The Higher Descriptive Set Theory of Isomorphism 2010

1.-2.Vorlesungen

Introduction

These lectures relate two approaches to classifying countable first-order theories.

The "standard" way of doing this is through Shelah's stability theory, which divides theories into the *classifiable* ones, for which a structure theory exists for the uncountable models, and the *unclassifiable* ones, for which chaos takes over. Roughly speaking, the classifiable theories are the ones for which there are fewer than the maximum number 2^κ of models of size κ for some uncountable κ. (This is not quite true; the classifiable *deep* theories still have the maximum number of models in all uncountable cardinals.) Shelah's "Main Gap" analysis shows that classifiability is equivalent to a conjunction of very absolute model-theoretic conditions: superstable without the DOP (dimensional order property) and without the OTOP (omitting types order property).

An alternative classification arises from Higher Descriptive Set Theory. Assume GCH and suppose that κ is uncountable and regular. Then the Baire space ω^ω generalises nicely to κ^κ, where points are functions $f : \kappa \to \kappa$ and basic open sets are of the form $N_p = \{f \mid p \subseteq f\}$ for some p in $\kappa^{<\kappa}$. The (generalised) Borel sets are obtained from the basic open sets by closing under unions of size κ and complements. A function $f : X_0 \to X_1$ between Borel sets is Borel iff the pre-image under f of each Borel set is Borel. As in the classical case, one can also introduce a Borel-reducibility $\leq_B$ between equivalence relations E_0, E_1 on κ^κ:

$E_0 \leq_B E_1$ iff for some Borel f, xE_0y iff $f(x)E_1f(y)$ for all x, y

For a first-order theory T let Mod_T^κ denote the set of models of T with universe κ; this can be viewed as a Borel set. Also let Isom_T^κ denote the isomorphism relation on Mod_T^κ; this is a Σ_1^1 equivalence relation. Our second way of classifying theories T is to look at the complexity of the equivalence relation Isom_T^κ under Borel reducibility $\leq_B$.

Our main goal is to establish the following result, which relates the model-theoretic and set-theoretic methods of classification to each other. Missing definitions will be supplied later.

Theorem 1 *(F-Hyttinen-Kulikov) Assume GCH and let κ be uncountable and regular.*
1. T is classifiable and shallow (in Shelah's sense) iff $Isom_T^\kappa$ is Borel.
2. In L, T is classifiable iff $Isom_T^\kappa$ is Δ_1^1.
3. T is classifiable iff for all regular $\lambda < \kappa$, $E_\lambda^\kappa \not\leq_B Isom_T^\kappa$, where E_λ^κ is the equivalence relation of equality modulo λ-nonstationarity.

It is important that κ be uncountable in this theorem. For example, the theory DLO (dense linear orders without endpoints) is unclassifiable, but the equivalence relation Isom_T^ω has only one equivalence class and therefore is trivially Borel! Martin Koerwien found an example in the other direction: There is a theory which is very nice model-theoretically (ω-stable of "depth" 2, and therefore very classifiable) such that Isom_T^ω is *not* Borel. So from now on, κ is always assumed to denote an *uncountable* regular cardinal.

Higher Descriptive Set Theory

Before bringing in the model theory, we'll first take a look at descriptive set theory on the generalised Baire space κ^κ for an uncountable regular cardinal κ. There are strong differences between this theory and classical descriptive set theory on the classical Baire space ω^ω. Also a number of simple questions are undecidable in ZFC and there remain many open problems. We assume GCH throughout.

We have defined the basic open sets in κ^κ to be the sets of the form $N_p = \{f \in \kappa^\kappa \mid p \subseteq f\}$, where $p \in \kappa^{<\kappa}$. The Borel sets are obtained by closing the class of basic open sets under complements and unions of size κ.

A nice way to think of Borel sets is as follows: By a *tree* we mean a partial order in which the predecessors of any point are wellordered. Let T be a tree of size κ which is wellfounded (i.e., has no infinite branch) and let l be a function which assigns a basic open set to each terminal node of T. Now for any $x \in \kappa^\kappa$ consider the game $G(T, l, x)$ with two players I and II, in which player I begins by choosing a minimal node of T and then the players take turns choosing immediate successors of the nodes previously chosen until a terminal node s is reached. Then Player II wins iff x belongs to the basic open set $l(s)$. A set B is Borel iff for some wellfounded tree T of size κ, we have: $x \in B$ iff II has a winning strategy in the game $G(T, l, x)$. Indeed, if B is of this form then B is Borel, as is shown by induction on the rank of the tree T. Conversely, the collection of sets B of this form contains all basic open sets, is closed under complementation (add a new bottom node, replace each basic open set by its complement, an open set, and switch players) and is closed under intersections of size κ.

The product spaces $(\kappa^\kappa)^n$ are defined in the usual way, using the corresponding product topologies. The ("boldface") Σ^1_1 sets are the projections of closed sets, i.e., the sets of the form $\{f \in \kappa^\kappa \mid (f, g) \in C \text{ for some } g\}$ where C is a closed subset of $(\kappa^\kappa)^2$. A set is Π^1_1 iff its complement is Σ^1_1 and is Δ^1_1 iff it is both Σ^1_1 and Π^1_1.

Proposition 2 *Borel sets are Δ^1_1, but not conversely.*

Proof. If B is Borel then for some wellfounded tree T of size κ and labelling l of its terminal nodes by basic open sets, x belongs to B iff there exists a winning strategy for Player II in the game $G(T, l, x)$. It is not difficult to write the latter condition on x as the projection of a closed set. This shows that Borel sets are Σ^1_1 and therefore also Δ^1_1.

The converse fails because there is a Δ^1_1 set $D \subseteq (\kappa^\kappa)^2$ which is *universal* for the Borel subsets of κ^κ, i.e., the Borel sets are exactly the sets of the form

$$D_\xi = \{\eta \mid (\eta, \xi) \in D\}$$

for ξ ranging over κ^κ. Given such a universal set, we obtain a Δ^1_1 non-Borel set D^0 through diagonalisation: $D^0 = \{\eta \mid (\eta, \eta) \notin D\}$.

To obtain D, we choose a coding $((T_\xi, l_\xi) \mid \xi \in \mathcal{C})$, where $\mathcal{C} \subseteq \kappa^\kappa$ is a Borel set of "codes", of all pairs (T, l) where T is a wellfounded tree of size κ and l is a labelling of its terminal nodes by basic open sets. Without the requirement of wellfoundedness this is easy to do, even for $\kappa = \omega$. The fact that κ is uncountable is needed to know that the set of $\xi \in \kappa^\kappa$ which code a *wellfounded* binary relation on κ forms the set of branches of length κ through a subtree of κ^κ and is therefore a closed set. Then we define:

$(\eta, \xi) \in D$ iff
$\xi \in \mathcal{C}$ and η belongs to the Borel set determined by the pair (T_ξ, l_ξ) iff
$\xi \in \mathcal{C}$ and Player II has a winning strategy in $G(T_\xi, l_\xi, \eta)$ iff
$\xi \in \mathcal{C}$ and Player I does not have a winning strategy in $G(T_\xi, l_\xi, \eta)$.

As the set of (codes for) winning strategies is a closed set, D is the desired Δ^1_1 set which is universal for Borel sets. □

Proposition 3 *For $A \subseteq \kappa^\kappa$ the following are equivalent:*
1. A is Σ^1_1, i.e. the projection of a closed set in $(\kappa^\kappa)^2$.
2. A is the projection of a Borel set.
3. A is the projection of a Σ^1_1 set.
4. A is the continuous image of a closed set.

Proof. As Closed $\subseteq$ Borel $\subseteq \Sigma^1_1$ and the projection map is continuous, we need only check $4 \to 1$. But the image of a continuous map f on a closed set is the projection of the graph of f and the graph of f is closed. □

The collection of Σ^1_1 sets is very rich, as is illustrated by the next proposition. Again the key fact is that wellfoundedness is a Borel and hence Δ^1_1 property for uncountable κ.

Proposition 4 *A subset of κ^κ is Σ^1_1 iff it is definable over $H(\kappa^+)$ by a Σ_1 formula (with parameters).*

Proof. The direction from left to right is clear. Conversely, suppose that $\varphi(\xi)$ is a Σ_1 formula with parameters in $H(\kappa^+)$ and let $\xi_0 \in \kappa^\kappa$ code the parameters. Then $\varphi(\xi)$ holds iff there exists $\eta \in \kappa^\kappa$ which codes a wellfounded transitive model M of ZF^- containing ξ, ξ_0 such that M satisfies $\varphi(\xi)$. The latter condition on ξ is the projection of a Borel set and therefore is Σ^1_1. □

Remark. It can also be shown that B is Borel iff for some fixed transitive ZF^- model M containing κ and formula $\varphi(\xi)$ with parameters from M, $\xi \in B$ iff $M[\xi] \vDash \varphi(\xi)$.

3.-4.Vorlesungen

Regularity Properties

The three classical regularity properties are the BP (Baire Property), the PSP (Perfect Set Property) and LM (Lebesgue Measurability). As we know of no good analogue of LM, we consider just BP and PSP.

The Baire Property

A set is *nowhere dense* iff its closure has no interior (i.e., has no nonempty open subset). A set is *meager* iff it is the union of κ-many nowhere dense sets. A *comeager* set is the complement of a meager set.

Proposition 5 *(Baire Category Theorem) Comeager sets are dense.*

Proof. Chasing the definitions, this says that the intersection of κ-many open dense sets is dense. This is easy to prove: List the open dense sets as $(U_i \mid i < \kappa)$ and below a given $p \in \kappa^{<\kappa}$ successively extend to $p_i \in \kappa^{<\kappa}$ so that N_{p_i} is contained in U_i and the p_i's converge to an $x \in \kappa^\kappa$. Then x belongs to all of the U_i's. $\square$

A set X *has the BP* iff its symmetric difference with some open set is meager.

Proposition 6 *Borel sets have the BP.*

Proof. The collection of sets with the BP contains all basic open sets, is closed under complement and is closed under unions of size κ. $\square$

Theorem 7 *(Halko-Shelah) There are Σ^1_1 sets without the BP.*

Proof. Let CUB denote the filter of closed unbounded sets, which can be viewed as a subset of κ^κ by considering the set of ξ such that $X_\xi = \{\alpha < \kappa \mid \xi(\alpha) = 0\}$ contains a closed unbounded set. Clearly CUB is Σ^1_1. If CUB had the BP then it would be either meager or comeager on some basic open set N_p. But consider any κ-intersection $\bigcap_{i<\kappa} D_i$ of dense open subset D_i of N_p. Extend p κ times to $(p_i \mid i < \kappa)$ of lengths $(\alpha_i \mid i < \kappa)$ where $p_0 = p$, $N_{p_{i+1}}$ is contained in D_i and p_{i+1} extends p_i, and for limit $\lambda < \kappa$, p_λ extends the p_i, $i < \lambda$ and $p_\lambda(\alpha_\lambda) = 1$. Then $\xi =$ the union of the p_i's belongs to each D_i yet X_ξ is nonstationary. This shows that CUB is not comeager on N_p. Similarly, by replacing 1 by 0 we have that CUB is not meager on N_p. $\square$

The above argument also shows that for any regular $\lambda < \kappa$, CUB_λ does not have the BP where the latter denotes all subsets of κ which contain λ-closed unbounded subsets.

Do Δ^1_1 sets have the BP?

Theorem 8 *(a) In L, there is a Δ^1_1 set without the BP.*
(b) It is consistent that all Δ^1_1 sets have the BP.

Proof. (a) In L there is a Δ^1_1 wellorder of κ^κ: $\xi <_L \eta$ iff ξ appears before η in the L-hierarchy iff there exists a transitive model M of ZF^- of with $\xi, \eta \in M$ such that $M \vDash \xi <_L \eta$. Now using this wellorder it is easy by diagonalisation to construct in κ^+ steps a Δ^1_1 set A such that for each p and comeager subset C of N_p, both $C \cap A$ and $C \setminus A$ are nonempty. It follows that A does not have the BP.
(b) Add κ^+-many κ-Cohen's to L with a size $< \kappa$ supported product. Then in this model BP holds for Δ^1_1: Let the extension be $L[G]$. Now suppose that φ_0, φ_1 are Σ^1_1 formulas with parameters in $L[G(< i)]$ which in $L[G]$ define a partition of κ^κ into two disjoint pieces, $i < \kappa^+$. Note that G factors as $G(< i) \times G[i, \kappa^+)$. Let x be $G(i)$, which is κ-Cohen over $L[G(< i)]$. Then for some $j < \kappa^+$, either $\varphi_0(x)$ or $\varphi_1(x)$ holds in $L[G(< i)][x][y]$ where $y = G[i+1, j)$ is equivalent to a κ-Cohen over $L[G(< i)][x]$. Assume that it is $\varphi_0(x)$ that holds. But $L[G(< i)][x]$ is Σ^1_1 elementary in $L[G(< i)][x][y]$: if p forces σ to name a witness to a Σ^1_1 formula $\varphi(x)$ in $L[G(< i)][x][y]$ then $L[G(< i)][x]$ contains a y' extending p which is κ-Cohen over an $L_\alpha[G(< i)][x]$ which is large enough to contain the name σ. So $\varphi(x)$ holds in $L[G(< i)][x]$. By the same argument, $\varphi_0(x')$ holds in $L[G(< i)][x']$ for all x' extending some condition p which are sufficiently κ-Cohen generic over $L[G(< i)]$, and this forms a comeager set below p in $L[G]$. By the persistence of Σ^1_1 formulas, $\varphi_0(x')$ holds in $L[G]$ for all such x'. So the original Δ^1_1 set defined by φ_0, φ_1 is comeager on some open set and therefore has the BP. $\square$

Remark. In L, CUB_λ is not Δ^1_1 for any regular $\lambda < \kappa$. But it is consistent that CUB_λ is Δ^1_1 for all regular $\lambda < \kappa$, giving another proof of (b) above.

The Perfect Set Property

T is a *perfect subtree of* $2^{<\kappa}$ iff each node of T splits and T is closed under increasing sequences of length $< \kappa$. Equivalently: T is the downward closure of the range of an injective, order-preserving and continuous function from the full tree $2^{<\kappa}$ into itself.

Lemma 9 *Suppose T is perfect. Then for CUB-many $\alpha < \kappa$, T has card(2^α) nodes of length α.*

Proof. Let T be the downward closure of the range of $f : 2^{<\kappa} \to 2^{<\kappa}$ where f is injective, order-preserving and continuous. If κ is inaccessible then for CUB-many $\alpha < \kappa$, $f(\sigma)$ has length less than α whenever σ has length less than α and therefore by continuity $f(\sigma)$ has length α whenever σ has length α. This gives card(2^α) many nodes in T of length α.

If $\kappa = \lambda^+$ then we must show that there are $2^\lambda = \kappa$ nodes of T of some fixed length $\alpha < \kappa$. If not, then let $\alpha_0 < \alpha_1 < \cdots$ be a λ-sequence such that every node of T of length α_i has incompatible extensions of length α_{i+1}. If α is the supremum of the α_i's then T has 2^λ nodes of length α. $\square$

For any subtree T of $2^{<\kappa}$ we let $[T]$ denote the set of κ-branches of T, a closed subset of 2^κ (where 2^κ is viewed as a subpace of κ^κ). A subset of 2^κ is *perfect* iff it is of the form $[T]$ for some perfect subtree T of $2^{<\kappa}$.

Perfect sets are nonempty, closed and contain no isolated points. However, as we see below, the converse can fail.

$X \subseteq 2^\kappa$ has the *perfect set property (PSP)* iff it either has size at most κ or contains a perfect set, i.e. a set of the form $[T]$ for some perfect tree T. Trivially, open sets have the PSP, as they are either empty or contain a basic open set, and basic open sets are perfect.

The PSP can fail for closed sets. A *quasi Kurepa tree* on κ is a subtree T of $2^{<\kappa}$ with the following properties:

1. Every node of T splits and can be extended to a κ-branch.
2. T has more than κ many κ-branches.
3. For stationary many $\alpha < \kappa$, T has at most α-many nodes of length α.

If T is quasi Kurepa then $[T]$ does not have the PSP by Lemma 9.

Theorem 10 *In L, there is a quasi Kurepa tree on κ for every regular uncountable κ.*

Proof. Assume $V = L$. For each singular *ordinal* $\alpha < \kappa$ let $\beta(\alpha)$ denote the least limit ordinal $\beta > \alpha$ such that α is singular in L_β. Then $\beta(\alpha)$ is defined and less than α^+ for singular α. Now let T be the tree of all $s \in 2^{<\kappa}$ such that:

1. $C(s) = \{\alpha < |s| \mid s(\alpha) = 1\}$ is closed in $|s|$.
2. If $\alpha < |s|$ is a singular limit point of $C(s)$ then $s \restriction \alpha$ belongs to $L_{\beta(\alpha)}$.
3. If $\kappa = \lambda^+$ for some cardinal λ, then $s(\alpha) = 0$ for all $\alpha < \lambda$.

Then $[T]$ consists of all (characteristic functions of) closed subsets C of κ such that $C \cap \alpha$ belongs to $L_{\beta(\alpha)}$ for all singular limit points α of C and if $\kappa = \lambda^+$ for some cardinal λ, then $C \cap \lambda = \emptyset$.

Clearly each node of T splits and can be extended to a κ-branch. If κ is inaccessible then for all singular cardinals $\alpha < \kappa$, T has at most card(α) nodes of length α, as each of these nodes belongs to $L_{\beta(\alpha)}$. If $\kappa = \lambda^+$ for some cardinal λ then for all $\alpha < \kappa$, T has at most card(α) nodes of length α, as for $\alpha \leq \lambda$ there is at most one such node and for α in (λ, κ) all such nodes belong to $L_{\beta(\alpha)}$ by induction.

So to verify that T is a quasi Kurepa tree we need only show that T has κ^+ many branches.

For any α in (κ, κ^+) let C_α be the CUB set of $\bar{\kappa} < \kappa$ such that $\bar{\kappa} = H(\bar{\kappa}) \cap \kappa$, where $H(\bar{\kappa})$ is the Σ_1 Skolem hull of $\bar{\kappa} \cup \{\kappa, \alpha\}$ in $L_{\beta(\alpha)}$. If $\bar{\kappa}$ is a singular limit point of C_α then $\bar{\kappa}$ is regular in the transitive collapse of $H(\bar{\kappa})$ and $C_\alpha \cap \bar{\kappa}$ is definable over that transitive collapse; it follows that $C_\alpha \cap \bar{\kappa}$ belongs to $L_{\beta(\bar{\kappa})}$. So C_α is a branch through T. As each CUB set in $L_{\beta(\alpha)}$ almost contains C_α, it follows that C_α does not belong to $L_{\beta(\alpha)}$ and therefore there are κ^+ many distinct such C_α's. $\Box$

5.Vorlesung

Kurepa trees are defined just like quasi Kurepa trees but with the stronger requirement that $\{\alpha < \kappa \mid T$ has at most α-many nodes of length $\alpha\}$ contain all infinite ordinals less than κ. Silver showed long ago that to kill Kurepa trees on κ one needs an inaccessible, so to get the PSP for closed sets one needs an inaccessible.

Theorem 11 *Starting with an inaccessible above κ , one can force the PSP to hold for all Σ^1_1 subsets of 2^κ.*

Proof. We use the same model that Silver used to kill Kurepa trees. Start with L where $\kappa < \lambda$ is inaccessible and let $P = \text{Coll}\ (\kappa, < \lambda)$ denote the forcing that makes λ into κ^+ with conditions of size $< \kappa$. Thus a condition is a function $p : \text{Dom}\ (p) \to \lambda$ where Dom (p) is a size $< \kappa$ subset of $\lambda \times \kappa$ and $p(\alpha, \gamma) < \alpha$ for each (α, γ) in Dom (p).

Suppose now that $X \subseteq 2^\kappa$ in the resulting model $L[G]$ is Σ^1_1 with parameters coming from the inner model $L[G|\alpha]$, $\alpha < \lambda$, α regular in L. Factor $L[G]$ as $L[G|\alpha][G[\alpha, \lambda)]$ where $G|\alpha$ is generic for $P|\alpha$ and $G[\alpha, \lambda)$ is generic over $L[G|\alpha]$ for $P[\alpha, \lambda)$. We may assume that X has more than κ many elements and therefore can choose an element a of X in $L[G] \setminus L[G|\alpha]$. Also choose an L-regular β between α and λ so that the membership of a in X is witnessed inside $L[G|\beta]$. Let $\dot{a}$ be a $P[\alpha, \beta)$-name for a in the ground model $L[G|\alpha]$ and p a condition in $P[\alpha, \beta)$ which forces that $\dot{a}$ is an element of X not in $L[G|\alpha]$.

Now in $L[G]$ build a perfect tree U of conditions below p in the κ-closed forcing $P[\alpha, \beta)$ such that each κ-branch of U hits each maximal antichain of $P[\alpha, \beta)$ which belongs to $L[G|\alpha]$ (there are only κ-many) and distinct branches of U force different interpretations of the name $\dot{a}$. Thus each κ-branch x through U yields a pair (g_x, a_x) where g_x is $P[\alpha, \beta)$-generic over $L[G|\alpha]$, a_x is the interpretation of $\dot{a}$ given by g_x and the membership of a_x in X is witnessed in $L[G_\alpha][g_x]$. The set of such a_x's forms a perfect subset of X. □

Can the PSP hold for all Π^1_1 sets? The answer is not known.

Borel Reducibility

As mentioned earlier, to obtain a provable characterisation of classifiable theories in terms of the complexity of their isomorphism relations on uncountable models we must use a generalisation of the theory of Borel reducibility. I'll introduce now such a generalisation and prove some basic facts about it.

Suppose that X_0, X_1 are Borel subsets of κ^κ. Then $f : X_0 \to X_1$ is a *Borel function* iff $f^{-1}[Y]$ is Borel whenever Y is Borel. This implies that the graph of f is Borel, as (x, y) belongs to the graph of f iff for all $s \in \kappa^{<\kappa}$, either y does not belong to N_s or x belongs to $f^{-1}[N_s]$.

If E_0, E_1 are equivalence relations on Borel sets X_0, X_1 respectively then we say that E_0 is *Borel reducible to* E_1, written $E_0 \leq_B E_1$, iff for some Borel $f : X_0 \to X_1$:

$$x_0 E_0 y_0 \text{ iff } f(x_0) E_1 f(x_1).$$

Now recall the following picture from the classical case:

$$1 <_B 2 <_B \cdots <_B \omega <_B \mathrm{id} <_B E_0$$

forms an initial segment of the Borel equivalence relations under $\leq_B$ where n denotes an equivalence relation with n classes for $n \leq \omega$, id denotes equality on ω^ω and E_0 denotes equality modulo finite on ω^ω.

At κ we easily get the initial segment

$$1 <_B 2 <_B \cdots <_B \omega <_B \omega_1 <_B \cdots <_B \kappa$$

where for each nonzero cardinal $\lambda \leq \kappa$ we identify λ with the $\equiv_B$ class of Borel equivalence relations with exactly λ-many classes. What happens above these equivalence relations? We might hope for:

Silver Dichotomy The equivalence relation id (equality on κ^κ) is the strong successor of κ under $\leq_B$, i.e., if a Borel equivalence relation E has more than κ classes then id is Borel-reducible to E.

6.Vorlesung

Theorem 12 *(a) The Silver Dichotomy implies the PSP for Borel sets. Therefore it fails in L and its consistency requires at least an inaccessible cardinal.*
(b) The Silver Dichotomy is false with Borel replaced by Δ^1_1.

Proof. (a) Assume the Silver Dichotomy and suppose that B is Borel with more than κ elements. Consider the equivalence relation xEy iff $x = y$ or $x, y \notin B$. Applying the Silver Dichotomy we get a Borel reduction of id to E; as all but one point in the range of this reduction belongs to B, we get a Borel, injective map from κ^κ into B and therefore also a Borel, injective map from 2^κ into B. If this map were continuous then its range would be a perfect subset of B, verifying the PSP for B. We finish the proof as follows:

Lemma 13 *(a) Any Borel function $f : \kappa^\kappa \to \kappa^\kappa$ is continuous on a comeager set.*
(b) Any comeager set contains a perfect subset.

Proof. (a) Recall that each Borel set has a *Borel code* (T, l), where T is a wellfounded tree of size κ and l is a labelling of the terminal nodes of T by basic open sets. As f is Borel it has a Borel graph and so there is a Borel code for its graph. Now let M be a transitive ZF^- model of size κ containing $H(\kappa)$ and this Borel code. Let X be the set of $x \in \kappa^\kappa$ which are generic over M for the forcing $\kappa^{<\kappa}$ (equivalent to κ-Cohen forcing). Then X is comeager as it is the intersection of the κ-many open dense sets coded in M. And the function f is continuous on X as for any $s \in \kappa^{<\kappa}$, any true statement "$s \subseteq f(x)$" is forced by a condition $t \subseteq x$ and therefore $s \subseteq f(y)$ holds for all $y \in X$ extending t, as such y are generic over M.
(b) This is just like the proof that comeager sets are dense. Let $(D_i \mid i < \kappa)$ be open dense and build a perfect tree T so that if σ is on the i-th splitting level of T then N_σ is contained in D_j for all $j < i$. This is possible using the density of the D_i's. Then every cofinal branch through T belongs to the intersection of the D_i's. $\square$

This completes the proof of (a) of the Theorem.

(b) Consider the equivalence relation xEy iff x, y code the same ordinal or x, y do not code ordinals. Then E has κ^+ many equivalence classes. Also E is Δ^1_1: We have seen (using $\kappa > \omega$) that wellfoundedness is Borel; for x, y coding ordinals we have xEy iff there exists order-preserving maps between the ordinals coded by x, y iff there is no order-preserving map from the ordinal coded by x into a proper initial segment of the ordinal coded by y (or vice-versa).

If id Borel-reduces to E then as above we get a perfect set $[T]$ such that distinct elements of $[T]$ code distinct ordinals. Let M be a transitive ZF^- model of size κ containing $H(\kappa)$ and the perfect tree T and suppose that x is a branch through T which is generic over M (for the forcing whose conditions are the nodes of T). Then the ordinal coded by x belongs to M as $M[x]$ satisfies ZF^-. But there are 2^κ many such x's, contradicting the fact that different such x's must code different ordinals. $\square$

Is the Silver Dichotomy consistent? This question remains open.

We can also consider what happens above id. In the case $\kappa = \omega$ we have:

Classical Glimm-Effros Dichotomy $E_0 =$ (equality mod finite) is the strong successor of id, i.e., if a Borel equivalence relation E is not Borel-reducible to id (i.e., E is not *smooth*) then E_0 Borel-reduces to E.

At κ, what shall we take E_0 to be? For infinite regular $\lambda \leq \kappa$, define $E_0^{<\lambda} =$ equality for subsets of κ modulo sets of size $< \lambda$.

Proposition 14 *For $\lambda < \kappa$, $E_0^{<\lambda}$ is Borel bireducible with id.*

So we can forget about $E_0^{<\lambda}$ for $\lambda < \kappa$ and set $E_0 = E_0^{\kappa}$, equality modulo bounded sets.

Proof. For each ordinal $\alpha < \kappa$ consider the equivalence relation on subsets of α given by: $x \equiv_\alpha y$ iff $x \triangle y$ has size $< \lambda$ (where $\triangle$ denotes symmetric difference). For each $\alpha < \kappa$ choose a selector f_α for $\equiv_\alpha$, i.e. a (noneffective) function reducing $\equiv_\alpha$ to equality on 2^α. So for x, y subsets of α, $f_\alpha(x)$ equals $f_\alpha(y)$ iff $x \equiv_\alpha y$. This is possible by the axiom of choice. The sequence $(f_\alpha \mid \alpha < \kappa)$ is just one big parameter, i.e. can be coded by a fixed subset of κ.

Now define a reduction of $E_0^{<\lambda}$ to id ($=$ equality on 2^κ) by sending x to a subset of κ coding the sequence $f(x) = (f_\alpha(x \cap \alpha) \mid \alpha < \kappa)$. If $x \triangle y$ has size $< \lambda$ then $f(x) = f(y)$. If $x \triangle y$ has size at least λ then there is $\alpha < \kappa$ such that $(x \cap \alpha) \triangle (y \cap \alpha)$ has size at least λ and hence $f(x)$ does not equal $f(y)$. The reduction is clearly continuous.

Conversely, id is continuously reducible to $E_0^{<\lambda}$: Choose an injection $i : 2^{<\kappa} \to \kappa$ and reduce id to $E_0^{<\lambda}$ by: $f(x) = \{i(x \cap \alpha) \mid \alpha < \kappa\}$. $\square$

7.-8.Vorlesungen

As in the classical case we have:

Proposition 15 $E_0 = E_0^{<\kappa}$ *is* not *Borel-reducible to id.*

Proof. Suppose that f were a Borel reduction of E_0 to equality. Let M be a transitive ZF$^-$ model of size κ containing $H(\kappa)$ and a Borel-code for f. Consider κ-Cohen forcing over M and let $\dot{G}$ denote the κ-Cohen generic. Choose a condition p forcing that $f(\dot{G})$ is defined. Then for each $\alpha < \kappa$, any two κ-Cohen conditions extending p must force the same value for $f(\dot{G})(\alpha)$: Otherwise, choose two conditions p_0, p_1 of equal length extending p and forcing different values of $f(\dot{G})(\alpha)$. Let x be κ-Cohen generic over M extending p_0 and let y be obtained from x by replacing p_0 by p_1. Then xE_0y but $f(x)(\alpha) \neq f(y)(\alpha)$, contradicting the assumption that f is a reduction. So $f(x)$ is constant on all x which are κ-Cohen generic and extend p. But that is impossible, because there are κ-Cohen generics x, y extending p which disagree on an unbounded subset of κ. $\square$

Other versions of E_0: For regular $\lambda < \kappa$ define $E^\kappa_\lambda =$ equality modulo the ideal of λ-nonstationary sets.

These equivalence relations are key for connecting Shelah Classification with Higher Descriptive Set Theory.

How do the relations E^κ_λ compare to each other under Borel reducibility for different λ? For simplicity, consider the special case $\kappa = \omega_2$.

Theorem 16 *(SDF-Hyttinen-Kulikov) (a) It is consistent that $E^{\omega_2}_\omega$ and $E^{\omega_2}_{\omega_1}$ are incomparable under Borel reducibility. (b) Relative to a weak compact it is consistent that $E^{\omega_2}_\omega$ is Borel-reducible to $E^{\omega_2}_{\omega_1}$.*

It is not known if it is consistent for $E^{\omega_2}_{\omega_1}$ to be Borel-reducible to $E^{\omega_2}_\omega$.

Proof of (a). We begin with the following general result. As always assume GCH. Fix κ uncountable and regular, not the successor of a singular cardinal (to handle this case we would need Shelah's approachability property on κ). For regular $\lambda < \kappa$, S^κ_λ denotes the set of ordinals less than κ of cofinality λ. For arbitrary stationary $X \subseteq \kappa$, E_X denotes the equivalence relation defined by $A\ E_X\ B$ iff $(A \triangle B) \cap X$ is nonstationary.

Lemma 17 *Suppose that $\mu_1, \mu_2 < \kappa$ are regular, $X \subseteq S^\kappa_{\mu_1}$ is stationary, $Y \subseteq S^\kappa_{\mu_2}$ is stationary and $X \cap \alpha$ is not stationary in α for $\alpha \in Y$ (automatic if $\mu_1 \geq \mu_2$). Also suppose that X, Y are disjoint (automatic if $\mu_1 \neq \mu_2$). Suppose that (T, l) is a code for a Borel reduction from E_X to E_Y. Then (T, l) is not such a code in some generic extension by a κ-closed forcing of size κ.*

Proof of Lemma. First add a κ-Cohen set G. If (T, l) does not code a Borel reduction from E_X to E_Y in $V[G]$ then we are done. Otherwise let f be the Borel reduction coded by (T, l) in $V[G]$. Note that $G \cap X$ is stationary, as κ-Cohen sets have stationary intersection with all ground model stationary sets (this already uses the hypothesis that κ is not the successor of a singular, or alternatively the approachability property on κ). So $(f(G \cap X) \triangle f(\emptyset)) \cap Y = Z$ is stationary. Now consider the forcing $\mathcal{Q}$ for killing the stationarity of $G \cap X$: A condition is a closed, bounded subset c of κ such that $c \cap (G \cap X)$ is empty.

Claim. $\mathcal{Q}$ preserves the stationarity of Z.

Proof. First we need:

Subclaim. Choose a large $H(\theta)$, θ regular and also a parameter x in $H(\theta)$. Then there is a club C in κ such that for all $\alpha \in C$ of cofinality μ_2 there exists an elementary submodel M of $H(\theta)$ containing the parameter x which has size μ_2, is μ_2-closed and satisfies $\sup(M \cap \kappa) = \alpha$.

Proof of Subclaim. First choose a continuous increasing chain $(M_i \mid i < \kappa)$ of elementary submodels of $H(\theta)$ such that x belongs to M_0 and for each $i < \kappa$, M_{i+1} is μ_2-closed (this is possible as we have assumed GCH and κ is not the successor of a singular cardinal). Let C consist of all $M_\lambda \cap \kappa$ for limit $\lambda < \kappa$, a club in κ. Then if λ has cofinality μ_2 the model M_λ is μ_2-closed and therefore it is easy to build an increasing chain $(N_j \mid j < \mu_2)$ of elementary submodels of M_λ where x belongs to N_0, each N_j has size less than μ_2 and the union M of the N_j's is μ_2-closed and has cofinal intersection with $M_\lambda \cap \kappa$. $\square$ (Subclaim)

Now suppose that the condition $q \in \mathcal{Q}$ forces that the $\mathcal{Q}$ name $\dot{C}$ is a club in κ and choose a large $H(\theta)$. containing $q, \dot{C}, \kappa, G \cap X$ as elements. Using the Subclaim choose an elementary submodel M of $H(\theta)$ which has size μ_2 and is μ_2-closed, and is such that $\alpha = \sup(M \cap \kappa)$ belongs to Z. This is possible because Z is stationary and the set of possibilities for $\sup(M \cap \kappa)$ contains all ordinals of cofinality μ_2 in some closed unbounded set. As α belongs to $Z \subseteq Y$, $X \cap \alpha$ is not stationary in α and therefore $G \cap X \cap \alpha$ is not stationary in α. Let D be closed unbounded in α and disjoint from $G \cap X$. Now we can build a continuous sequence of conditions $q = q_0 \geq q_1 \geq \cdots$ of length $\mu_2 + 1$ so that for $i < \mu_2$, q_i belongs to M, q_{i+1} forces an ordinal greater than $\max(q_i)$ to belong to $\dot{C}$ and there is an element of D between $\max(q_i)$ and

$\max(q_{i+1})$. As α does not belong to X it follows that $q^* = q_{\mu_2} \cup \{\alpha\}$ is a condition extending q with $\max(q^*) = \alpha \in Z$ and q^* forces α to belong to $\dot{C}$. So Z remains stationary. $\square$ (Claim)

9.-10.Vorlesungen

Now to finish the proof of the Lemma, note that we have shown that $\mathcal{C} * \mathcal{Q}$ forces that the Borel reduction f coded by (T, l) does not reduce E_X to E_Y because it forces $G \cap X$ to be nonstationary on X and its image $f(G \cap X)$ to be stationary on Y. It remains to show that $\mathcal{Z} = \mathcal{C} * \mathcal{Q}$ is equivalent to a κ-closed forcing of size κ. But this forcing is equivalent to forcing with pairs (p, q) where p is a κ-Cohen condition and q is a κ-Cohen condition of the same length α such that $\{\beta \mid q(\beta) = 1\}$ is closed in α and disjoint from $\{\beta \mid p(\beta) = 1\}$. This forcing is κ-closed because if (p_i, q_i), $i < \lambda$, is a descending sequence for some limit $\lambda < \kappa$ we obtain a lower bound by taking p, q to be the unions of the p_i, q_i and setting $p(\alpha) = 0$, $q(\alpha) = 1$ where α is the sup of the lengths of the p_i, q_i's. $\square$

Now to prove (a) of the Theorem, choose a model in which some stationary subset X of $S^{\omega_2}_{\omega}$ has nonstationary intersection with each ordinal in $S^{\omega_2}_{\omega_1}$. For example, take a model in which $\square_{\omega_1}$ holds. Then by the Lemma applied to the pair $X, S^{\omega_2}_{\omega_1}$, we can force with a κ-closed forcing of size κ to kill any code for a Borel reduction of E_X to $E^{\omega_2}_{\omega_1}$. The same holds for the pair $S^{\omega_2}_{\omega_1}, S^{\omega_2}_{\omega}$. Now perform a κ-closed iteration of length κ^+, successively killing codes for Borel reductions of E_X to $E^{\omega_2}_{\omega_1}$ and of $E^{\omega_2}_{\omega_1}$ to $E^{\omega_2}_{\omega}$. The result is a model in which E_X is not Borel reducible to $E^{\omega_2}_{\omega_1}$ and $E^{\omega_2}_{\omega_1}$ is not Borel reducible to $E^{\omega_2}_{\omega}$. So the proof of (a) ends with the following

Fact. Suppose $X \subseteq Y$. Then E_X is Borel reducible to E_Y.

Proof of Fact. Send A to $A \cap X$. If $(A \triangle B) \cap X$ is nonstationary then so is $((A \cap X) \triangle (B \cap X)) \cap Y$ because it equals $(A \triangle B) \cap X$. Conversely, if $(A \triangle B) \cap X$ is stationary then so is $((A \cap X) \triangle (B \cap X)) \cap Y$ for the same reason. $\square$

Now we begin the proof of (b), which says that relative to a weak compact, it is consistent that $E^{\omega_2}_{\omega}$ is Borel reducible to $E^{\omega_2}_{\omega_1}$.

We obtain Borel reductions using "$\diamondsuit$ reflection". Suppose that X, Y are subsets of a regular κ and Y consists of ordinals of uncountable cofinality. We say that X *$\diamondsuit$-reflects to* Y iff there is a sequence $(D_\alpha \mid \alpha \in Y)$ such that:

1. D_α is a stationary subset of α for each $\alpha \in Y$.
2. If $Z \subseteq X$ is stationary then $\{\alpha \in Y \mid D_\alpha = Z \cap \alpha\}$ is stationary.

Lemma 18 *Suppose that X $\diamondsuit$-reflects to Y. Then E_X is Borel reducible to E_Y.*

Proof. Let $(D_\alpha \mid \alpha \in Y)$ witness the hypothesis. For $A \subseteq \kappa$ set $f(A) = \{\alpha \in Y \mid A \cap X \cap D_\alpha$ is stationary in $\alpha\}$. Note that f is continuous. We claim that f is the desired reduction.

Suppose that $(A \triangle B) \cap X$ is nonstationary and choose a club C disjoint from it. Then for α in $Y \cap \mathrm{Lim}\ C$, $C \cap \alpha$ is club in α and so $A \cap X \cap D_\alpha \cap C$ is stationary in α iff $B \cap X \cap D_\alpha \cap C$ is stationary in α. Thus $(f(A) \triangle f(B)) \cap Y$ is nonstationary, as desired.

Conversely, suppose that $(A \triangle B) \cap X$ is stationary and without loss of generality assume that $(A \setminus B) \cap X$ is stationary. By hypothesis, $S = \{\alpha \in Y \mid (A \setminus B) \cap X \cap \alpha = D_\alpha\}$ is stationary. For α in S we have that $A \cap X \cap D_\alpha$ is stationary in α and and $B \cap X \cap D_\alpha$ is empty. So $(f(A) \triangle f(B)) \cap Y$ is stationary, as desired. $\Box$

Suppose that κ is weak compact. Then κ satisfies Π^1_1 reflection: If $(H(\kappa), A)$ satisfies the Π^1_1 sentence $\varphi(A)$ then $(H(\alpha), A \cap H(\alpha))$ satisfies $\varphi(A \cap H(\alpha))$ for some $\alpha < \kappa$.

Lemma 19 *Suppose that κ is weak compact and $X \subseteq \kappa$ is stationary. Then for stationary many regular $\alpha < \kappa$, $X \cap \alpha$ is stationary in α.*

Proof. If C is a club in κ then $(H(\kappa), X, C)$ satisfies the Π^1_1 sentence saying that κ is regular (there is no cofinal function from an ordinal into the ordinals), X is stationary and C is club. By reflection there is a regular α so that $C \cap \alpha$ is club in α and $X \cap \alpha$ is stationary in α. As C is closed, α belongs to C. $\Box$

Lemma 20 *Suppose that κ is weak compact and $V = L$. Then all stationary subsets of κ $\diamondsuit$ reflect to $Reg \cap \kappa$ = the set of regular cardinals less than κ. In particular, E_X is Borel reducible to $E_{Reg\cap\kappa}$ for all stationary $X \subseteq \kappa$.*

Proof. Let Y denote $\mathrm{Reg}\cap\kappa$. We define the desired D_α's inductively as follows. Given α in Y, let (Z, C) be the L-least pair of subsets of α such that Z is a stationary subset of $X\cap\alpha$, C is club in α and D_β does not equal $Z\cap\beta$ for β in $Y\cap C$ (if there is no such pair then set $D_\alpha=\emptyset$). Then take D_α to be Z. For future use also define C_α to be C.

If the above sequence of D_α's does not work then let (Z, C) be the L-least pair of subsets of κ such that Z is a stationary subset of X, C is club in κ and D_β does not equal $Z\cap\beta$ for β in $Y\cap C$. Let M be an elementary submodel of some large $H(\theta)$ which contains X as an element such that $\alpha=M\cap\kappa$ is an element of Y and $Z\cap\alpha$ is stationary in α. Such an M exists by Lemma 19. Let $\bar{M}$ be the transitive collapse of M. Then $(Z\cap\alpha, C\cap\alpha)$ is the L-least pair (z, c) in $\bar{M}$ such that z is a subset of $X\cap\alpha$ which is stationary in $\bar{M}$, c is club in α and D_β does not equal $z\cap\beta$ for β in $Y\cap c$. As $Z\cap\alpha$ really is stationary in α (and not just stationary with respect to clubs in $\bar{M}$), the pair $(Z\cap\alpha, C\cap\alpha)$ is in fact equal to the pair (D_α, C_α) defined above. But this contradicts the choice of the pair (Z, C), as α belongs to $Y\cap C$ and D_α does equal $Z\cap\alpha$. $\square$

Finally, to get a consistent Borel reducibility from $E^{\omega_2}_{\omega}$ to $E^{\omega_2}_{\omega_1}$, we start with a weak compact κ in L and force with an ω-closed Lévy collapse to make κ into ω_2. We show that in the extension $L[G]$, $S^{\omega_2}_{\omega}$ $\diamondsuit$-reflects to $Y=$ the L-regular cardinals less than κ. Note that Y is stationary in $L[G]$ as it is stationary in L and any club in κ in $L[G]$ contains a club in κ in L, by the κ-cc of the forcing. As Y is a stationary subset of $S^{\omega_2}_{\omega_1}$ in $L[G]$ it follows that $S^{\omega_2}_{\omega}$ also $\diamondsuit$ reflects to $S^{\omega_2}_{\omega_1}$ in $L[G]$.

In $L[G]$ we define the sequence $(D_\mu\mid\mu\in Y)$ as follows. Let $f^G:\kappa\times\omega_1\to\kappa$ be the generic function added by G; i.e., f^G maps $\{\mu\}\times\omega_1$ onto μ for each $\mu<\kappa$ and is forced with countable conditions. For each L-regular $\mu\in[\omega_2^L,\kappa)$ let $(\sigma^\mu_i\mid i<(\mu^+)^L)$ enumerate the nice $P(<\mu)$-names for subsets of μ. Then we take D_μ to be the interpretation of the $P(<\mu)$-name $\sigma^\mu_{f^G((\mu^+)^L,0)}$ via the $P(<\mu)$-generic $G\cap P(<\mu)$ (provided this is stationary in μ; $D_\mu=\mu$ otherwise).

We claim that the sequence $(D_mu\mid\mu\in Y)$ works. Indeed, suppose that the condition p forces $\dot{S}$ to be a nice P-name for a stationary subset of $S^{\omega_2}_{\omega}$ and C is a club in κ; we show that p has an extension q forcing $\dot{S}\cap\mu=D_\mu$ for some μ in $Y\cap C$. This suffices, as any club subset of $\kappa=\omega_2$ in $L[G]$

contains a club subset in L. Apply Π^1_1 reflection to get $\mu \in Y \cap C$ so that p belongs to $P(< \mu)$ and p forces that $\dot{S} \cap \mu$ is stationary in μ. Now extend p to q by choosing $q((\mu^+)^L, 0)$ to be i where $\dot{S} \cap \mu$ is the $P(< \mu)$-name σ^μ_i. $\square$

11.-12.Vorlesungen

Clinton's Question

I'll give a partial answer to a question of Clinton Conley: What is the relationship between E_0 and E^κ_λ?

Theorem 21 *It is consistent that under Borel reducibility, $E^{\omega_2}_{\omega_1}$ and $E^{\omega_2}_{\omega}$ are incomparable and neither reduces to E_0.*

Theorem 22 *In L, E_0 reduces to both $E^{\omega_2}_{\omega}$ and $E^{\omega_2}_{\omega_1}$.*

Proof of Theorem 21. First recall the basic technical lemma that we used to get the incomparability of $E^{\omega_2}_{\omega_1}$ and $E^{\omega_2}_{\omega}$:

Lemma 23 *Suppose that $X \subseteq S^{\omega_2}_{\omega}$ is stationary but has nonstationary intersection with α for all $\alpha \in S^{\omega_2}_{\omega_1}$. Suppose that (T, l) is a code for a Borel reduction from E_X to $E^{\omega_2}_{\omega_1}$. Then (T, l) is not such a code in some generic extension by an ω_2-closed forcing of size ω_2.*

This was proved as follows: First add an ω_2-Cohen set G. If (T, l) does not code a Borel reduction from E_X to $E^{\omega_2}_{\omega_1}$ in $V[G]$ then we are done. Otherwise let f be the Borel reduction coded by (T, l) in $V[G]$ and note that $G \cap X$ is stationary. As f is a reduction it follows that $(f(G \cap X) \triangle f(\emptyset)) \cap S^{\omega_2}_{\omega_1} = Z$ is stationary. Now force to kill the stationarity of $G \cap X$. Using the hypothesis that X is a "nonreflecting" stationary set, this forcing preserves the stationarity of Z and therefore "kills" the reduction (T, l). Finally, the two-step iteration ω_2-Cohen $*$ (Kill stationarity of $G \cap X$) is ω_2-closed.

The existence of X as in the Lemma follows from $\square_{\omega_1}$ and therefore holds in L. By an ω_3-iteration one can use the Lemma to kill all Borel codes for reductions of E_X to $E^{\omega_2}_{\omega_1}$. As E_X is Borel reducible to $E^{\omega_2}_{\omega}$, it follows that in the resulting model, $E^{\omega_2}_{\omega}$ is not Borel-reducible to $E^{\omega_2}_{\omega_1}$.

There is a similar lemma for killing reductions from $E^{\omega_2}_{\omega_1}$ to $E^{\omega_2}_{\omega}$, using the fact that the entire $S^{\omega_2}_{\omega_1}$ trivially has "nonstationary" intersection with each $\alpha \in S^{\omega_2}_{\omega}$.

Now the point is that the same technique shows how to kill reductions from $E^{\omega_2}_{\omega}$ or $E^{\omega_2}_{\omega_1}$ to E_0.

Lemma 24 *Suppose that (T,l) is a code for a Borel reduction from either $E^{\omega_2}_{\omega}$ or $E^{\omega_2}_{\omega_1}$ to E_0. Then in an ω_2-closed forcing extension, (T,l) is no longer such a code.*

Proof. Suppose that (T,l) codes a Borel reduction from $E^{\omega_2}_{\lambda}$ to E_0 ($\lambda = \omega$ or ω_1) and add an ω_2-Cohen set G. If this kills the code (T,l) then we are done. Otherwise, let f be the reduction coded by (T,l) in $V[G]$. As $G \cap S^{\omega_2}_{\omega} = Z$ is stationary and f is a reduction, we have that $f(G) \triangle f(\emptyset)$ is unbounded. Now force to kill the stationarity of Z. As before, the two-step iteration ω_2-Cohen $*$ (Kill the stationarity of Z) is equivalent to an ω_2-closed forcing. As $f(G) \triangle f(\emptyset)$ trivially remains unbounded, (T,l) no longer codes a Borel reduction in the extension. $\square$

Proof of Theorem 22. Let $(D_\alpha \mid \alpha < \omega_2)$ be the canonical $\diamondsuit$ sequence in L at ω_2. Then for each $X \subseteq \omega_2$, both of the following sets are stationary:

$$\{\alpha \in S^{\omega_2}_{\omega} \mid D_\alpha = X \cap \alpha\}$$
$$\{\alpha \in S^{\omega_2}_{\omega_1} \mid D_\alpha = X \cap \alpha\}$$

Now to reduce E_0 to $E^{\omega_1}_{\lambda}$ ($\lambda = \omega$ or ω_1) send X to $f(X) = \{\alpha \in S^{\omega_2}_{\lambda} \mid D_\alpha \setminus \bar{\alpha} = (X \cap \alpha) \setminus \bar{\alpha}$ for some $\bar{\alpha} < \alpha\}$. If $X \triangle Y$ is bounded by α it follows that $f(X)$, $f(Y)$ agree on all ordinals in $S^{\omega_2}_{\lambda}$ greater than α. If $X \triangle Y$ is unbounded it follows that $f(X)$, $f(Y)$ disagree on all cofinality λ limit points α of $X \triangle Y$ where $D_\alpha = X \cap \alpha$. So we get the desired reductions. $\square$

Remark. It follows by work of Shelah that assuming GCH, E_0 is Borel reducible to E^{κ}_{λ} except for the case $\kappa = \mu^+$, $\lambda =$ the cofinality of μ. What happens in this case remains open.

Another application of "code-killing"

Another special case of our general lemma for killing codes for Borel reductions is the following.

Lemma 25 *Suppose that S,T are disjoint stationary subsets of S^{κ}_{λ} and (T,l) is a code for a Borel reduction from E_S to E_T. Then in a forcing extension by a κ-closed forcing of size κ, (T,l) is no longer such a code.*

Again, the key point in the proof is that $S \cap \alpha$ is "nonstationary" for all $\alpha \in T$, simply because the ordinals in S and T all have the same cofinality.

Now we use this to prove:

Theorem 26 *Consistently: There is an injective, order-preserving embedding from $(\mathcal{P}(\kappa), \subseteq)$ into the partial order of Σ^1_1 equivalence relations under Borel reducibility.*

It won't be proved here, but one can in fact get the embedding to have range included in the Δ^1_1 equivalence relations. The case of Borel equivalence relations is open:

Question. Are there Borel equivalence relations which are incomparable under Borel reducibility (κ uncountable)? Is this at least consistent?

Proof of Theorem 26. Let $(S_i \mid i < \kappa)$ be a partition of S^κ_λ into κ-many pairwise disjoint stationary sets. For any $A \subseteq \kappa$ let $E(A)$ be the equivalence relation $E_{X(A)}$ where $X(A) = \bigcup_{i \in A} S_i$. If A is a subset of B then $E(A)$ is Borel reducible to $E(B)$ because $X(A)$ is a subset of $X(B)$. Now suppose that A is not a subset of B and choose $\alpha \in A \setminus B$. Of course $E(\{\alpha\})$ is Borel reducible to $E(A)$ and $E(B)$ is Borel reducible to $E(\kappa \setminus \{\alpha\})$. So to get the desired conclusion that $E(A)$ is not Borel reducible to $E(B)$ it would suffice to know that that $E(\{\alpha\})$ is not Borel reducible to $E(\kappa \setminus \{\alpha\})$.

By Lemma 25 there is for each α a κ-closed forcing of size κ which kills a code for a Borel reduction of $E(\{\alpha\})$ to $E(\kappa \setminus \{\alpha\})$. By iteration, we can kill all such codes for all α and thereby obtain the desired embedding. $\square$

Shelah Classification and Higher Descriptive Set Theory

Let T be countable, complete and first-order. Then T is *classifiable* iff there is a "structure theory" for its models. Example: Algebraically closed fields (transcendence degree).

T is *unclassifiable* otherwise. Example: Dense linear orderings.

Shelah's Characterisation (Main Gap): T is classifiable iff T is superstable without the OTOP and without the DOP.

A classifiable T is *deep* iff it has the maximum number of models in all uncountable powers. Example: Acyclic undirected graphs, every node has infinitely many neighbours.

Another way of classifying theories uses the higher descriptive set theory that we have developed. For simplicity assume GCH and $\kappa = \lambda^+$ where λ is uncountable and regular. Isom_T^κ is the isomorphism relation on the models of T of size κ.

Theorem 27 *(SDF-Hyttinen-Kulikov)*
(a) T is classifiable and shallow iff $Isom_T^\kappa$ is Borel.
(b) T is classifiable iff for all regular $\mu < \kappa$, $E_{S_\mu^\kappa}$ is not Borel reducible to $Isom_T^\kappa$.
(c) In L, T is classifiable iff $Isom_T^\kappa$ is Δ_1^1.

The proof uses Ehrenfeucht-Fraissé games:

The Game $EF_t^\kappa(\mathcal{A}, \mathcal{B})$

$\mathcal{A}$, $\mathcal{B}$ are structures of size κ, t is a tree. Player I chooses size $< \kappa$ subsets of $A \cup B$ and nodes along an initial segment of a branch through t; player II builds a partial isomorphism between $\mathcal{A}$ and $\mathcal{B}$ which includes the sets that player I has chosen. Player II wins iff he survives until a cofinal branch is reached.

The tree t *captures $Isom_T^\kappa$* iff for all size κ models $\mathcal{A}$, $\mathcal{B}$ of T, $\mathcal{A} \simeq \mathcal{B}$ iff Player II has a winning strategy in $\mathrm{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$.

Now there are 4 cases:

Case 1: T is classifiable and shallow.

Then Shelah's work shows that some well-founded tree captures Isom_T^κ. We use this to show that Isom_T^κ is Borel.

Case 2: T it classifiable and deep.

Then Shelah's work shows that no fixed well-founded tree captures Isom_T^κ. We use this to show that Isom_T^κ is not Borel.

Shelah's work also shows that $L_{\infty\kappa}$ equivalent models of T of size κ are isomorphic. This means that the tree $t = \omega$ (with a single infinite branch) captures Isom^κ_T. As the games $\mathrm{EF}^\kappa_\omega(\mathcal{A}, \mathcal{B})$ are determined, this shows that Isom^κ_T is Δ^1_1.

We must also show: $E_{S^\kappa_\mu}$ (equality modulo the μ-nonstationary ideal) is not Borel reducible to Isom^κ_T for any regular $\mu < \kappa$. This is because (in this case) Isom^κ_T is absolutely Δ^1_1, whereas μ-stationarity is not.

Now we look at the unclassifiable cases. Recall: Classifiable means superstable without DOP and without OTOP.

Case 3: T is unstable, superstable with DOP or superstable with OTOP.

Work of Hyttinen-Shelah and Hyttinen-Tuuri shows that in this case no tree of size κ without branches of length κ captures Isom^κ_T. This can be used to show Isom^κ_T is not Δ^1_1.

But $E_{S^\kappa_\lambda} \le_B \mathrm{Isom}^\kappa_T$ is harder. Following Shelah, there is a Borel map $S \mapsto \mathcal{A}(S)$ from subsets of κ to Ehrenfeucht-Mostowski models of T built on linear orders so that $\mathcal{A}(S_0) \simeq \mathcal{A}(S_1)$ iff $S_0 = S_1$ modulo the λ-nonstationary ideal.

Case 4: T is stable but not superstable.

This is the hardest case and requires some new model theory. Hyttinen replaces Ehrenfeucht-Mostowski models built on linear orders with primary models built on trees of height $\omega+1$ to show $E_{S^\kappa_\omega} \le_B \mathrm{Isom}^\kappa_T$. (We don't know if $E_{S^\kappa_\lambda} \le_B \mathrm{Isom}^\kappa_T$ or if Isom^κ_T could be Δ^1_1 in this case.)

Now we have all we need to prove the Theorem mentioned earlier:

(a) T is classifiable and shallow iff Isom^κ_T is Borel.

We showed that if T is classifiable and shallow then Isom^κ_T is Borel and if it is classifiable and deep it is not. If T is not classifiable then some $E_{S^\kappa_\mu}$ Borel reduces to Isom^κ_T, so the latter cannot be Borel.

(b) T is classifiable iff for all regular $\mu < \kappa$, $E_{S^\kappa_\mu}$ is not Borel reducible to Isom^κ_T.

We showed that if T is not classifiable then $E_{S^\kappa_\mu}$ is Borel reducible to Isom^κ_T where μ is either λ or ω. We also showed that if T is classifiable and deep then no $E_{S^\kappa_\mu}$ is Borel reducible to Isom^κ_T, by an absoluteness argument. When T is classifiable and shallow there is no such reduction as Isom^κ_T is Borel.

(c) In L, T is classifiable iff Isom^κ_T is Δ^1_1.

We showed that if T is classifiable then Isom^κ_T is Δ^1_1, in ZFC. If T is not classifiable then $E_{S^\kappa_\mu}$ Borel reduces to Isom^κ_T for some μ, and in L, $E_{S^\kappa_\mu}$ is not Δ^1_1.

13.-14.Vorlesungen

Definability of Isomorphism relations

We assume GCH and fix κ to be the successor of a regular and at least ω_2. Then the definability of Isom^κ_T for a countable, complete first-order theory T is closely related to the stability-theoretic properties of T. To establish this we first have to understand the definability of Isom^κ_T in terms of Ehrenfeucht-Fraissé games. In what follows we refer to the relation Isom^κ_T simply as "isomorphism for T". Recall that the tree t *captures isomorphism for T* iff for any models $\mathcal{A}, \mathcal{B}$ of T of size κ, $\mathcal{A}$ and $\mathcal{B}$ are isomorphic iff II has a winning strategy in the game $\mathrm{EF}^\kappa_t(\mathcal{A}, \mathcal{B})$.

Theorem 28 *Isomorphism for T is Borel iff some well-founded, rooted tree t of size κ captures isomorphism for T.*

Theorem 29 *(a) If isomorphism for T is Δ^1_1 then some rooted tree t of size κ without branches of length κ captures isomorphism for T.*
(b) Let t^*_ω be the ill-founded tree consisting of a single branch of ordertype ω. Then t^*_ω captures isomorphism for T iff any two $L_{\infty\kappa}$ equivalent models of T of size κ are isomorphic, and in this case isomorphism for T is Δ^1_1.

Proof of Theorem 28. First suppose that some well-founded, rooted tree t of size κ captures isomorphism for T; we show that isomorphism for T is Borel. This means that for some well-founded, rooted tree u of size κ and labelling h of the terminal nodes of u by open subsets of $\kappa^\kappa \times \kappa^\kappa$, $\mathcal{A}_\eta, \mathcal{A}_\xi$ are isomorphic iff player II has a winning strategy in the game $\mathcal{G}(u, h, (\eta, \xi))$, where $\eta, \xi \in \kappa^\kappa$ are codes for the models $\mathcal{A}_\eta, \mathcal{A}_\xi$ of T with universe κ.

We take u to be the tree of all sequences $((p_0, A_0), f_0, \cdots, (p_n, A_n), f_n)$ which are valid positions in a game $\mathrm{EF}_t^\kappa(\mathcal{A}, \mathcal{B})$ where $\mathcal{A}, \mathcal{B}$ have universe κ. I.e., we require:

1. $p_i \in t$ and $A_i \subseteq \kappa$, $\mathrm{card}(A_i) < \kappa$.
2. f_i is a partial function of size $< \kappa$ from κ to κ whose domain and range includes A_i.
3. The p_i's form an initial segment of a branch through t, the A_i's and f_i's are increasing.

Then for the labelling h we take the value of h on the terminal node $((p_0, A_0), f_0, \cdots, (p_n, A_n), f_n)$ of u to be the set of pairs (η, ξ) such that f_n is a partial isomorphism from $\mathcal{A}_\eta$ to $\mathcal{A}_\xi$. This is an open set because it uses only boundedly much information about the structures $\mathcal{A}_\eta$, $\mathcal{A}_\xi$.

It is easy to convert a winning strategy for II in the game $\mathrm{EF}_t^\kappa(\mathcal{A}_\eta, \mathcal{A}_\xi)$ into a winning strategy for II in the game $\mathcal{G}(u, h, (\eta, \xi))$ and vice-versa. By hypothesis isomorphism for T is captured by the tree t and therefore we conclude that isomorphism for T is Borel.

For the converse we need a theorem of Lopez-Escobar. We say that $B \subseteq \kappa^\kappa$ is *closed under permutations* iff whenever $\mathcal{A}_\eta$ is isomorphic $\mathcal{A}_\xi$, η belongs to B iff ξ belongs to B.

Theorem 30 *A subset B of κ^κ is Borel and closed under permutations iff for some sentence φ of $L_{\kappa^+\kappa}$, $B = \{\eta \mid \mathcal{A}_\eta \models \varphi\}$.*

Given this, argue as follows. A basic connection between Ehrenfeucht-Fraissé games and infinitary languages is the following.

Lemma 31 *Let $\mathcal{A}$, $\mathcal{B}$ be models of size κ. Then $\mathcal{A}$ and $\mathcal{B}$ are $L_{\kappa^+\kappa}$ equivalent iff II has a winning strategy in $EF_t^\kappa(\mathcal{A}, \mathcal{B})$ for each wellfounded tree t of size κ. More exactly, $\mathcal{A}$, $\mathcal{B}$ satisfy the same sentences of quantifier-rank $< \alpha$ iff II has a winning strategy in the game $EF_{t_\alpha}^\kappa(\mathcal{A}, \mathcal{B})$ where t_α is the tree of finite descending sequences through the ordinal α.*

Now suppose that isomorphism for T is Borel. Then by the (2-dimensional version of the) Lopez-Escobar Theorem there is a sentence φ of $L_{\kappa^+\kappa}$ such that $\mathcal{A}_\eta$, $\mathcal{A}_\xi$ are isomorphic iff the model $(\mathcal{A}_\eta, \mathcal{A}_\xi)$ satisfies φ. If we let α be greater than the quantifier rank of φ then we claim that t_α captures isomorphism for models of T (of size κ): Suppose that II has a winning strategy in $\mathrm{EF}^{\kappa}_{t_\alpha}(\mathcal{A},\mathcal{B})$. Then clearly II also has a winning strategy in $\mathrm{EF}^{\kappa}_{t_\alpha}((\mathcal{A},\mathcal{B}),(\mathcal{A},\mathcal{A}))$. It follows by the choice of α that $(\mathcal{A},\mathcal{B})$ and $(\mathcal{A},\mathcal{A})$ are isomorphic and therefore $\mathcal{A}$ is isomorphic to $\mathcal{B}$.

Proof of Theorem 30. The direction from right to left is immediate, as one shows by induction on the rank of φ that for any parameter choice $\vec{a} = (a_i \mid i < \alpha)$ from κ, the set of η such that $\mathcal{A}_\eta \models \varphi(\vec{a})$ is Borel.

Now we turn to the harder direction, using "Vaught transforms". Note that the set S_κ of permutations of κ is Borel. For any $u \in \kappa^{<\kappa}$ we let $\bar{u}$ denote the set of permutations $p \in \kappa^\kappa$ such that $u \subseteq p$. Then for any $A \subseteq 2^\kappa$ and $u \in \kappa^{<\kappa}$ define

$A^{*u} = \{\eta \mid \{p \in \bar{u} \mid \eta \circ p \in A\}$ is comeager in $\bar{u}\}$.

We let $\mathcal{Z}$ denote the set of Borel $A \subseteq 2^\kappa$ such that A^{*u} is $L_{\kappa^+\kappa}$ definable for all $u \in \kappa^{<\kappa}$. We show that all Borel sets belong to $\mathcal{Z}$. The desired result then follows by setting $u = \emptyset$, as $A^{*\emptyset} = A$ for sets A closed under permutations.

It is straightforward to show that basic open sets belong to $\mathcal{Z}$. For intersections of size κ use the formula

$$\bigcap_{i<\kappa}(A_i^{*u}) = (\bigcap_{i<\kappa} A_i)^{*u}.$$

For complements we use:

$$(*)\ \eta \in (\sim A)^{*u} \text{ iff for all } v \supseteq u,\ \eta \notin A^{*v}.$$

Proof of $(*)$: Here we use the property of Baire for Borel sets. Suppose $\eta \notin (\sim A)^{*u}$. Then
$B = \{p \in \bar{u} \mid \eta \circ p \in A\}$ is not meager in $\bar{u} \rightarrow$
There is a $\bar{v} \subseteq \bar{u}$ such that B is comeager in $\bar{v} \rightarrow$
There exists $v \supseteq u$, $\eta \in A^{*v}$.

Conversely, if $\eta \in (\sim A)^{*u}$ then

$\{p \in \bar{u} \mid \eta \circ p \in A\}$ is meager $\rightarrow$
For all $v \supseteq u$, $\{p \in \bar{v} \mid \eta \circ p \in A\}$ is meager $\rightarrow$
For all $v \supseteq u$, $\eta \notin A^{*v}$.

This completes the proof of Theorem 30 and therefore also of Theorem 28.

Proof of Theorem 32 (b) Note that the games $\mathrm{EF}^{\kappa}_{t^*_\omega}(\mathcal{A},\mathcal{B})$ are closed games for player II and therefore determined. If t^*_ω captures isomorphism for T and $\mathcal{A},\mathcal{B}$ are not isomorphic then I must have a winning strategy in $\mathrm{EF}^{\kappa}_{t^*_\omega}(\mathcal{A},\mathcal{B})$ and therefore for some α, I will have a winning strategy in the game $\mathrm{EF}^{\kappa}_{t_\alpha}(\mathcal{A},\mathcal{B})$; it follows that $\mathcal{A}$, $\mathcal{B}$ cannot be $L_{\kappa^+\kappa}$ equivalent. And again, as the games $\mathrm{EF}^{\kappa}_{t^*_\omega}(\mathcal{A},\mathcal{B})$ are determined, it follows that isomorphism for T is Δ^1_1.

15.-16.Vorlesungen

Theorem 32 *(a) If isomorphism for T is Δ^1_1 then some tree t of size κ without branches of length κ captures isomorphism for T.*
(b) Let t^*_ω be the ill-founded tree consisting of a single branch of ordertype ω. Then t^*_ω captures isomorphism for T iff any two $L_{\infty\kappa}$ equivalent models of T of size κ are isomorphic, and in this case isomorphism for T is Δ^1_1.

We proved (b); we now turn to (a). First we need a lemma.

Lemma 33 *Let Z be Σ^1_1 and write*

$$\xi \in Z \textit{ iff } t(\xi) \textit{ has a } \kappa\textit{-branch}$$

where t is a tree on $\kappa^{<\kappa} \times \kappa^{<\kappa}$ and $t(\xi) = \{p \in \kappa^{<\kappa} \mid (\xi \restriction |p|, p) \in t\}$. If Z is Δ^1_1 then there is a tree t' of size κ with no κ-branch such that

$$\xi \in Z \textit{ iff } t(\xi) \not\leq t'$$

(where for trees u and v, $u \leq v$ means that u embeds *into v, i.e., there is a strictly order-preserving map from u to v).*

Proof. If Z is Δ^1_1 then choose $\bar{t}$ to be a tree of size κ such that

$$\xi \notin Z \text{ iff } \bar{t}(\xi) \text{ has a } \kappa\text{-branch.}$$

Now form the tree t' consisting of triples (p,q,r) where p,q,r have the same length, (p,q) belongs to t and (p,r) belongs to $\bar{t}$. Clearly t' has no κ-branch, else the first components of that branch would yield an element of both Z

and its complement. If ξ does not belong to Z then choose a branch η through $\bar{t}(\xi)$ and embed $t(\xi)$ into t' by sending q to $(\xi \restriction |q|, q, \eta \restriction |q|)$. $\square$ (Lemma 33)

Proof of Theorem 32 (a). Suppose that isomorphism is Δ_1^1. We want to produce a tree u of size κ with no κ-branch that captures isomorphism for T.

Let t be the tree of partial isomorphisms from $\mathcal{A}_\eta$ to $\mathcal{A}_\xi$. (Recall that $\mathcal{A}_\xi$ is the model coded by ξ.) As isomorphism is Δ_1^1 we can apply Lemma 33 to get a tree u of size κ without κ-branches such that $\mathcal{A}_\eta \simeq \mathcal{A}_\xi$ iff $t(\eta, \xi) \not\leq u$. We show that u captures isomorphism for T. Suppose not and choose ξ_1, ξ_2 such that $\mathcal{A}_{\xi_1}$ and $\mathcal{A}_{\xi_2}$ are not isomorphic yet II wins the game $\mathrm{EF}_u^\kappa(\mathcal{A}_{\xi_1}, \mathcal{A}_{\xi_2})$.

Claim 1. Let v be a tree and let σv denote the tree of downward-closed linear subsets of v. Then there is no embedding of σv into v.

Proof. Suppose that g were such an embedding and define

$x_0 = g(\emptyset)$
$x_\alpha = g(\{y \in v \mid y \leq_v x_\beta \text{ for some } \beta < \alpha\})$

Then the x_α's yield a branch of length the ordinals through v, contradiction. $\square$

Claim 2. There is an embedding of σu into $t(\xi_1, \xi_2)$.

It follows from the Claims that $t(\xi_1, \xi_2)$ does not embed into u, contradicting the fact that $\mathcal{A}_{\xi_1}$ is not isomorphic to $\mathcal{A}_{\xi_2}$ and the choice of u.

Proof of Claim 2. Let τ be a winning strategy for II in $\mathrm{EF}_u^\kappa(\mathcal{A}_{\xi_1}, \mathcal{A}_{\xi_2})$. We define an embedding g of σu into $t(\xi_1, \xi_2)$ inductively. If s belongs to σu and g is defined for $s' <_{\sigma u} s$, then play EF_u^κ up to $\sup(s) \in u$. If s does not contain its sup then put $g(s) = \cup_{s' <_{\sigma u} s} g(s')$. Otherwise apply II's strategy to produce a partial isomorphism ν_α of length α extending $\cup_{s' <_{\sigma u} s} g(s')$ and set $g(s) = \nu_\alpha$. $\square$

Theorem 32 (b) is used to show that isomorphism is Δ_1^1 for classifiable theories. Theorem 32 (a) shows that it is not Δ_1^1 for most unclassifiable theories; the exceptional case is that of "strictly stable" theories, i.e., theories which are stable but not superstable.

It is open whether isomorphism is provably not Δ^1_1 for strictly stable theories. However, for all unclassifiable theories there is some $\lambda < \kappa$ such that E^κ_λ is Borel reducible to isomorphism, so we get a provable characterisation of classifiability via the next result.

Theorem 34 *If T is classifiable then E^κ_λ is not Borel reducible to isomorphism for $\lambda < \kappa$.*

What we use about classifiability in the following proof is the following characterisation due to Shelah: T is classifiable iff the tree t^*_ω captures isomorphism for T, i.e., models $\mathcal{A}$, $\mathcal{B}$ of T of size κ are isomorphic iff II wins the game $\mathrm{EF}^\kappa_{t^*_\omega}(\mathcal{A}, \mathcal{B})$. Note that the latter is equivalent to I not having a winning strategy in $\mathrm{EF}^\kappa_{t^*_\omega}(\mathcal{A}, \mathcal{B})$, as this game is closed for II.

Proof. Suppose that $r : 2^\kappa \to 2^\kappa$ were a Borel reduction of E^κ_λ to Isom^κ_T. So for each $\eta \in 2^\kappa$, $r(\eta)$ is a model of T and $\eta E^\kappa_\lambda \xi$ iff $\mathcal{A}_{r(\eta)}$ is isomorphic to $\mathcal{A}_{r(\xi)}$. Let M be a transitive, κ-closed model of ZF^- containing a code for r and consider $P = \kappa$-Cohen forcing over M. Choose a condition $p \in P$ which either forces the models $\mathcal{A}_{r(\vec{0})}$ and $\mathcal{A}_{r(\dot{g})}$ to be isomorphic or forces them to not be isomorphic, where $\vec{0}$ is the constant sequence with value 0 and $\dot{g}$ denotes the κ-Cohen generic. Note that since T is classifiable, II has a winning strategy in $\mathrm{EF}^\kappa_{t^*_\omega}(\mathcal{A}_{r(\vec{0})}, \mathcal{A}_{r(\dot{g})})$ when they are isomorphic, and otherwise I does.

But we can choose γ_0, γ_1 below p to be κ-Cohen generic over M such that, identifying them with subsets of κ, γ_0 is nonstationary and γ_1 contains a club in the real world. If p forces $\mathcal{A}_{r(\vec{0})}$ and $\mathcal{A}_{r(\dot{g})}$ to be isomorphic then we get a contradiction using γ_1, as then $\mathcal{A}_{r(\vec{0})}$ and $\mathcal{A}_{r(\gamma_1)}$ are isomorphic in $M[\gamma_1]$ and therefore in the real world, whereas $\vec{0}$ and γ_1 are clearly not E^κ_λ equivalent. If p forces $\mathcal{A}_{r(\vec{0})}$ and $\mathcal{A}_{r(\dot{g})}$ to not be isomorphic then we get a contradiction using γ_0, as then $\mathcal{A}_{r(\vec{0})}$ and $\mathcal{A}_{r(\gamma_0)}$ are not isomorphic in $M[\gamma_1]$, hence I wins $\mathrm{EF}^\kappa_{t^*_\omega}(\mathcal{A}_{r(\vec{0})}, \mathcal{A}_{r(\gamma_0)})$ in $M[\gamma_0]$, hence I wins $\mathrm{EF}^\kappa_{t^*_\omega}(\mathcal{A}_{r(\vec{0})}, \mathcal{A}_{r(\gamma_0)})$ in the real world (as M is closed under $< \kappa$ sequences) and hence $\mathcal{A}_{r(\vec{0})}$ and $\mathcal{A}_{r(\gamma_0)}$ are not isomorphic in the real world, contradicting the fact that $\vec{0}$ and γ_0 are E^κ_λ equivalent. □