base case is where $\kappa$ equals $\omega_1$ of $M[0^\#]$; clearly $\kappa$ is a cardinal of $M$ since $M$ is contained in $M[0^\#]$. The result follows immediately by induction if $\kappa$ is of the form $i_{\omega_1 \cdot \lambda}$, $\lambda$ limit. So we may assume that $\kappa$ is of the form $i_{\omega_1 \cdot \alpha}$, $\alpha > 0$. Now $\kappa$ is $L$-regular and also at least $\omega^M_2$, since by Lemma 6 the latter is the $L$-cardinal successor to $\omega_1$. Therefore by Lemma 7, the cardinality of $\kappa$ in $M$ is equal to its cofinality in $M$. In particular, the cardinality of $\kappa$ in $M$ is regular in $M$ and has cofinality $\omega_1$ in $M[0^\#]$. Now assume that $\kappa$ is not a cardinal of $M$. Then $\gamma = the$ $M$-cardinality of $\kappa$ is an $L$-regular ordinal in the interval $[i_{\omega_1 \cdot \alpha}, \kappa)$ of uncountable $M[0^\#]$-cofinality. By Lemma 6, $\gamma$ is a limit of indiscernibles, but every limit of indiscernibles in this interval, with the possible exception of $i_{\omega_1 \cdot \alpha}$, has $L[0^\#]$-cofinality $\omega$. It follows that $\gamma$ equals $i_{\omega_1 \cdot \alpha}$, and that the latter has uncountable $M[0^\#]$-cofinality. But then Lemma 6 implies that the $L$-cardinal successor to $i_{\omega_1 \cdot \alpha}$ is an $M$-cardinal, and this is a contradiction. So $\kappa$ is a cardinal of $M$.

(b) We use reverse Easton forcing. To simplify notation, assume that $V = L[0^\#]$. First consider the reverse Easton iteration with Easton supports, where $P(\leq \alpha) = P(< \alpha) * P(\alpha)$ and at an $L$-regular stage $\alpha$, $P(\alpha)$ is the forcing with finite conditions for collapsing $\alpha$ to $\omega$. Thus $P = P(< \omega_1)$
makes every ordinal less than $\omega_1$ countable in the generic extension. We
claim that there exists a $P$-generic. Notice that for any indiscernible $i < \omega_1$
there does exist a $P(< i)$-generic, because $i^+$ of $L$ is countable. And if
$j_0 < j_1 < \cdots$ is an increasing $\omega$-sequence of indiscernibles with supremum $j$,
$G_n$ is $P(< j_n)$-generic over $L$ and $G_n \subseteq G_{n+1}$ for each $n$, then $G = \cup \{ G_n \mid n \in \omega \}$ is $P(< j)$-generic over $L$, since the Mahloness of $j$ in $L$ implies that
constructible antichains in $P(< j)$ have $L$-cardinality less than $j$. It is now
straightforward to build a generic for $P$ as the union of generic
$P(< j_n)$ for countable indiscernibles $j$. In the resulting generic extension $M$,
$\omega_1^M = \omega_1$ and $\omega_2^M$ is the $L$-cardinal successor to $\omega_1$.

Now repeat the same forcing construction on the interval between the $L$-
cardinal successor to $\omega_1$ and $i \omega_1 + \omega_1$, collapsing every $L$-regular cardinal in this interval
to the $L$-cardinal successor to $\omega_1$, using conditions of size $\leq \omega_1$. As
this forcing is $\leq \omega_1$-distributive and for each indiscernible $i$ between $\omega_1$ and
$i \omega_1 + \omega_1$, $i^+$ of $L$ is the countable union of sets of $L$-cardinality $\omega_1$, there exists a $P(< i)$-generic for each indiscernible $i$ between $\omega_1$ and $i \omega_1 + \omega_1$. By taking the
union of such generics, we get a $P(< i \omega_1 + \omega_1)$-generic, which ensures that in
the resulting extension $M$, $\omega_3^M = i \omega_1 + \omega_1$ and $\omega_4^M$ is the $L$-cardinal successor
to $i \omega_1 + \omega_1$.

Continue in this way to obtain, for any fixed finite $m$, a generic extension
of $L$ in which $\omega_{2n+1} = i \omega_1 \cdot (n+1)$ and $\omega_{2n+2}$ is the $L$-cardinal successor to
$i \omega_1 \cdot (n+1)$ for each $n \leq m$. The fact that GCH holds in these models follows by
standard techniques. (One cannot achieve this for all finite $n$ simultaneously,
because defining the forcing would need an infinite sequence of indiscernibles
as a parameter, not available in the ground model $L$).

Proof of Theorem 2. (a) Let $\lambda$ be of the form $i_{\omega_1 \cdot \alpha}$, $\alpha$ limit. In the proof of
Theorem 1 we showed that each $i_{\omega_1 \cdot \alpha}$, $\alpha > 0$ is an $M$-cardinal, and therefore
$\lambda$ is a limit $M$-cardinal. As $\lambda$ is regular in $L$, it is also regular in $M$ by the
Covering Theorem, and therefore $\lambda$ is inaccessible in $M$.

(b) Assume that $V = L[\alpha]$. We show that there is a generic extension $M$ of
$L$ such that $\kappa = i_{\omega_1 \cdot \omega}$ is the least $M$-inaccessible.

First we force a CUB subset $C$ of $\kappa$, containing the $i_{\omega_1 \cdot n}$’s for $0 < n \in \omega$,
whose limit points are $L$-singular. Consider the forcing $P$ whose conditions
are bounded closed subsets of $\kappa$ with $L$-singular limit points. This forcing is
$< \kappa$-distributive. And $\kappa^+$ of $L$ is the countable union of constructible sets
\( X_n, n \in \omega \) of \( L \)-cardinality less than \( \kappa \). Moreover we can choose \( X_n \) to be \( L \)-definable from \( \kappa_n = i_{\omega_1 (n+1)} \) together with \( n \) indiscernibles \( \geq \kappa \). Thus we can build an \( \omega \)-sequence \( p_0 \geq p_1 \geq \cdots \) of conditions in \( P \) such that the union of the \( p_n \)'s is \( P \)-generic, the maximum of \( p_n \) is less than \( \kappa_{n+1} \) and \( \kappa_n \) is an element of \( p_n \). It follows that \( C = \) the union of the \( p_n \)'s is a CUB subset of \( \kappa \) containing each \( \kappa_n \) whose limit points are \( L \)-singular.

Now over this generic extension as ground model, force to collapse each \( L \)-regular \( \beta < \kappa \) not in \( C \) to \( \alpha \) of \( L \) where \( \alpha \) is \( \omega \cup \) the maximum of \( C \cap \beta \). A generic for this reverse Easton iteration can be obtained almost as in the proof of part (b) of Theorem 1, by successively choosing generics for the \( \leq \kappa_n \)-distributive part of the iteration between \( \kappa_n \) and \( \kappa_{n+1} \). However we now need a new argument to ensure that the union of the generics below the \( \kappa_n \)'s produces a generic for the entire forcing, since the Mahloness of \( \kappa \) has been destroyed. So proceed as follows: Let \( S_n \) consist of all dense constructible subsets of this forcing which are definable in \( L \) from \( n \) indiscernibles \( \geq \kappa \) together with parameters less than \( \kappa_n \). At stage \( n \) of the construction, extend what has already been chosen to a generic for the forcing up to \( \kappa_n \), and also meet all dense sets in \( S_n \). After \( \omega \) steps the result is the generic for the entire forcing.

Now let \( M \) be the generic extension of \( L \) resulting from this last forcing. Then in \( M \), the only limit cardinals less than \( \kappa \) are limit points of \( C \) and hence are \( L \)-singular. It follows that \( \kappa \) is the least \( M \)-inaccessible. By a similar construction, we can obtain, for any fixed finite \( m \), a model \( M \), generic over \( L \), in which the first \( m \) inaccessibles are the ordinals \( i_{\omega_1 \cdot n} \) for \( 1 \leq n \leq m \). Again the fact that GCH holds in these models follows by standard techniques.

**Proof of Theorems 3, 4.** (a) Note that part (a) of Theorem 3 implies part (a) of Theorem 4. We prove part (a) of Theorem 3 by induction on \( \alpha \). If \( \alpha = 0 \), we must show that \( i_{\omega_1, \beta} \) is Mahlo in \( M \) when \( \beta \) has uncountable cofinality in \( M[0^\#] \). Suppose that \( C \in M \) is CUB in \( i_{\omega_1, \beta} \); then \( C \) has an element of the form \( i_{\omega_1, \gamma} \), \( \gamma \) limit, using the uncountable \( M[0^\#]-\)cofinality of \( \beta \). By part (a) of Theorem 2, this element of \( C \) is inaccessible in \( M \), and therefore we have established the Mahloness of \( i_{\omega_1, \beta} \) when \( \beta \) has uncountable \( M[0^\#]-\)cofinality. If \( \alpha \) is a limit ordinal, then (a) follows easily by induction. Finally, suppose that \( \alpha = \alpha' + 1 \) and \( \beta \) has \( M[0^\#]-\)cofinality at least \( \omega_{\omega+2} \); we must show that \( i_{\omega_1, \beta} \) is \( \alpha' + 1 \)-Mahlo in \( M \). If \( C \in M \) is CUB in \( i_{\omega_1, \beta} \) then \( C \) has an element of the form \( i_{\omega_1, \gamma} \), where \( \gamma \) has \( M[0^\#]-\)cofinality at least
\(\omega_{\alpha'+1}\); it follows by induction that this element of \(C\) is \(\alpha'\)-Mahlo in \(M\), and therefore we have shown that \(i_{\omega_1,\beta}\) is \(\alpha'+1\)-Mahlo in \(M\), as desired.

(b) Assume \(V = L[0^\#]\). We first consider the case \(\alpha = 0\) of Theorem 3 and begin by constructing a generic extension \(M\) of \(L\) in which \(\kappa = i_{\omega_1,\omega_1}\) is the least Mahlo cardinal of \(M\). We would like to perform a reverse Easton iteration of length \(\kappa\), in which the Mahloness of cardinals less than \(\kappa\) is destroyed; unfortunately it is not possible to obtain a generic for the natural such iteration at ordinal stages of uncountable cofinality. Our solution is to first add a generic CUB \(C \subseteq \kappa\) containing no such ordinal as a limit point, ensure that all limit cardinals less than \(\kappa\) are limit points of \(C\) and finally kill the Mahloness of elements of \(\text{Lim } C\) by a reverse Easton iteration indexed by \(\text{Lim } C\).

To add \(C\), use the reverse Easton iteration \(P\) where at an \(L\)-regular stage \(\alpha \leq \kappa\), \(P(\alpha)\) either forces a closed unbounded subset of \(\alpha\) using bounded closed conditions or chooses a closed bounded subset of \(\alpha\). Then build a \(P\)-generic \(G\) meeting the following requirements:

1. At an indiscernible \(i\) not of the form \(i_{\omega_1,\alpha}\), \(\alpha > 0\), \(G(i)\) chooses the empty set.
2. At an indiscernible \(i\) of the form \(i_{\omega_1,\alpha}\), \(\alpha\) limit, \(G(i)\) is the union of \(G(i_{\omega_1,\beta})\), \(\beta < \alpha\).
3. At an indiscernible \(i\) of the form \(i_{\omega_1,(\lambda+n)}\), \(\lambda\) limit or 0, \(n \in \omega\), \(G(i)\) is taken to be the least condition \(p\) in the forcing \(P(i)\) for adding a CUB subset of \(i\) such that: (a) \(p\) extends \(G(i_{\omega_1,(\lambda+n)}) \cup \{i_{\omega_1,(\lambda+n)}\}\). (b) \(p\) meets all dense subsets of \(P(i)\) which are definable in \(L[G(<i)]\) from ordinals less than or equal to \(i_{\omega_1,(\lambda+n)}\) together with the first \(n+1\) indiscernibles greater than or equal to \(i\).

Notice that in Step 3 the maximum of \(G(i)\) is less than the least indiscernible greater than \(i_{\omega_1,(\lambda+n)}\). The result is that if \(C = G(\kappa)\) then \(\text{Lim } C \cap I\) consists of the indiscernibles of the form \(i_{\omega_1,\alpha}\), \(\alpha\) limit, \(\alpha < \omega_1\). Notice that the Mahloness of these indiscernibles, as well as of \(\kappa\), has been preserved.

Now force to collapse each \(\beta < \kappa\) not in \(C\) to \(\alpha^+\) of \(L\) where \(\alpha\) is the maximum of \(C \cap \beta\). (This is as in the proof of part (b) of Theorem 2, except it is easier here, since the Mahloness of \(\kappa\) has been preserved). The result is that each limit cardinal less than \(\kappa\) belongs to \(\text{Lim } C\), and again the Mahloness of elements of \(\text{Lim } C \cap I\) has been preserved.
Finally we force to kill the Mahloness of elements of \( \text{Lim } C \). This uses the reverse Easton iteration where at regular stages \( \alpha < \kappa \) in \( \text{Lim } C \), one forces a CUB subset of \( \alpha \) consisting of singular cardinals. It is possible to inductively choose generics \( G(\leq i) \) for \( i \in \text{Lim } C \cap I \), using the Mahloness of \( i \) to obtain the genericity of \( G(< i) \) and the countable cofinality of \( i \) to select \( G(i) \). The result is that no cardinal less than \( \kappa \) is Mahlo, and therefore \( \kappa \) is the least Mahlo cardinal in the final model \( M \). A similar argument produces a generic extension \( M \) of \( L \) (contained in \( L[0^\#] \)) in which \( \iota_{\omega_1 \cdot \omega_1}, \iota_{\omega_1 \cdot \omega_1 \cdot 2}, \ldots, \iota_{\omega_1 \cdot \omega_1 \cdot n} \) are the first \( n \) Mahlo cardinals of \( M \). The GCH can be easily verified in these models. This proves part (b) of Theorem 3 when \( \alpha = 0 \).

We next treat the case \( \alpha = 1 \) of Theorem 3. Let \( \kappa \) now denote \( \omega_2 \). We show that there is a generic extension \( M \) of \( L \) satisfying GCH such that \( \kappa \) is the least 1-Mahlo in \( M \). A similar argument will give the existence of a generic extension \( M \) of \( L \) contained in \( L[0^\#] \) and satisfying GCH in which \( \kappa, \iota_{\kappa \cdot 2}, \ldots, \iota_{\kappa \cdot n} \) are the first \( n \) 1-Mahlo cardinals, which will therefore establish part (b) of Theorem 3 when \( \alpha = 1 \).

We would like to use a reverse Easton iteration to kill the 1-Mahloness of each cardinal less than \( \kappa \), by adding CUB sets of non-Mahlos. Once again, we have difficulty choosing a generic \( G(i) \) at an indiscernible \( i < \kappa \) of uncountable cofinality. The solution is to use a \( \square_{\omega_1} \)-sequence to obtain \( G(i) \) as the union of generic \( G(\bar{i}) \), for \( \bar{i} < i \) of countable cofinality. However, we must be sure that such \( \bar{i} \) are not Mahlo, before they can be included in \( G(i) \). Thus we must also kill the Mahloness of indiscernibles of countable cofinality.

We perform the following reverse Easton iteration of length \( \kappa \): At an \( L \)-regular stage \( \alpha \) we either add only a CUB subset \( D_1(\alpha) \) of \( \alpha \) consisting of ordinals which are not Mahlo, or both \( D_1(\alpha) \) and \( D_0(\alpha) \), a CUB subset of \( \alpha \) consisting of ordinals which are not regular. Our intention is to add only \( D_1(i) \) if \( i \) is an indiscernible of uncountable cofinality and both \( D_0(i) \) and \( D_1(i) \) if \( i \) is an indiscernible of countable cofinality. To build a generic for this forcing, we make use of a \( \square_{\omega_1} \)-sequence \( (C_\alpha | \omega_1 < \alpha < \omega_2, \alpha \text{ limit}) \). Thus, \( C_\alpha \) is CUB in \( \alpha \) of ordertype at most \( \omega_1 \) and \( C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha} \) if \( \bar{\alpha} \) is a limit point of \( C_\alpha \). Also assume that if \( \alpha \) is a limit of indiscernibles then \( C_\alpha \) consists only of indiscernibles. Now inductively build generics \( G(\leq i), i \in I \cap \kappa \) with the property that if \( \bar{i} > \omega_1 \) and \( \bar{i} \) is a limit point of \( C_i \) then \( D_1(\bar{i}) \) extends \( D_1(i) \). Note that if \( C_i \) is undefined or does not have unboundedly many limit points then \( i \) has cofinality \( \omega \) and therefore it is straightforward to build \( G(i) \).
We must of course guarantee in this case that \( D_1(i) \) extends \( D_1(\bar{i}) \) where \( \bar{i} \) is the largest limit point of \( C_i \). But as \( \bar{i} \) is not Mahlo in \( G(<i) \) it can be included as an element of \( D_1(i) \). Also note that we obtain a generic \( G(<i) \) at limit stages \( i \) (and a generic \( G(\leq i) \) when \( C_i \) is unbounded in \( i \)) using the fact that for any constructible dense \( D \subseteq G(\leq i) \), there is \( \bar{i} < i \) such that \( D \cap G(\leq \bar{i}) \) is dense on \( P(\leq \bar{i}) \).

The final result is that \( \kappa \) is the least 1-Mahlo in the resulting generic extension \( M \). \( M \) can be shown to satisfy the GCH using standard techniques.

New problems arise when \( \alpha = 2 \). We must construct a generic extension \( M \) of \( L \) in which \( \kappa = \omega_3 \) is the least 2-Mahlo. As in the previous case, we can kill the Mahloness of indiscernibles of countable cofinality. We would also like to kill the 1-Mahloness of indiscernibles of cofinality \( \omega_1 \) using \( \square \); the difficulty is that the ordertype of the usual \( \square \)-sequence at an ordinal of cofinality \( \omega_1 \) can be greater than \( \omega_1 \), which makes it impossible to cohere generics along this sequence when forcing to kill 1-Mahloness. The solution is to use a different form of \( \square \).

\[ \square^\text{cof}\leq\lambda \] \( (\lambda \leq \kappa \) infinite cardinals):

There exists \( \langle C_\alpha \mid \kappa < \alpha < \kappa^+, \alpha \text{ limit, cof } \alpha \leq \lambda \rangle \) such that \( C_\alpha^* \) is CUB in \( \alpha \) of ordertype \( \leq \lambda \) and \( C_\alpha^* = C_\alpha^* \cap \alpha \) if \( \bar{\alpha} \) is a limit point of \( C_\alpha^* \).

**Lemma 8** Assume Global \( \square \): There exists \( \langle C_\alpha \mid \alpha \text{ a singular limit ordinal} \rangle \) such that \( C_\alpha \) is CUB in \( \alpha \) of ordertype \( < \alpha \) and \( C_\bar{\alpha} = C_\alpha \cap \bar{\alpha} \) if \( \bar{\alpha} \) is a limit point of \( C_\alpha \). Then \( \square^\text{cof}\leq\lambda \) holds for each \( \lambda \leq \kappa \).

This is proved as follows. For a limit ordinal \( \alpha > \lambda \) of cofinality \( \leq \lambda \), define: \( \alpha_0 = \alpha \), \( \alpha_{k+1} \) = ordertype of \( C_{\alpha_k} \) (if \( \alpha_k \) is greater than \( \lambda \)) and let \( k(\alpha) \) be least so that \( \alpha_{k+1} \leq \lambda \). Let \( f_l : \alpha_{l+1} \rightarrow \alpha_l \) be the increasing enumeration of \( C_{\alpha_l} \) for each \( l \leq k(\alpha) \). Then define \( C_\alpha^* \) to be the range of \( f_0f_1\cdots f_k \) on \( \{ \gamma < \alpha_{k+1} \mid f_k(\gamma) > \lambda \} \) if \( k(\alpha) > 0 \) and \( C_\alpha^* = C_\alpha \) otherwise. Note that the value of \( k(\alpha) \) is the same for elements of \( C_\alpha^* \) as it is for \( \alpha \). It is straightforward to verify that the \( C_\alpha^* \)'s for \( \kappa < \alpha < \kappa^+ \) provide a \( \square^\text{cof}\leq\lambda \)-sequence, using the coherence properties of the given Global \( \square \)-sequence and the fact that \( k(\alpha) = k(\bar{\alpha}) \) for \( \bar{\alpha} \) in \( C_\alpha^* \).

As Global \( \square \) holds in \( L[R] \) for each real \( R \), it follows that \( \square^\text{cof}\leq\lambda \) holds in \( V = L[0^\#] \) for each pair of infinite cardinals \( \lambda \leq \kappa \).
Now we present the construction of the desired model $M$ in which $\kappa = \omega_3$ is the least 2-Mahlo, using a $\square_{\omega_2}^{\omega_2}$-sequence. Let $\langle C_\alpha^* \mid \omega_1 < \alpha < \omega_3, \alpha \text{ limit, cof } \leq \omega_1 \rangle$ be such a sequence, and for limit $\alpha \in (\omega_1, \omega_2)$ let $C_\alpha^*$ be $C_\alpha$, where $\langle C_\alpha \mid \alpha \text{ limit, } \omega_1 < \alpha < \omega_2 \rangle$ witnesses $\square_{\omega_1}$. Also let $\langle C_\alpha \mid \alpha \text{ limit, } \omega_2 < \alpha < \omega_3 \rangle$ witness $\square_{\omega_2}$. We assume that if both $C_\alpha^*$ and $C_\alpha$ are defined then $C_\alpha^* \subseteq C_\alpha$, and if $i$ is a limit of indiscernibles then $C_i^*$ and $C_i$ (when defined) consist only of indiscernibles.

At an $L$-regular stage $\alpha$ we either add only $D_2(\alpha)$, a CUB subset of $\alpha$ consisting of ordinals which are not 1-Mahlo, or both $D_2(\alpha)$ and $D_1(\alpha)$, a CUB subset of $\alpha$ consisting of ordinals which are not Mahlo, or $D_2(\alpha)$, $D_1(\alpha)$ and $D_0(\alpha)$, a CUB subset of $\alpha$ consisting of ordinals which are not regular. Our intention at an indiscernible stage $i$ is to add $D_0(i)$ only if $i$ has countable cofinality and $D_1(i)$ only if $i$ has cofinality at most $\omega_1$. To build a generic for this forcing, we respect the coherence properties: If $\tilde{i}$ is a limit point of $C_i^*$ then $D_1(i)$ and $D_2(i)$ extend $D_1(\tilde{i})$ and $D_2(\tilde{i})$, respectively; if $\tilde{i}$ is a limit point of $C_i$ then $D_2(i)$ extends $D_2(\tilde{i})$. We must guarantee that when $C_i^*$ has a largest limit point $\tilde{i}$, then $\tilde{i}$ can be included in $D_1(i)$ and $D_2(i)$. But $\tilde{i}$ has countable cofinality and therefore is not Mahlo in $G(< i)$; it follows that $\tilde{i}$ can be put into $D_1(i)$ and $D_2(i)$. We must also ensure that when $C_i$ has a largest limit point $\tilde{i}$, then $\tilde{i}$ can be included in $D_2(i)$. But in this case $\tilde{i}$ has cofinality $\leq \omega_1$ and therefore is not 1-Mahlo in $G(< i)$; it follows that $\tilde{i}$ can be put into $D_2(i)$. As before we obtain generics at limit stages, and in the resulting model $M$, $\kappa$ is the least 2-Mahlo cardinal.

The general case of Theorem 3 is based upon the previous one. First assume that there is no inaccessible in $V = L[\theta]$. Define the function $F$ on indiscernibles by $F(i) = \alpha$ where $\omega_\alpha$ is the cofinality of $i$. Note that $F(i) < i$ is either 0 or a successor ordinal for every $i$, as we have assumed that there are no inaccessibles.

Now we describe a reverse Easton iteration designed to guarantee that $i$ is not $F(i)$-Mahlo for each indiscernible $i$. At an $L$-regular stage $\alpha$ of this iteration, we choose $\beta(\alpha) \in \{-1\} \cup \alpha$ and let $P(\alpha, \beta)$ be the forcing to add a CUB subset $C(\alpha, \beta)$ of $\alpha$ consisting of ordinals which are not $\beta$-Mahlo, for each $\beta$ in the interval $[\beta(\alpha), \alpha)$. (When $\beta$ is $-1$, we interpret $\beta$-Mahlo to mean regular.) Then $P(\alpha)$ is the forcing obtained by taking the product of these $P(\alpha, \beta)$’s, with $< \alpha$ support. Our intention is to choose $\beta(\alpha)$ to be $F(i) - 1$ at each indiscernible $i$. 

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We now build a generic for this iteration, with the desired choices of \( F(i) \) for \( i \in I \). Let \( \langle C_\alpha \mid \alpha \text{ limit} \rangle \) be a Global \( \square \)-sequence such that \( C_\alpha \) has ordertype at most the cardinality of \( \alpha \) for each \( \alpha \) and if \( i \) is a limit of indiscernibles then \( C_i \) consists only of indiscernibles. Now as in the proof of Lemma 8, if \( i \) is an indiscernible which is singular, define \( i_0 = i \), \( i_{k+1} = \text{ordertype } C_{i_k} \) if \( i_k \) is singular, \( k(i) \) the least \( k \) such that the ordertype of \( C_{i_k} \) is equal to \( i \). (The \( i_k \)'s for \( k > 0 \) need not be indiscernibles.) Let \( C_0^i \) be \( C_i \) and for \( k < k(i) \), let \( f_k : i_{k+1} \rightarrow i_k \) be the increasing enumeration of \( C_{i_k} \) and \( C^k_{i+1} \) the image of \( C_{i+1} \) under the composition \( f_0 f_1 \cdots f_k \). Also for \( k \leq k(i) \) let \( \alpha_k(i) \) be the \( \alpha \) such that the ordertype of \( C_{i_k} \) has cardinality \( \omega_\alpha \).

Our generics \( G(\leq i) \) for \( P(\leq i) \) are defined by induction on \( i \in I \) to have the following coherence property: If \( \tilde{i} \) is a limit point of \( C^k_i \) then \( G(i, \beta) \) extends \( G(\tilde{i}, \beta) \) for \( \alpha_k(i) \leq \beta < \tilde{i} \). To choose \( G(i) \), we must first require that \( G(i, \beta) \) extend \( G(\tilde{i}, \beta) \) for \( \tilde{i} \) a limit point of \( C_i \) and \( \alpha_0(i) \leq \beta < \tilde{i} \). For this to be possible we need to know that \( \tilde{i} \) is not \( \beta \)-Mahlo in \( L[G(< \tilde{i})] \); but \( \tilde{i} \) is not \( F(\tilde{i}) \)-Mahlo in this model, and therefore not \( \beta \)-Mahlo since \( F(\tilde{i}) \leq \alpha_0(i) \leq \beta \). For the same reason we may require that \( G(i, \beta) \) extend \( G(\tilde{i}, \beta) \) for \( \tilde{i} \) a limit point of \( C^k_i \) and \( \alpha_k(i) \leq \beta < \tilde{i} \), for each \( k \leq k(i) \). Therefore the desired \( G(i) \) can be chosen, when some \( C^k_i \) has a largest limit point. If no \( C^k_i \) has a largest limit point then by induction \( G(i) \) can be chosen as the union of the \( G(\tilde{i}) \), for \( \tilde{i} \) a limit point of \( C_i \) (or arbitrarily if \( C_i \) has ordertype \( \omega \)).

So there exists a generic for this iteration. If we restrict this iteration to \( \omega_{\gamma+1} \), where \( \gamma \) is not inaccessible and the \( \beta(\alpha) \) are required to take values less than \( \gamma \), then we produce a model \( M \) in which \( \omega_{\gamma+1} \) is the least \( \gamma \)-Mahlo. By a similar argument we can arrange that \( \omega_{\gamma+1}, \omega_{\gamma+1}, \ldots, \omega_{\gamma+1}, \ldots, \omega_{\gamma+1}, n \) are the first \( n \) \( \gamma \)-Mahlos. This proves part (b) of Theorem 3. If there is no inaccessible, then we have obtained a generic extension \( M \) of \( L \) in which no cardinal \( \kappa \) is \( \kappa \)-Mahlo. And if the above iteration is restricted to the least inaccessible, then we obtain a model \( M \) such that the least \( \kappa \) which is \( \kappa \)-Mahlo in \( M \) is the least inaccessible. By a similar argument we can arrange that the first \( n \) cardinals \( \kappa \) which are \( \kappa \)-Mahlo in \( M \) are the first \( n \) incomparables. This proves part (b) of Theorem 4. \( \square \)

**Proof of Theorem 5.** Suppose that \( M \) is a generic extension of \( L \) obtained by the forcing \( P \), and \( M \) is contained in \( L[0^\#] \). If \( \varphi \) is true in \( M \), then \( \varphi \) is forced by some condition in \( P \); we may assume that \( \varphi \) is forced by the weakest condition of \( P \). By assumption, there is an \( \omega + \omega \)-Erdős cardinal...
in $L[0#] = L[M, 0#]$. It follows from the results of [1] that $P$ has a generic relative to which a periodic subclass of $I$ is a class of indiscernibles, and this generic can be generically coded by a real which belongs to a set-generic extension of $L[0#]$. Since $\varphi$ is a parameter-free sentence there is such a real in $L[0#]$, by absoluteness. It follows that $\varphi$ is true in an inner model of $L[R]$ for some real $R$ such that $R#$ exists. □

**References**
