Generalisations of Gödel's L

Desirable features of Gödel's L:

a. Definable wellordering (strong form of AC)
b. GCH
c. Jensen's $\diamondsuit$, $\square$ and Morass

The theory $\text{ZFC} + V = L$ is *mathematically strong*

Problem. For many interesting φ in set theory:

$$\text{ZFC} + \varphi \text{ proves Con(ZFC)}$$

But $\text{ZFC} + V = L$ does NOT prove Con(ZFC)

$\text{ZFC} + V = L$ is *consistency weak*

Large cardinal axioms (LC's): Inaccessible, measurable, strong, Woodin, superstrong, ...

Empirical fact: ZFC+LC's is *consistency strong:* For any φ

$$\text{Con}(\text{ZFC} + \text{LC}) \rightarrow \text{Con}(\text{ZFC} + \varphi)$$

for some large cardinal axiom LC. In fact:

$\text{Con}(\text{ZFC} + \text{LC} + \epsilon) \rightarrow \text{Con}(\text{ZFC} + \varphi) \rightarrow \text{Con}(\text{ZFC} + \text{LC})$,

for some large cardinal axiom LC and small ϵ

Q1. Can we combine the *mathematical* power of $V = L$ with the *consistency* power of LC's?

Q2. Are large cardinals needed solely for the analysis of consistency strength, or do they follow from basic logical principles?

Q1: Large cardinals and L-like models

Inner model program: Show that any model with large cardinals has an L-like inner model with large cardinals.

Contributors: Gödel, Silver, Dodd, Jensen, Mitchell, Steel, Neeman and others.

Example 1: Inaccessible cardinals

Easy: If κ is inaccessible, then
$L \vDash \kappa$ inaccessible.

Example 2: Measurable cardinals

Scott: $L \vDash$ There is no measurable cardinal

What inner model shall we use?

Relativised L: $\mathcal{L}^E_\alpha = (L^E_\alpha, \in, E_\alpha)$, $\alpha \in \mathrm{Ord}$

$\mathcal{L}^E_0 = (\emptyset, \emptyset, \emptyset)$
$\mathcal{L}^E_{\alpha+1} = (\mathrm{Def}(\mathcal{L}^E_\alpha), \in, E_{\alpha+1})$ (in fact $E_{\alpha+1} = \emptyset$)
$\mathcal{L}^E_\lambda = (\bigcup_{\alpha<\lambda} L^E_\alpha, \in, E_\lambda)$,

Desired inner model is $L[\langle E_\alpha \mid \alpha \in \mathrm{Ord}\rangle] = L[E]$. But what is E?

First: What is a measurable cardinal?

$\exists$ measurable iff $\exists\ j : V \to M$

[j is an elementary embedding from $(V, \in)$ to $(M, \in)$ for some inner model M, j is not the identity]

Idea: Approximate the *class* embedding $j : V \to M$ by *set* embeddings E_λ.

Theorem 1. Suppose that there is a measurable cardinal. Then there exists $E = (E_\alpha \mid \alpha \in \text{Ord})$ such that:

1. For limit λ, E_λ is either empty or an embedding $E_\lambda : L_\alpha^E \to L_\lambda^E$ for some $\alpha < \lambda$.
2. $L[E] \models$ There is a measurable cardinal.
3. E is definable over $L[E]$.
4. *Condensation:* With mild restrictions, $M \prec \mathcal{L}_\alpha^E$ implies M is isomorphic to some $\mathcal{L}_{\bar{\alpha}}^E$.
5. $L[E] \models \diamondsuit$, $\square$ and (gap 1) Morass

3 $\to$ definable wellordering
4 $\to$ GCH

Theorem 1 has been generalised after great effort to stronger large cardinal properties.

Why is the Inner Model Program so difficult?

Condensation: $M \prec \mathcal{L}^E_\alpha = (L^E_\alpha, \in, E_\alpha)$ implies M is isomorphic to some $\mathcal{L}^E_{\bar\alpha} = (L^E_{\bar\alpha}, \in, E_{\bar\alpha})$.

With Gödel's methods, M is isomorphic to some $\mathcal{L}^F_{\bar\alpha} = (L^F_{\bar\alpha}, \in, F_{\bar\alpha})$

Goal: $\mathcal{L}^F_{\bar\alpha} = \mathcal{L}^E_{\bar\alpha}$

Only known technique: *Comparison method*

Let $\bar M$, $\bar N$ denote $\mathcal{L}^F_{\bar\alpha}$, $\mathcal{L}^E_{\bar\alpha}$. Construct chains of embeddings

$$\bar M = \bar M_0 \to \bar M_1 \to \bar M_2 \to \cdots \to \bar M_\lambda$$
$$\bar N = \bar N_0 \to \bar N_1 \to \bar N_2 \to \cdots \to \bar N_\lambda$$

until $M_\lambda = N_\lambda$. Then conclude that $\bar M = \bar N$.

Where do the embeddings come from?

$\bar{M} = (L^F_{\bar{\alpha}}, \in, F_{\bar{\alpha}})$, where $F = \langle F_\beta \mid \beta < \bar{\alpha} \rangle$

Choose $\beta \leq \bar{\alpha}$. Then
$F_\beta : L^F_{\bar{\beta}} \to L^F_\beta$.

Extend F_β to
$F^*_\beta : L^F_{\bar{\alpha}} \to L^{F^*}_{\bar{\alpha}^*}$.

Now adjoin the predicate $F_{\bar{\alpha}}$ to get
$F^*_\beta : \bar{M} = (L^F_{\bar{\alpha}}, \in, F_{\bar{\alpha}}) \to (L^{F^*}_{\bar{\alpha}^*}, \in, F^*_{\bar{\alpha}^*}) = \bar{M}^*$.

$F^*_\beta : \bar{M} \to \bar{M}^*$ is the *ultrapower embedding*
of $\bar{M}$ via F_β.

Thus the chains

$\bar{M} = \bar{M}_0 \to \bar{M}_1 \to \bar{M}_2 \to \cdots \to \bar{M}_\lambda$
$\bar{N} = \bar{N}_0 \to \bar{N}_1 \to \bar{N}_2 \to \cdots \to \bar{N}_\lambda$

are obtained by taking *iterated ultrapowers*

Key question: Is $\bar{M}$ iterable, i.e., are the models $\bar{M} = \bar{M}_0 \to \bar{M}_1 \to \bar{M}_2 \to \cdots \to \bar{M}_\lambda$ well-founded?

If so, comparison works and Condensation can be proved!

Iterability problem. Show that there are iterable structures $M = (L^E_\alpha, \in, E_\alpha)$ which contain large cardinals.

Still open; solved only up to a Woodin limit of Woodin cardinals.

Outer model program. Show that any model with large cardinals has an L-like outer model with large cardinals.

Are there any (proper) outer models?

Treat V as a *countable* transitive model of GB (Gödel-Bernays class theory)

Outer model of V = a countable transitive model of GB which contains all the sets and classes of V

By forcing, V has many outer models

The *inner model program* has reached Woodin limits of Woodin cardinals.

But the *outer model program* has gone *all the way!*

Theorem 2. Suppose that there is a superstrong cardinal. Then there exists an outer model $L[A]$ of V (obtained by forcing) such that:

1. A is a class of ordinals.
2. $L[A] \models$ There is a superstrong cardinal.
3. A is definable over $L[A]$.
4. *Condensation:* With mild restrictions, $M \prec (L_\alpha[A], \in, A \cap \alpha)$ implies M is isomorphic to some $(L_{\bar{\alpha}}[A], \in, A \cap \bar{\alpha})$.
5. $L[A] \models \diamondsuit$, $\square$ and (gap 1) Morass

3 $\longrightarrow$ definable wellordering

4 $\longrightarrow$ GCH

What is a superstrong cardinal?

Suppose $j : V \to M$.

Critical point of j = least ordinal κ such that $j(\kappa) \neq \kappa$.

j is α-strong iff $V_\alpha \subseteq M$

Superstrong = $j(\kappa)$-strong
Hyperstrong = $j(\kappa) + 1$-strong
n-superstrong = $j^n(\kappa)$-strong
ω-superstrong = $j^\omega(\kappa)$-strong
$j^\omega(\kappa) + 1$-strong is inconsistent!

ω-superstrong is at the edge of inconsistency

κ is n-superstrong iff j is n-superstrong
(similarly for hyperstrong, ω-superstrong)

Hyperstrong $\rightarrow \square$ fails

Theorem 3. With $\square$ omitted, Theorem 2 holds for ω-superstrong

Conclusion:
L-*like* is consistent with superstrong
L-*like without* $\square$ is consistent with all large cardinals

Q2: The inner model hypothesis

Inner model hypothesis. If a sentence φ holds in an inner model of some outer model of V (i.e., in some model compatible with V), then it already holds in some inner model of V.

The IMH implies that there are no large cardinals in V:

Theorem 4. The IMH implies that for some real R, there is no transitive set model of ZFC containing R. In particular, there are no inaccessible cardinals and the Singular Cardinal Hypothesis is true.

The IMH implies however that there are large cardinals in inner models:

Theorem 5. The IMH implies the existence of an inner model with measurable cardinals of arbitrarily large Mitchell order.

The IMH is consistent relative to large cardinals:

Theorem 6. The consistency of the IMH follows from the consistency of a Woodin cardinal with an inaccessible cardinal above it.

The strong inner model hypothesis

Fact: The IMH with arbitrary ordinal parameters or with arbitrary real parameters is inconsistent.

The parameter p is *(globally) absolute* iff there is a parameter-free formula which has p as its unique solution in all outer models of V with the same cardinals as V up to hcard (p), the cardinality of the transitive closure of p

Strong inner model hypothesis. Suppose that p is absolute, V^* is an outer model of V with the same cardinals $\leq$ hcard (p) as V and φ is a sentence with parameter p which holds in an inner model of V^*. Then φ holds in an inner model of V.

The SIMH solves the continuum problem:

Theorem 7. Assume the SIMH. Then CH is false. In fact, $2^{\aleph_0}$ cannot be absolute and therefore cannot be $\aleph_\alpha$ for any ordinal α which is countable in L.

The SIMH implies that there are very large cardinals in inner models:

Theorem 8. The SIMH implies the existence of an inner model with a strong cardinal.

Is the SIMH consistent relative to large cardinals?

Gödel

Referring to *maximum principles* in set theory, Gödel said:

"I believe that the basic problems of abstract set theory, such as Cantor's continuum problem, will be solved satisfactorily only with the help of axioms of *this* kind."

I think that Gödel would have liked the Inner Model Hypothesis!

But will the IMH be adopted by the set theory community?

Time will tell...