The Hyperuniverse

The Hyperuniverse $\mathcal{H}$ is the collection of all countable transitive models of ZFC

The Hyperuniverse is interesting for 2 reasons:

(Mathematical)

- Much of set theory is about building transitive models of ZFC.
- By Löwenheim-Skolem, the first-order properties of these models all appear in models of the Hyperuniverse.
- The Hyperuniverse is closed under all techniques for building new countable transitive models from old ones and therefore provides the broadest range of possibilities for natural interpretations of set theory.

The Hyperuniverse

(Philosophical)
The Hyperuniverse can be used to formulate principles of set-theoretic truth (*The Hyperuniverse Program*):

- Elements of the Hyperuniverse provide *possible pictures of V* which mirror all possible first-order properties of V.
- We can formulate natural criteria for *preferred* elements of the Hyperuniverse based on their status within the Hyperuniverse as a whole.
- Under the assumption that first-order properties of the real universe are mirrored by preferred elements of the Hyperuniverse, we can regard the first-order properties shared by these preferred universes as being “true” in V.

The Hyperuniverse

This tutorial will deal primarily with the mathematical, but also touch upon the philosophical, aspects of the Hyperuniverse.

This work raises numerous issues in forcing, definability, large cardinals, determinacy and infinitary logic.

But before this study can begin we have to clear up one point:

Consistently with ZFC, the Hyperuniverse is empty!

Of course set theory is now sufficiently advanced that we can safely make the following:

Assumption: Every real belongs to a transitive model of ZFC.

This gives us elements of the Hyperuniverse of arbitrarily large countable height and through the methods of forcing leads to a rich variety of universes within the Hyperuniverse.

Part I: The Structure of the Hyperuniverse

Structural Features of the Hyperuniverse

A. Notions of inner model.

The first questions to ask concern the relation of inclusion between universes.
Let M, N be universes (i.e., elements of the Hyperuniverse) of the same ordinal height.

M is an *inner model* of N iff N contains M
M is a *strong inner model* of N iff in addition (N, M) models ZFC
M is a *definable inner model* of N iff in addition M is N-definable

Clearly the 1st and 3rd of these notions are transitive.

1. The notion of "strong inner model" is not transitive, and therefore the three notions of inner model are distinct.

Part I: The Structure of the Hyperuniverse

1. The notion of "strong inner model" is not transitive.

Proof Sketch: Start with $V_0 \models V = L$.
Let C_0, C_1 be generic over V_0 for ∞-Cohen, the forcing that adds a Cohen class of ordinals. Also arrange that C_0, C_1 agree except on a cofinal subset of $\mathrm{Ord}(V_0)$ of ordertype ω.
Force over (V_0, C_0) to add an $\aleph_{\alpha\times 2+1}$-Cohen generic for α in C_0, using an Easton product. This is the model V_1.
Then force over (V_1, C_0) to add an $\aleph_{\alpha\times 2+1}$-Cohen generic for all α not in C_0; this is the model V_2.
Finally, force over (V_2, C_1) to add an $\aleph_{\alpha\times 2+2}$-Cohen generic for α in C_1; this is the model V_3.

Then (V_2, V_1) and (V_3, V_2) are models of ZFC but both C_0 and C_1 are definable in (V_3, V_1) so the latter is not a model of ZFC.

Part I: The Structure of the Hyperuniverse

B. The inner model partial order

We look now at universes of a fixed ordinal height α under the (plain) inner model ordering.

2. There is a smallest universe.

This is L_α.

Universes M, N of height α are *compatible* iff they have a common outer model.

3. There are incompatible universes of height α.

Proof: Let C be a real coding α.
Build reals A, B which are Cohen generic over L_α and have the following property:
Let $(k_n \mid n \in \omega)$ enumerate the places where A, B differ in increasing order; then $A(k_n) = 0$ iff n belongs to C.
Then $L_\alpha[A], L_\alpha[B]$ are incompatible universes, as $C \leq_T (A, B)$.

Part I: The Structure of the Hyperuniverse

4. There are universes V_0, V_1 of height α which are compatible but have no least common outer model.

Proof: This is similar to the proof of intransitivity for the notion of strong outer model.
Let C_0, C_1 be ∞-Cohen generics over L_α whose union is the complement of a cofinal subset of α of ordertype ω.
Let V_0 be generic over (L_α, C_0) for adding an $\aleph_{\beta+1}$-Cohen set for β in C_0, via an Easton product.
Define V_1 in the same way, using C_1 instead of C_0.
Then any common outer model of V_0, V_1 must have $\aleph_{\beta+1}$-Cohen generics on a final segment of the $\aleph_{\beta+1}$'s for β not in the union of C_0, C_1.
There is no least such common outer model.

5. There are two universes of height α with no largest common inner model.

Proof: Modify the previous proof by choosing C_0, C_1 to be ∞-Cohen generics over L_α whose intersection is a cofinal subset of α or ordertype ω.
Define V_0, V_1 by adding $\aleph_{\beta+1}$-Cohen generics for all β and then taking only those generics for β in C_0, C_1, respectively.
Then the intersection of V_0, V_1 is the non-ZFC model V_2 generated by $\aleph_{\beta+1}$-Cohen generics for a cofinal set of β's of ordertyper ω.
There is no largest ZFC-model contained in V_2.

Part I: The Structure of the Hyperuniverse

6. There is an increasing ω-chain of universes of height α with no least upper bound.

Proof: Take $(R_n \mid n \in \omega)$ to be mutually Cohen over L_α and let V_n be $L_\alpha[R_0, \ldots, R_n]$.
Now let V^* be the choiceless model $L_\alpha(A \cup \{A\})$ where $A = \{R_n \mid n \in \omega\}$.
There are two models of ZFC whose intersection is V^*.
It follows that any least upper bound of the V_n's must be contained in V^*.
But in V^* there is no wellordered sets of reals containing A so no inner model of V^* satisfying choice can contain all of the V_n's.

Part I: The Structure of the Hyperuniverse

7. There is an increasing ω-chain of universes of height α with a least upper bound.

Proof: Let $L_\alpha[G]$ be obtained by adding a β-Cohen set over L_α for each regular β.
Let $(\alpha_n \mid n < \omega)$ be an ω-sequence cofinal in α and define $V_n = L[G \restriction \alpha_n]$.
Then the union of the V_n's is the ZFC-model $L[G]$, which is clearly the least upper bound.

Remark. Using Jensen coding, it is in fact possible to get an increasing ω-chain of universes, each of the form $L_\alpha[R_n]$ for some real R_n, with a least upper bound.

8. There is a decreasing ω-chain of universes of height α with a greatest lower bound.

Proof: Let $(R_i \mid i \in \omega)$ be mutually Cohen over L_α and V_n the model $L_\alpha[(R_i \mid i > n)]$.
Then the intersection of the V_n's is L_α:
Suppose that σ_n is a $P(> n)$-name for each n, where P is the forcing to add the R_i's and $P(> n)$ is the forcing to add the R_i's for $i > n$.
Suppose that $p \in P$ forces that the σ_n's are all equal.
If p does not force the σ_n's to belong to L_α then there are p_0, p_1 extending p which force different values of σ_0 at some number k and therefore different values of σ_n at k for each n.
But this is impossible if n is beyond the support of p_0, p_1.

9. There is a decreasing ω-chain of universes of height α with no greatest lower bound.

Proof: Let G be generic for adding an $\aleph_{\beta+1}$-Cohen set over L_α for each β, choose a sequence $(\alpha_i \mid i \in \omega)$ cofinal in α and let V_n be $L[G_n]$ where G_n is G without the $G(\alpha_i)$, $i < n$.
Then the intersection of the V_n's is $L[G']$ where G' is G without all of the $G(\alpha_i)$, $i < \omega$ and this non-ZFC model has no greatest ZFC submodel.

C. Jensen coding and minimality

Universes have outer models of a special form.

Theorem

(Jensen) Suppose that M is a universe of height α. Then M has an outer model of the form $L_\alpha[R]$ for some real R. Moreover, if M satisfies GCH then $H(\gamma)^M$ is definable over $L_\gamma[R]$ for each cardinal γ of M.

A universe M is *minimal over a real* iff for some real R, M is the least universe (of any ordinal height) containing R.

10. Every universe has an outer model which is minimal over a real.

Proof: In light of Jensen's theorem we may assume that M is of the form $L_\alpha[R]$.
Now force a club C of cardinals γ such that $L_\gamma[R]$ does not satisfy ZFC.
Then collapse cardinals to ensure that all limit cardinals belong to C and apply Jensen's theorem again.
The result is a model of the form $L_\alpha[R']$ in which ZFC fails in $L_\gamma[R']$ for all cardinals γ.

Now use:

Part I: The Structure of the Hyperuniverse

Theorem

(R.David-SDF) Suppose that $N = L_\alpha[R]$ is a model of ZFC, φ is a Σ_1 formula with parameter R and $N \vDash \varphi(\gamma)$ for every cardinal γ of N. Then for some real S, $L_\alpha[S]$ is an outer model of N satisfying $\varphi(\delta)$ for every S-admissible δ.

Apply this to the model $L_\alpha[R']$ and the formula $\varphi(\gamma) \equiv (L_\gamma[R] \nvDash \text{ZFC})$.
This gives a real S such that ZFC fails in $L_\delta[R]$ for all S-admissible δ and therefore $L_\alpha[S]$ is the least universe containing the reals R, S.

D. Nodes in the Hyperuniverse

A universe M of height α is a *node for comparability* iff every universe of height α is comparable with M, i.e., either contains M or is contained in M.

M is a *node for compatibility* iff every universe of height α is compatible with M.

Obviously L_α is a node for comparability.

11. Suppose that M is a universe of height α which is a node for comparability. Then M equals L_α.

Proof: There is an uncountable set of reals X such that any two distinct elements of X are mutually Cohen over L_α.

If M is contained in $L_\alpha[R]$ for two distinct R in X then $M = L_\alpha$.

Otherwise M must contain all but one element of X, contradicting its countability.

Part I: The Structure of the Hyperuniverse

Open Question: Is L_α the only node for compatibility of height α? I.e., if M is a universe of height α which is compatible with all universes of height α, must M equal L_α?

12. Suppose that M has height α and contains a function $f : \omega \to \omega$ that is not dominated by such a function in L_α. Then M is not a node for compatibility.

Proof: Using f we can build a Cohen real R so that R codes any real (such as a code for α) on the range of f.

Then $L_\alpha[R]$ and M are incompatible universes.

It can also be shown that if M is $L_\alpha[S]$ where S is Sacks-generic over L_α then again M is not a node for compatibility.

E. Characterisable universes

A universe M of height α is α-*characterisable* iff for some sentence φ, M is the unique universe of height α satisfying φ.

13. Suppose that M is α-characterisable. Then M is an element of L_β where β is the least admissible greater than α. Therefore if α is a cardinal in L_β, M must equal L_α.

Proof: Let φ witness that M is α-characterisable.

Let $\mathcal{L}_\beta$ denote the admissible fragment of $L_{\omega_1\omega}$ determined by the admissible set L_β.

Let ψ be the sentence in this fragment given by:

ZFC $+ \varphi$

$\forall x(x$ is an ordinal iff $\bigvee_{\gamma<\alpha} x = \underline{\gamma})$

Then ψ is consistent and complete, and therefore has a model which is an element of L_β; this is the unique model of φ of height α.

Part I: The Structure of the Hyperuniverse

A universe M is *characterisable* iff for some sentence φ, M is the unique universe satisfying φ (of any height).

14. (Must be checked!) There is a characterisable M which does not satisfy $V = L$.

Proof Sketch: This uses ideas from the construction of a Π^1_2 singleton which is class-generic over L.
Let L_α be the minimal model of ZFC.
Associate to each ordinal $\gamma < \alpha$ a "guess" $(\gamma_2, \ldots, \gamma_n)$ where $\gamma_i < \gamma$ is least so that L_{γ_i} is Σ_i elementary in L_γ (and γ_{n+1} does not exist).
Using the Recursion Theorem let $\gamma \mapsto p(\gamma)$ be a Σ_1-definable procedure which produces a P-generic when applied to the $\gamma^*_n =$ the least γ such that L_γ is Σ_n elementary in the full L_α.

Part I: The Structure of the Hyperuniverse

Define a suborder of ∞-Cohen $*$ MacAloon coding, where the latter kills GCH at $\aleph_{\alpha+1}$ for α in the ∞-Cohen generic.
Inductively define the conditions of length γ in this forcing as follows:
At stage γ, take all conditions which force that $p(\gamma)$ belongs to the generic and whose first coordinate is Σ_k-generic for P_γ for all k such that γ is Σ_k-admissible; also take those which force that $p(\gamma)$ is not in the generic and whose first coordinate is Δ_2-definable over L_γ.
Then there is only one possible generic for the resulting forcing P as no generic can be Δ_2-definable at an increasing chain of γ_n's such that L_{γ_n} is Σ_n elementary in L_α, and therefore any generic must agree with the generic produced by the conditions $p(\gamma_n^*)$, $n \in \omega$.
The characterising sentence φ says that the universe is P-generic over L_α for antichains which belong to L_α.

Part II: Genericity in the Hyperuniverse

A. Notions of Genericity

We have been talking about arbitrary outer models.
But sometimes we want to only consider outer models which are obtained by some type of forcing.

Let M be a universe of height α.

G is *set-generic over M* iff for some forcing $P \in M$, G is a pairwise compatible, upward-closed subset of P which meets every dense subset of P in M.

N is a *set-generic outer model of M* iff $N = M[G]$ for some G which is set-generic over M.

Part II: Genericity in the Hyperuniverse

G is *definable class-generic* over M iff for some M-definable forcing P, G is a pairwise compatible, upward-closed subset of P which meets every M-definable dense subset of P.

N is a *definable class-generic outer model of* M iff $N = M[G]$ for some G which is definable class-generic over M.

Further notions of genericity make use of models of class theory.

$(M, \mathcal{C})$ is a *model of GB* iff $\mathcal{C}$ is a collection of subsets of M (the members of $\mathcal{C}$ are "the classes") such that:
i. For any $A_1, \ldots, A_n$ from $\mathcal{C}$, $(M, A_1, \ldots, A_n)$ is a model of ZFC (with the A_i's as additional predicates)
ii. Any subset of M definable over $(M, A_1, \ldots, A_n)$ belongs to $\mathcal{C}$.

Part II: Genericity in the Hyperuniverse

Now let $(M, \mathcal{C})$ be a model of GB.

G is *class-generic over* $(M, \mathcal{C})$ iff for some forcing P in $\mathcal{C}$, G is a pairwise compatible, upward-closed subset of P which meets every dense subset of P in $\mathcal{C}$.

A model $(N, \mathcal{D})$ of GB is a *class-generic outer model of* $(M, \mathcal{C})$ iff for some G which is class-generic over $(M, \mathcal{C})$, $N = M[G]$ and $\mathcal{D}$ consists of those subsets of $M[G]$ which are definable over $(M[G], G, A)$ for some $A \in \mathcal{C}$.

G is *definable hyperclass-generic over* $(M, \mathcal{C})$ iff for some $(M, \mathcal{C})$-definable forcing $P \subseteq \mathcal{C}$, G is a pairwise compatible, upward-closed subset of P which meets every dense subset of P which is definable over $(M, \mathcal{C})$.
(Note that G is not a "class", i.e. subset of M, but a "hyperclass", i.e., subset of $\mathcal{C}$.)

Part II: Genericity in the Hyperuniverse

To discuss definable hyperclass-generic extensions it is necessary to assume more than GB in the ground model $(M, \mathcal{C})$.

We need a strengthened form of Morse-Kelley class theory:

*Axioms of MK**

a. GB.

b. $\{x \mid \varphi(x)\}$ (where x ranges over sets) is a class, even if φ quantifies over classes and has class parameters.

c. If for all sets x there is a class A such that $\varphi(x, A)$, then there is a fixed class B such that for all x, $\varphi(x, (B)_y)$ holds for some y, where $(B)_y = \{z \mid (y, z) \in B\}$.

(Again, φ may quantify over classes and include class parameters.)

Our next aim is to define the structure $(M, \mathcal{C})[G]$ when G is definable hyperclass-generic over $(M, \mathcal{C})$.

This is best done by translating the theory MK* into a first-order set theory called SetMK.

Part II: Genericity in the Hyperuniverse

The axioms of SetMK are:

a. ZF^- (ZF minus Power Set).
b. There is a strongly inaccessible cardinal κ (in particular V_κ exists and models choice).
c. Every set can be mapped injectively into V_κ.
(We don't require that V_κ can be wellordered.)

15. (a) If $(M, \mathcal{C})$ is a model of MK^* where M has height α then there is a unique model M^* of SetMK with largest cardinal α such that $M = V_\alpha^{M^*}$ and the elements of $\mathcal{C}$ are the subsets of M in M^*.
(b) Conversely, if M^* is a model of SetMK with largest cardinal α then $(M, \mathcal{C})$ is a model of MK^*, where $M = V_\alpha^{M^*}$ and $\mathcal{C}$ consists of the subsets of M in M^*.

Proof Sketch. (a) Given $(M, \mathcal{C})$, take M^* to be the union of all transitive sets isomorphic to some structure (M, R) where R is a binary relation on M in $\mathcal{C}$.
The fact that $(M, \mathcal{C})$ models MK^* implies that M^* models bounding and comprehension principles, hence all of ZF^-.
It is straightforward to check the other axioms of SetMK and the uniqueness of M^*.
(b) This is also straightforward, using the bounding principle in M^* to verify the 3rd axiom of MK^*.

Part II: Genericity in the Hyperuniverse

Now assuming that $(M, \mathcal{C})$ models MK* we can define the generic extension $(M, \mathcal{C})[G]$ for definable hyperclass-generic G.
Let P be the $(M, \mathcal{C})$-definable forcing for which G is P-generic.

Let M^* be the model of SetMK associated to $(M, \mathcal{C})$ and inductively define P-names in M^* in the usual way, after Kunen:

A P-name in M^* is a set in M^* consisting of pairs (τ, p) where τ is a P-name in M^* and p belongs to P.

For P-names σ, σ^G denotes $\{\tau^G \mid p \in G \text{ for some } (\tau, p) \in \sigma\}$.

Then $M^*[G]$ is the set of all such τ^G and $(M, \mathcal{C})[G]$ is the model of class theory derived from $M^*[G]$, whose sets are the elements of $V_\alpha^{M^*[G]}$ and whose classes are the subsets of $V_\alpha^{M^*[G]}$ in $M^*[G]$.

Now if $(M, \mathcal{C})$ is a model of MK^* then a model $(N, \mathcal{D})$ of MK^* is a *definable hyperclass-generic outer model of* $(M, \mathcal{C})$ iff for some G which is definable hyperclass-generic over $(M, \mathcal{C})$, $(N, \mathcal{D}) = (M, \mathcal{C})[G]$, as defined above.

B. Tameness

Unlike for set-forcing, generics for definable class-forcing, class-forcing or definable hyperclass-forcing do not necessarily yield models of ZFC, GB and MK^*, respectively.

But under the right circumstances (when the forcings are "tame"), they do:

Part II: Genericity in the Hyperuniverse

First recall the proof that set-forcing preserves ZFC:
Let M denote the ground model and P the set-forcing in question.

a. The forcing relation is definable: For each $\varphi(x_1, \ldots, x_n)$ the relation $p \Vdash \varphi(\sigma_1, \ldots, \sigma_n)$ is an M-definable relation of $(p, \sigma_1, \ldots, \sigma_n)$ (where the σ_i's range over P-names in M).

b. Using the definability of the forcing relation, the Truth Lemma holds:
For G which are P-generic over M, $M[G] \vDash \varphi(\sigma_1, \ldots, \sigma_n)$ iff $p \Vdash \varphi(\sigma_1, \ldots, \sigma_n)$ for some $p \in G$.

Part II: Genericity in the Hyperuniverse

c. To verify the Bounding Principle, argue as follows:
If $p \Vdash \forall x \in \sigma \exists y \varphi(\sigma, y)$ then for each P-name σ_0 of rank $<$ rank(σ), the collection of $q \leq p$ such that $q \Vdash \varphi(\sigma_0, \tau)$ for some τ is dense below p.
Now apply the Bounding Principle in M to obtain a set T such that this is still true if we restrict τ to belong to T.
Then use T and the definability of the forcing relation to form a name Σ so that $p \Vdash \forall x \in \sigma \exists y \in \Sigma \varphi(\sigma, y)$.
d. Separation ($=$ Comprehension) and AC follow easily from Separation and AC in M, using the definability of the forcing relation.
e. To verify the Power Set axiom, use the fact that for any ordinal α, each subset of α has a nice name of the form $\bigcup_{\beta<\alpha}\{(\check{\beta}, p) \mid p \in A_\beta\}$ where each A_β is a subset of P, and the set of all such names forms a set in M.

Part II: Genericity in the Hyperuniverse

Now suppose that P is a definable class-forcing in M.
Can we repeat steps (a)-(e) to show that $M[G]$ is a model of ZFC for P-generic G?

Counterexamples:
Collapsing the universe to ω: Consider $P = \{p : \text{Dom}(p) \to M \mid \text{Dom}(p)$ a finite subset of $\omega\}$. A P-generic adds a map from ω onto M, so kills the Bounding Principle.
Too many reals: Consider $P = \{p : \text{Dom}(p) \to 2 \mid \text{Dom}(p)$ a finite subset of $\text{Ord}(M) \times \omega\}$. A P-generic adds reals R_β, $\beta \in \text{Ord}(M)$, so kills the Power Set axiom.

To obtain ZFC-preservation we need to worry about:
(a) Definability of the forcing relation.
(c) The Bounding Principle
(e) The Power Set axiom.

The proof of the Bounding Principle for set-forcing immediately suggests the following:

Pretameness Condition. Suppose that $a \in M$, $p \in P$, $D \subseteq a \times P$ is M-definable and $(D)_i = \{p \mid (i, p) \in D\}$ is dense below p for each $i \in a$.

Then there exists $d \subseteq D$, $d \in M$ and $q \leq p$ in P such that each $(d)_i$ is predense below q (i.e., each $r \leq q$ is compatible with some element of $(d)_i$).

Given Pretameness and the Definability of the forcing relation, it is straightforward to verify the Bounding Principle in P-generic extensions.

Part II: Genericity in the Hyperuniverse

Fortunately, Pretameness is also sufficient for the Definability of the forcing relation:

16. Suppose that P is pretame. Then the forcing relation is definable.

Proof Sketch: The key step is to show that there is an "effective" way to extend any $p \in P$ to a $q \in P$ which decides $p \Vdash \sigma \in \tau$ for P-names σ, τ (and similarly for $p \Vdash \sigma = \tau$).

By induction we have an effective way to extend any $q \leq p$ to r deciding $\sigma = \tau_0$ for P-names τ_0 of rank less than the rank of τ. This gives us a definable class D such that $(D)_{\tau_0}$ is a dense class of conditions deciding $\sigma = \tau_0$ for each such τ_0.

Now apply Pretameness to effectively extend p either to force $\sigma = \tau_0$ for some τ_0 or to force $\sigma \neq \tau_0$ for each τ_0 and therefore force $\sigma \notin \tau$.

Part II: Genericity in the Hyperuniverse

To preserve the Power Set axiom it is necessary to show that for each $a \in M$ and $p \in P$, there is $q \leq p$ and a set $S \in M$ such that it is dense below q to force any P-name for a subset of a to be equal to a P-name in S.

In practice this happens in one of two ways:

i. For each $a \in M$, P factors as $P_0 * P_1$ where P_1 does not add subsets of a and P_0 is a set-forcing.

ii. For each $a \in M$, P factors as $P_1 * P_0$ where P_1 does not add subsets of a and P_0 is a set forcing.

The first option is typical of reverse Easton iterations and the second of forcings which resemble Easton products, like Jensen codings.

Part II: Genericity in the Hyperuniverse

Note that Pretameness already gives the Definability of the forcing relation, so the preservation of Power Set is a first-order condition on M, expressed by "P forces the Power Set axiom".
So we can legitimately define:

Tameness Condition. P is pretame and forces the Power Set axiom.

Remarks. (a) Tameness gives us slightly more than ZFC-preservation, as it implies that for P-generic G, $(M[G], M)$ is a model of ZFC (with M as an additional predicate).
This is because the relation "$p \Vdash \sigma \in M$" is M-definable.
(b) Conversely, ZFC-preservation with M as a predicate implies Tameness.

Part II: Genericity in the Hyperuniverse

Preserving GB with class forcing is similar, with definable classes replaced by arbitrary classes of the given GB model:

Tameness for Class Forcing. A class forcing P in the GB model $(M, \mathcal{C})$ is *pretame* iff for any condition p, $a \in M$ and $D \subseteq a \times P$ in $\mathcal{C}$ such that $(D)_i = \{p \mid (i, p) \in D\}$ is dense below p for each $i \in a$, there exists $d \subseteq D$, $d \in M$ and $q \leq p$ in P such that each $(d)_i$ is predense below q.
P is *tame* iff it is pretame and forces the Power Set axiom.

And Tameness is equivalent to GB-preservation.
(As $(M, \mathcal{C})[G]$ includes M as a class, there is no need to adjoin M as an additional predicate, as was the case for definable class forcing.)

Part II: Genericity in the Hyperuniverse

Tameness for Definable Hyperclass forcing is obtained by translating pretameness for models of SetMK into class theory and looks like this:

A definable hyperclass forcing P in the MK* model $(M, \mathcal{C})$ is *pretame* iff for any condition p, and $(M, \mathcal{C})$-definable $D \subseteq M \times P$ such that $(D)_i = \{p \mid (i, p) \in D\}$ is dense below p for each $i \in M$, there exists $d \in \mathcal{C}$ such that for each $i, j \in M$, $(i, (d)_{i,j}) \in D$ (where $(d)_{i,j} = \{a \in M \mid (i, j, a) \in d\}$) and for eaech $i \in M$, $\{(d)_{i,j} \mid j \in M\}$ is predense below q.
P is *tame* iff it is pretame and forces the axioms of GB.

Tameness for Definable Hyperclass forcing is equivalent to MK*-preservation.
The proof passes through the associated model of SetMK and uses the fact that this model is $L_\beta(M)$ where β is the least ordinal not coded by an element of $\mathcal{C}$.

Part II: Genericity in the Hyperuniverse

C. Separating notions of genericity

a. Let M be a countable transitive model of ZFC.
Then there is a real R which belongs to a definable class-generic extension of M but to no set-generic extension of M.

Proof: Force to make GCH hold everywhere to get $M[G]$ and then add an ∞-Cohen class of ordinals A to get $M[G] = L[A]$.
Finally, use Jensen coding to add a real R so that A is definable in $L[R]$.
Then R belongs to no set-generic extension of M as otherwise A would be definable in M.

Part II: Genericity in the Hyperuniverse

b. Let $(M, \mathcal{C})$ satisfy GB where $\mathcal{C}$ includes the satisfaction predicate $\mathrm{Sat}(M)$ for M ($\mathrm{Sat}(M) = \{(\varphi, x) \mid M \vDash \varphi(x)\}$).
Then there is a real R which belongs to a class-generic extension of $(M, \mathcal{C})$ but to no definable class-generic extension of M.

Proof: Use Jensen coding over $(M, \mathcal{C})$ to make $\mathrm{Sat}(M)$ definable from a real R.
Then R is in a class-generic extension of $(M, \mathcal{C})$ but in no definable class-generic extension of M as otherwise by the Truth Lemma, $\mathrm{Sat}(M[R])$ would be definable over $(M, \mathrm{Sat}(M))$ and hence over $M[R]$, contradicting Tarski.

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17. Suppose that $(M, \mathcal{C})$ satisfies MK^*.
Then there is a real R which belongs to a definable hyperclass-generic extension of $(M, \mathcal{C})$ but to no class-generic extension of $(M, \mathcal{C}_0)$ for any GB model $(M, \mathcal{C}_0)$ where $\mathcal{C}_0 \subseteq \mathcal{C}$.

(Note that the conclusion is stronger than saying that R belongs to no class-generic extension of $(M, \mathcal{C})$.)

Proof Sketch: Let M^* be the model of SetMK corresponding to $(M, \mathcal{C})$.
Now consider the following M^*-definable forcing:
By a variant of almost disjoint forcing, we add a subset X of κ (the largest cardinal of M^* = the ordinal height of M) such that for any subset A of κ in M^*, $A^{(\omega)}$ is definable over (M, A, X), where $A^{(\omega)}$ is the satisfaction predicate for (M, A).
This is a definable hyperclass forcing over $(M, \mathcal{C})$.

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Then X cannot belong to any class-generic extension of any GB model $(M, \mathcal{C}_0)$ for $\mathcal{C}_0 \subseteq \mathcal{C}$: If $P \in \mathcal{C}_0$ and $G \subseteq P$ witnessed this then via the Truth Lemma, $(X, G)^{(\omega)}$ would be definable over $(M, P^{(\omega)}, G)$ and therefore by the choice of X over (M, X, G), in contradiction to Tarski's undefinability of the satisfaction predicate. Finally use Jensen coding to code X by a real.

Part II: Genericity in the Hyperuniverse

D. Genericity and inner models

Recall:
M is an *inner model* of N iff N contains M
M is a *strong inner model* of N iff in addition (N, M) models ZFC
M is a *definable inner model* of N iff in addition M is N-definable

Now suppose that N is a generic outer model of M in some sense of generic; must M be a strong or even definable inner model of N?

We have seen that for definable class-forcing, M will be a strong inner model of N.
This is vacuously true for class-forcing and definable hyperclass-forcing as when extending $(M, \mathcal{C})$ to $(M, \mathcal{C})[G]$ we include M itself as a class and therefore ZFC holds relative to it.
So we focus on the question of whether M must be a definable inner model.

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19. (Laver) Suppose that N is a set-generic extension of M. Then M is a definable inner model of N.

Proof: Choose a V-regular κ so that P belongs to $H(\kappa)^M$, where V is P-generic over M. We need three facts:

i. M κ-covers V: Any subset X of M in V of size $< \kappa$ in V is a subset of such a set in M.

This is because if f maps some ordinal $\alpha < \kappa$ onto X then for each $i < \alpha$ there are $< \kappa$ possibilities for $f(i)$, given by the $< \kappa$ different forcing conditions.

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ii. M κ-approximates V: If X is a subset of M in V all of whose size $< \kappa$ M-approximations (i.e., intersections with size $< \kappa$ elements of M) belong to M, then X also belongs to M.

This is because if $\dot{X}$ is forced not to be in M then we can choose for each condition a set in M whose membership in $\dot{X}$ is not decided by that condition; no condition can force the intersection of $\dot{X}$ with the resulting size $< \kappa$ set of elements of M to be in M.

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iii. If N is an inner model which κ-covers and κ-approximates V such that M, N have the same $H(\kappa^+)$ then $M = N$.

By κ-approximation it's enough to show that any set X of ordinals of size $< \kappa$ in M also belongs to N (and vice-versa). Build a κ-chain $X = X_0 \subseteq X_1 \subseteq \cdots$ of sets of size $< \kappa$ such that $X_{2\alpha+1}$ belongs to M and $X_{2\alpha+2}$ belongs to N. If Y is the union of the X_α's then by κ-approximation, Y belongs to $M \cap N$. But as M, N have the same $H(\kappa^+)$ they also have the same subsets of the ordertype of Y and therefore the same subsets of Y. It follows that X belongs to N.

Finally: All of this holds with M, V replaced by $H(\lambda)^M, H(\lambda)$ for V-regular cardinals $\lambda > \kappa^+$. So $H(\lambda)^M$ is definable in V from λ, $H(\kappa^+)^M$ uniformly in λ, so M is V-definable.

Part II: Genericity in the Hyperuniverse

20. There exists $M \subseteq N$ where N is a definable class-generic extension of M, such that M is not definable as an inner model of N.

Proof Sketch: Start with some L_α and let P be the Easton product that adds an α-Cohen set for each regular α.
Let (G_0, G_1) be generic for $P \times P$ (which is isomorphic to P) and let $M = L_\alpha[G_0]$, $N = L_\alpha[G_0, G_1]$.
Then N is a P-generic extension of M and as P is L_α-definable it is also M-definable.
But one can show that no formula defines M is an inner model of N, using the homogeneity of the forcing and the fact that any parameter in a potential definition of M is captured by a bounded part of the $P \times P$-generic.

Part II: Genericity in the Hyperuniverse

E. Criteria for genericity

Suppose that M is an inner model of N.
Is there a simple criterion that determines whether or not N is a set-generic extension of M? First observe:

21. Suppose that N is a set-generic extension of M.
Then M globally covers N: For some N-regular κ*, if* $f : \alpha \to M$ *belongs to N then there is* $g : \alpha \to M$ *in M such that* $f(i) \in g(i)$ *and* $g(i)$ *has N-cardinality* $< \kappa$ *for all* $i < \alpha$*.*

Proof: Define $g(i)$ to be the set of possible values of $f(i)$ given by the different forcing conditions. We can choose any κ so that the forcing is κ-cc.

Surprisingly, this provides a simple criterion for set-generic extensions:

Part II: Genericity in the Hyperuniverse

22. (Bukovsky) Suppose that M is a definable inner model which globally covers N.
Then N is a set-generic extension of M.

Proof: First suppose that $N = M[A]$ for some set of ordinals A; we'll get rid of this extra hypothesis later.

Fix a N-regular κ such that A is a subset of κ and M globally κ-covers N, i.e., if $f : \alpha \to M$ in N then there is $g : \alpha \to M$ in M so that $f(i) \in g(i)$ and $g(i)$ has N-cardinality $< \kappa$ for each $i < \alpha$.

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The languages $\mathcal{L}^{QF}_{\kappa}(M)$, $\mathcal{L}^{QF}_{\kappa^+}(M)$

The formulas of $\mathcal{L}^{QF}_{\kappa}(M)$ are defined inductively by:

1. *Basic formulas* $\alpha \in \dot{A}$, $\alpha \notin \dot{A}$ for $\alpha < \kappa$.
2. If $\Phi \in M$ is a size $< \kappa$ set of formulas then so are $\bigvee \Phi$ and $\bigwedge \Phi$.

Each formula can be regarded as an element of $H(\kappa)^M$. The set of formulas forms a κ-complete Boolean algebra in M, denoted by $\mathcal{B}^M_{\kappa}$.

$\mathcal{L}^{QF}_{\kappa^+}(M)$ is defined similarly, replacing "size $< \kappa$" by "size $\leq \kappa$".

$A \subseteq \kappa$ *satisfies* φ iff φ is true when $\dot{A}$ is replaced by A.

$T \vDash \varphi$ iff for all $A \subseteq \kappa$ (in a set-generic extension of M), if A satisfies all formulas in T then A also satisfies φ.

The above is expressible in M for T, φ in M.

Part II: Genericity in the Hyperuniverse

Quotients of $\mathcal{B}^M_\kappa$: Suppose that T is a set of formulas in $\mathcal{B}^M_{\kappa^+}$. Then $\mathcal{I}_T$ is the ideal of formulas in $\mathcal{B}^M_\kappa$ which are inconsistent with T.

Now we prove the genericity of A over M.

Recall that M globally κ-covers N. Let f be a function in N from subsets of $\mathcal{B}^M_\kappa$ in M to $\mathcal{B}^M_\kappa$ such that:

If A satisfies some $\psi \in \Phi$ then A satisfies $f(\Phi) \in \Phi$.

Using a wellorder of $H(\kappa^+)^M$ we can regard f as a function from some ordinal α into M. Apply global κ-covering to get g in M so that $g(\Phi) \subseteq \Phi$ has size $< \kappa$ and $f(\Phi) \in g(\Phi)$ for each Φ.

Consider the following set of formulas T in $\mathcal{B}^M_{\kappa^+}$:

$T = \{(\bigvee \Phi \to \bigvee g(\Phi)) \mid \Phi \subseteq \mathcal{B}^M_\kappa,\ \Phi \in M\}$.

Let P be the forcing $(\mathcal{B}^M_\kappa \setminus \mathcal{I}_T)/\mathcal{I}_T$ the set of T-consistent formulas modulo T-provability.

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Claim 1. $P = (\mathcal{B}^M_\kappa \setminus \mathcal{I}_T)/\mathcal{I}_T$ is κ-cc.

Proof. Suppose that Φ is a maximal antichain in P. We show that $g(\Phi) = \Phi$ (and therefore Φ has size $< \kappa$). It suffices to show that any $\varphi \in \Phi$ is T-consistent with some element of $g(\Phi)$. Choose any $B \subseteq \kappa$ which satisfies $T \cup \{\varphi\}$ (this is possible because φ is T-consistent). As T includes the formula $\bigvee \Phi \rightarrow \bigvee g(\Phi)$ it follows that B also satisfies $\bigvee g(\Phi)$ and therefore ψ for some $\psi \in g(\Phi)$. So φ is T-consistent with $\psi \in g(\Phi)$. $\square$

Claim 2. Let $G(A)$ be $\{[\varphi]_{\mathcal{I}_T} \mid \varphi$ belongs to $\mathcal{B}^M_\kappa$ and A satisfies $\varphi\}$. Then $G(A)$ is P-generic over M.

Proof. Suppose that Φ consists of representatives of a maximal antichain X of equivalence classes in P. Then $T \vDash \bigvee \Phi$, else the negation of $\bigvee \Phi$ represents an equivalence class violating the maximality of X. As A satisfies the theory T it follows that A satisfies some element of Φ and therefore $G(A)$ meets X. $\square$

Part II: Genericity in the Hyperuniverse

It now follows that $M[A]$ is a P-generic extension of M, as $M[A] = M[G(A)]$.

This proves Bukovsky's theorem assuming that $N = M[A]$ for some set of ordinals A.

But the same proof shows that $M[A]$ is a κ-cc generic extension of M for any set of ordinals $A \in N$.
Choose A so that $M[A]$ contains all subsets of $2^{<\kappa}$ in N. Then $M[A]$ must equal all of N:
Otherwise for some set B of ordinals in N, $M[A, B]$ is a nontrivial κ-cc generic extension of $M[A]$ and therefore adds a new subset of $2^{<\kappa}$ to $M[A]$.

Part II: Genericity in the Hyperuniverse

Bukovsky for class forcing?

Is there a similar criterion to Bukovsky's that characterises definable class-generic extensions?

I don't know, but there are two simple criteria, one of which is necessary and the other sufficient for definable class-genericity, and which are "fairly close" to each other.

Stability predicates

Suppose that N is a countable transitive model of ZFC.
Work inside N.
For an infinite cardinal α, $H(\alpha)$ is defined as usual.
Let C be the club of infinite cardinals β such that:
$\alpha < \beta \to H(\alpha)$ has size $< \beta$.

$(n > 0)$ α is *n-stable* in β iff $\alpha < \beta$ are limit points of C and:
$(H(\alpha), C \cap \alpha) \prec_{\Sigma_n} (H(\beta), C \cap \beta)$.

The *Stability Predicate* $S = \{(\alpha, \beta, n) \mid \alpha$ is n-stable in $\beta\}$.
For any club C, we also take $S \restriction C$ to be the Stability Predicate restricted to C:
$S \restriction C = \{(\alpha, \beta, n) \mid \alpha, \beta$ belong to C and α is n-stable in $\beta\}$.

23. Suppose that N is a countable transitive model of ZFC and (M, A) is a definable inner model of N satisfying ZFC. Let S be the Stability Predicate of N. If $S \restriction C$ is (M, A)-definable for some N-definable club C, then N is a definable class-generic extension of (M, A).

(Note that we have taken the liberty of extending the notion of "definable class-genericity" to ZFC models with predicates.)

The model $(L[S], S)$ where S is the Stability Predicate of N is called the *Stable Core of N*.

As a partial converse:

24. Again suppose that N is a countable transitive model of ZFC, (M, A) is a definable inner model of N satisfying ZFC and S is the Stability Predicate of N.
If N is a definable class-generic extension of (M, A) then there is an (M, A)-definable predicate S' such that for some N-definable club C, $S' \restriction C = S \restriction C$.

Thus the definability in (M, A) of the Stability Predicate of N is "close" to being equivalent to the statement that N is a definable class-generic extension of (M, A): If we could repalce "N-definable club C" in the above by "(M, A)-definable club C", then we would have an exact equivalence.

Part II: Genericity in the Hyperuniverse

A corollary of the results about the Stability Predicate is:

25. Let M be a countable transitive model of ZFC.
Then M is definable class-generic over (HOD^M, S) for an M-definable predicate S.

Part II: Genericity in the Hyperuniverse

F. Transcendence

To what extent can arbitrary outer models be captured by forcing?

26. Suppose that N is a countable transitive model which satisfies that $0^\#$ exists. Let L_α be the L of N and suppose that $(M, \mathcal{D})$ is a class-generic extension of $(L_\alpha, \mathcal{C})$ where the latter satisfies GB. Then N is not contained in M.

Proof Sketch: Otherwise the $0^\#$ of N is an element of M and therefore in M one can define the Silver indiscernibles for L_α.
But then via the forcing relation, one can define the set of ordinals $i < \alpha$ which are forced by some condition to belong to the Silver indiscernibles, and a final segment of these ordinals i have the property that $(L_i, A \cap i)$ is elementary in (L_α, A), where the forcing is (L_α, A)-definable.
This contradicts Tarski's undefinability of the satisfaction relation for the model (L_α, A).

Part II: Genericity in the Hyperuniverse

There are similar results showing that $0^\#$ is not generic over L via hyperclass forcing and there is no forcing method which is known to be able to capture all outer models of L in which $0^\#$ does not exist.

Part III: Maximality in the Hyperuniverse

The study of *Maximality* in the Hyperuniverse is motivated by the *Hyperuniverse Program*.
As mentioned before, this approach to the study of set-theoretic truth works as follows:

• Elements of the Hyperuniverse provide *possible pictures of V* which mirror all possible first-order properties of V.
• We can formulate natural criteria for *preferred* elements of the Hyperuniverse based on their status within the Hyperuniverse as a whole.
• Under the assumption that first-order properties of the real universe are mirrored by preferred elements of the Hyperuniverse, we can regard the first-order properties shared by these preferred universes as being "true" in V.

Key for the program is the choice of criteria for preferred universes.

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Criteria are to be based on motivating principles which arise from an unbiased look at the Hyperuniverse.

One such motivating principle is *Maximality*.

Maximality for the universe of sets is an old idea, tracing back to Gödel and Scott:

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Gödel (1964):

"From an axiom in some sense opposite to [V=L], the negation of Cantor's conjecture could perhaps be derived. I am thinking of an axiom which ... would state some maximum property of the system of all sets, whereas [V=L] states a minimum property. Note that only a maximum property would seem to harmonize with the concept of set ..."

Scott (1977):

"I see that there are any number of contradictory set theories, all extending the Zermelo-Fraenkel axioms; but the models are all just models of the first order axioms and first-order logic is weak. I still feel that it ought to be possible to have strong axioms which would generate these types of models as submodels of the universe, but where the universe can be thought of as something absolute ... But really pleasant axioms have not been produced by someone else or me, and the suggestion remains speculation. A new idea (or point of view) is needed, and in the meantime all we can do is to study the great variety of models."

Part III: Maximality in the Hyperuniverse

Question. What does it mean for a universe to be "maximal"?

We use *truth in inner models* to define maximality:

Also, for technical reasons, we work with *definable inner models* rather than with general inner models

$\mathcal{L}$ = language of ZFC
For a universe W:
$\Phi(W)$ = all sentences of $\mathcal{L}$ which are true in some definable inner model of W

Obviously:
V a definable inner model of $W \rightarrow \Phi(V) \subseteq \Phi(W)$

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V is *maximal* iff:
V a definable inner model of $W \to \Phi(V) = \Phi(W)$

27. (with Woodin) Assume PD. Then there are maximal universes.

Proof: Assume PD.
For each real R let $M(R)$ denote the smallest transitive model of ZFC containing R.
For each sentence φ there is a real R_φ such that either $M(R) \vDash \varphi$ for all $R \geq_T R_\varphi$ or $M(R) \nvDash \varphi$ for all $R \geq_T R_\varphi$.
Choose R^* so that $R_\varphi \leq_T R^*$ for all φ;
then $\mathrm{Thy}(M(R))$ is constant for $R \geq_T R^*$.

Claim. $M(R^*)$ is a maximal universe.

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Claim. $M(R^*)$ is a maximal universe.

We need to show that if $M(R^*)$ is a definable inner model of N and N has a definable inner model M satisfying some sentence φ, then also $M(R^*)$ has a definable inner model satisfying φ.

Apply Jensen coding to N to produce a real S such that $R^* \leq_T S$ and $M(S)$ has N (and therefore also M) as a definable inner model. If ψ defines M in $M(S)$ then $M(S)$ satisfies the sentence:
"ψ with some choice of parameters defines a transitive inner model of ZFC $+ \varphi$"
(Note that this sentence is first-order, as "ZFC" can be replaced by a finite subtheory)
But by choice of R^*, Thy($M(S)$) equals Thy($M(R^*)$), so also $M(R^*)$ has a definable inner model satisfying φ, as desired.

Remarks. (a) Full PD is not needed for the preceding proof; it is enough to have a bit more than lightface PD (a Woodin cardinal with an inaccessible above is enough).
(b) Welch and I showed that the existence of maximal universes gives the consistency of measurable cardinals of any Mitchell order.

The Maximality Paradox

The idea of maximality is that the universe should be "large"; but we have:

28. Suppose that M is a maximal universe.
Then in M there are no inaccessibles and some real has no $\#$.

Proof: We have seen that with Jensen coding one can produce a real R so that M is a definable inner model of $M(R)$.
In $M(R)$ there is a real (namely R) such that there is no transitive model of ZFC containing R.
So by maximality of M, this holds in a definable inner model of M.
But then this also holds in M, i.e., in M there is a real R such that there is no transitive model of ZFC containing R.
This implies that in M there are no inaccessibles and that $R^\#$ does not exist.

Thus Maximality kills the existence of inaccessible cardinals as well as boldface $\boldsymbol{\Pi}^1_1$ determinacy.

This is the *Maximality Paradox*.

I see two ways of resolving this paradox:

Part III: Maximality in the Hyperuniverse

Option 1: A re-examination of the roles of large cardinals and determinacy in set theory

Maximality as formulated above is compatible with:

i. The existence of large cardinals *in inner models*.
ii. The existence of $\#$'s for ordinal-definable reals.
iii. Determinacy for ordinal-definable sets of reals.

Thus one could adopt the following perspective:

a. Indeed large cardinals don't exist, they only exist in inner models. Their importance in set theory results from their existence in inner models and not from their existence in V.

b. PD is false, but determinacy for OD sets of reals is true. The importance of PD in set theory derives from its consequences for lightface-definable projective sets.

Part III: Maximality in the Hyperuniverse

Regarding the existence of large cardinals not in V but only in inner models:

The main use of large cardinals is for consistency upper and lower bound results:

a. *Consistency upper bounds:*
V with large cardinals *Force* $\rightarrow V[G] \vDash \varphi$

$V[G]$ only has large cardinals in inner models

b. *Consistency upper bounds:*
$V \vDash \varphi$ *Core model construction* $\rightarrow K$ with large cardinals

Large cardinals do not exist in V but only in the inner model K

Part III: Maximality in the Hyperuniverse

Regarding lightface versus boldface PD:

a. PD is sometimes "justified" through *extrapolation*:
ZFC gives the measurability of analytic sets, PD extends this to all projective sets so PD must be true.

But similar extrapolations lead to contradiction:
Shoenfield gives boldface Σ^1_2 absoluteness for arbitrary outer models;
but boldface Σ^1_3 absoluteness for arbitrary outer models is inconsistent (only by restricting to *set-generic* outer models is it consistent).

More reasonable is the extrapolation to Σ^1_3 absoluteness without parameters; this is consistent with (and indeed implied by) maximality.

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b. PD is also sometimes “justified” by the fact that it implies that the theory of HC cannot be changed by set-forcing.

But even the Σ_2 theory of HC can change when passing to more general outer models, even while preserving very large cardinals.

As with Σ^1_3 absoluteness, the restriction to *set-generic* outer models cannot yield a convincing principle.

Part III: Maximality in the Hyperuniverse

Returning to the Maximality Paradox, we have:

Option 2: #-Maximality

Note that so far maximality has focused on "horizontal" or "powerset" maximality, whereby M is maximal with respect to its outer models.

But what about "vertical" or "ordinal" maximality?

This is associated with the principle of "reflection", advocated by Gödel. In its basic form reflection says:

If V satisfies some property then so does some V_α

Reflection principles can be used to argue in favour of the existence of the types of large cardinals compatible with $V = L$:

inaccessible, Mahlo, weakly compact, ineffable, Π^1_n reflecting, ω-Erdős, $\cdots$

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It appears that one can maximize the amount of reflection compatible with $V = L$ by imposing the existence of $0^{\#}$:

$0^{\#}$ exists iff there is a closed unbounded class of indiscernibles for L

I would like to use this idea to maximise reflection for other transitive models of ZFC.

Part III: Maximality in the Hyperuniverse

Recall that a ZF^- model m is a model of SetMK iff m has a largest cardinal κ which is strongly inaccessible in m.

A *pre-#* is a countable structure (m, U) where m is a transitive model of SetMK with largest cardinal κ and U is an ultrafilter on the subsets of κ in m

(m, U) is a # iff (m, U) is iterable, i.e., by taking the ultrapower of m using u and repeating this through the ordinals, wellfoundedness is never lost.

If (m, U) is a # with iteration

$(m, U) = (m_0, U_0) \to (m_1, U_1) \to \cdots$

and critical points $(\kappa_i \mid i \in \mathrm{Ord})$ then $(m, U)_\infty$ denotes the union of the $V_{\kappa_i}^{m_i}$, a model of ZFC.

Part III: Maximality in the Hyperuniverse

A countable transitive model M of ZFC is *#-generated* iff for some # (m, U) with iteration

$(m, U) = (m_0, U_0) \to (m_1, U_1) \to \cdots,$

M equals $V_{\kappa_\lambda}^{m_\lambda}$ for some limit ordinal λ.

Clearly #-generated models enjoy a lot of "reflection" ("ordinal maximality")

Part III: Maximality in the Hyperuniverse

Now I make the informal

Conjecture. The statement that M is $\#$-generated can be formulated as a "reflection" ("vertical maximality") principle for M.

If the Conjecture is true then I propose the following alternative solution to the Maximality Paradox:

Reformulate maximality only with reference to $\#$-generated universes:

We say that a $\#$-generated universe M is $\#$-*maximal* iff:

M a definable inner model of N, N $\#$-generated $\rightarrow \Phi(M) = \Phi(N)$.

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29. Assume large cardinals plus PD. Then there is a $\#$-generated, $\#$-maximal universe with large cardinals.

Proof Sketch: For each real R let $M^{\#}(R)$ be the least $\#$-generated model of the form $L_\alpha[R]$. Use PD to get a real R such that $R \leq_T S$ implies that $\mathrm{Thy}(M^{\#}(R)) = \mathrm{Thy}(M^{\#}(S))$.
We claim that $M^{\#}(R)$ is $\#$-maximal:
Indeed, let M be a $\#$-generated outer model of $M^{\#}(R)$ with a definable inner model satisfying some sentence φ.
By a result of Jensen, M has a $\#$-generated outer model N satisfying $V = L[S]$ for some real S, which must then be $M^{\#}(S)$.
By the choice of R, $M^{\#}(R)$ also has a definable inner model satisfying φ. So $M^{\#}(R)$ is $\#$-maximal.
If M is any $\#$-generated outer model of an initial segment of $L[R]$ then again by Jensen's result, M is $\#$-maximal; assuming large cardinals there are such models M containing large cardinals.

Part III: Maximality in the Hyperuniverse

We end with a discussion of *Strong Maximality*.

Notice that maximality has no implications for CH, as any outer model of a maximal universe is still maximal.
Strong Maximality is a form of maximality with parameters that resolves CH.

To motivate Strong Maximality we consider the following forms of Lévy absoluteness:

LAbs: If a Σ_1 formula with no parameters holds in an outer model of M then it holds in M.
LAbs(ω_1): If a Σ_1 formula with parameter ω_1 holds in an ω_1-preserving outer model of M then it holds in M.
LAbs(ω_1, ω_2): If a Σ_1 formula with parameters ω_1, ω_2 holds in an ω_1- and ω_2-preserving outer model of M then it holds in M.

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30. (a) LAbs holds for all universes.
(b) Assuming PD, there is a universe satisfying LAbs(ω_1).
(c) Any universe satisfying LAbs(ω_1, ω_2) also satisfies not CH.

Proof: (a) follows from Lévy's absoluteness theorem.
(b) Any maximal universe satisfies LAbs(ω_1), using the fact that Jensen coding preserves ω_1.
(c) Apply LAbs(ω_1, ω_2) to an extension that results from adding ω_2 Cohen reals.

LAbs(ω_1, ω_2) is a special case of the following more general property:

Part III: Maximality in the Hyperuniverse

Strong Maximality for M. Suppose that p is absolute, i.e., has a definition uniform over all outer models of M which preserve cardinals up to the cardinality of the transitive closure of p. Then if $\varphi(p)$ holds in an inner model of such an outer model, it also holds in an inner model of M containing p.

Conjecture. Assuming large cardinals there are Strongly Maximal universes (which necessarily satisfy $\mathrm{LAbs}(\omega_1, \omega_2)$).
And the same is true for Strong $\#$-Maximality, which is defined just like Strong Maximality, but restricted to $\#$-generated universes.

Note that just as $\mathrm{LAbs}(\omega_1, \omega_2)$ implies not CH, Strong Maximality implies that the size of the continuum is rather enormous, greater for example than any $\aleph_\alpha$ where α is countable in L.
And Strong $\#$-Maximality, if consistent, should also be consistent with large cardinals.

Part III: Maximality in the Hyperuniverse

About possible solutions to the Continuum Problem

Note that even the version of LAbs(ω_1, ω_2) which refers only to ccc forcing extensions (and not to arbitrary outer models) is sufficient to infer not CH.

And this version of absoluteness is consistent: Just perform a ccc finite-support iteration, at each stage handling one of the ω-many instances of absoluteness.

Similarly one has the consistency of the version of Strong Absoluteness (or Strong $\#$-Absoluteness) which refers only to ccc forcing extensions.

Does this solve the Continuum Problem?

I don't think so: The problem is that the restriction to ccc forcing extensions (or for that matter to forcing extensions at all) is artificial.

Part III: Maximality in the Hyperuniverse

However the unrestricted version of LAbs(ω_1, ω_2) does not have this defect.

For this reason I feel that a consistency proof of this principle is a strong candidate for a compelling principle of absoluteness that resolves the Continuum Problem.

I very much hope that some day we will see such a consistency proof.