Δ_1-Definability

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Jensen's Coding Theorem provides a method for making a class of ordinals Δ_2-definable in a real. In this paper we explore the possibility of obtaining a Δ_1-definition. While this is not possible in general, we show that by assuming enough condensation, a class can be made Δ_1 in a real parameter in a class-generic extension. Our main application is the following.

Theorem *There is a real R class-generic over L such that $L - cof\,\omega = \{\alpha : L - cof(\alpha) = \omega\}$ is Δ_1 over $L[R]$ in the parameter R. Every L-cardinal, with the exception of $\aleph_1^L$ remains a cardinal in $L[R]$. Moreover the existence of such a real R follows from the existence of $O^\#$.*

Every L-definable class is Δ_1-definable over $L[O^\#]$. The following result shows that conversely if every L-definable class is Δ_1-definable in a real parameter than $O^\#$ exists.

Proposition 1 *Suppose L-card $= \{\alpha : \alpha$ is a cardinal in $L\}$ is Σ_1 in a real parameter. Then $O^\#$ exists.*

PROOF: $\aleph_{\omega+1} \in L$-card so if L-card were Σ_1 in a real, by reflection there would be unboundedly many $\alpha < \aleph_{\omega+1}$ also in L-card. So $\aleph_{\omega+1} > (\aleph_\omega)^{+L}$ and by Jensen's Covering Theorem, $O^\#$ exists. $\dashv$

An L-amenable class A is Σ_1-*complete* if every $\Sigma_1(L)$ class is $\mathbf{\Delta_1}\langle L, A\rangle$. An example of a Σ_1-complete class is L-card. It follows from Proposition 1

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that no Σ_1-complete class can be Δ_1 in a real, unless $O^\#$ exists. L-cof ω, the class of ordinals which have countable cofinality in L, is a nice example of a class which is neither $\Delta_1(L)$ nor Σ_1-complete.

Now we turn to a sufficient condition for an L-amenable class A to be Δ_1-definable in a real R not constructing $O^\#$.

Definition 1 Suppose x is an extensional set (i.e. x satisfies the axiom of extensionality) and let $\bar{x}$ denote the transitive collapse of x. For $A \subseteq ORD$ we say that x *preserves* A if $\langle \bar{x}, \in, A \cap \bar{x} \rangle$ is isomorphic to $\langle x, \in, A \cap x \rangle$.

Proposition 2 *Suppose A is L-amenable and Δ_1-definable in a real R not constructing $O^\#$ and κ is an uncountable cardinal. Suppose $\langle T, \in, \ldots \rangle \in L$ is a transitive structure of size at least κ for a countable language. Then there is $\langle x, \in, \ldots \rangle \prec \langle T, \in, \ldots \rangle$ such that card$(x) = \kappa$, $x \in L$, and x preserves A.*

PROOF: By the Strong Covering Lemma (see [Sh, chapter XIII], or [C]) we can choose y of cardinality κ such that $T \in y$, $\langle y, \in \rangle$ is Σ_1-elementary in $\langle L[R], \in \rangle$ and $x_0 = y \cap L \in L$. Then $x = x_0 \cap T$ is as desired. $\dashv$

Our sufficient condition is an elaboration of the necessary condition stated in Proposition 2.

Definition 2 If x is a set and δ is an ordinal then $x[\delta] = \{f(\gamma) : \gamma < \delta, f \in x, f \text{ a function}, \gamma \in \text{Dom}(f)\}$. We say that x *strongly preserves* A if $x[\delta]$ preserves A for each cardinal δ. A sequence $t_0, t_1, \ldots$ of sets is *tight* if it is continuous, $t_i \in t_{i+1}$ for each i and for each i, $\langle \bar{t}_j : j < i \rangle$ belongs to the least ZF$^-$-model which contains $\bar{t}_i$ as an element and correctly computes card$(\bar{t}_i)$, where as before $\bar{t}_i$ =the transitive collapse of t_i.

Condensation Condition. Suppose t is transitive, κ is regular, $\kappa \in t$, and $x \in t$. Then:

(a) There is a tight κ-sequence $t_0 \prec t_1 \prec \ldots \prec t$ such that card$(t_i) = \kappa$ all i, $x \in t_0$ and each t_i strongly preserves A.

(b) If κ is inaccessible then there exists $t_0 \prec t_1 \prec \ldots \prec t$ as above but where $\text{card}(t_i) = \aleph_i$.

Theorem 1 *Assume V=L and that $A \subseteq ORD$ obeys the Condensation Condition. Then A is Δ_1 in a real parameter in a cofinality preserving class-generic extension of L.*

PROOF: We describe a refinement of the forcing to code A by a real R found in Beller-Jensen-Welch [BJW] so as to obtain that in fact A is Δ_1 over $L[R]$. This work is similar in many respects to the work found in David [Da].

The key is to use a modified version of S_α = the "reshaped strings" $s : [\alpha, |s|) \to 2$, $\alpha \leq |s| < \alpha^+$. For s to belong to S_α we require:

1. Whenever $\alpha \leq 2\gamma < |s|$, $\gamma \in A$ if and only if $s(2\gamma) = 1$.
2. Whenever $L_\beta[A \cap \alpha, s \restriction \eta] \models \text{ZF}^- + \eta = \alpha^+$ (or $\beta = \eta$ and $L_\beta[A \cap \alpha, s \restriction \eta] \models \text{ZF}^- + \alpha$ is the largest cardinal) then $A \cap [\alpha, \beta) =$ Even Part (X) where $X \subseteq [\alpha, \beta)$ is decoded in $L_\beta[A \cap \alpha, s \restriction \eta]$ from $A \cap \alpha, s \restriction \eta$ as in Jensen Coding.

If $|s| = \alpha$ then $\mu_s^0 = \alpha$ and otherwise $\mu_s^0 = \bigcup\{\mu_{s\restriction\eta} : \eta < |s|\}$. For all $s, \mu_s =$ least $\mu > \mu_s^0$ such that $s \in L_\mu$, $\langle L_\mu, A\rangle$ is amenable and $\langle L_\mu, A\rangle \models \text{ZF}^- +$ $\text{card}(|s|) \leq \alpha$ + the Condensation Condition holds for A.

Define the successor coding R^s using the above definitions (in particular, $\mathcal{A}_s = \langle L_{\mu_s}, A\rangle$ where μ_s is as above). Note that it is easy to show that for any $s \in S_\alpha$ and $\delta < \alpha^+$ there exists t extending $s, t \in S_\alpha$, $|t| \geq \delta$; this is because we can use the freedom to kill models of ZF^- to prevent new instances of 2. from occurring, when extending from s to t. If $s \in S_{\alpha^+}$ then we must show that R^s is $\leq \alpha$-distributive in $\mathcal{A}_s$. Let $\kappa \leq \alpha$ be a regular cardinal, $t \in \mathcal{A}_s$, and $x = \{(u, \bar{u}), A \cap \alpha^+, s, \langle D_i : i < \kappa\rangle\} \in t$, where $(u, \bar{u}) \in R^s$ and open dense sets $D_i, i < \kappa$ are given. Apply the Condensation Condition in $\mathcal{A}_s$ to t, κ, x to obtain a tight κ-sequence $t_0 \prec t_1 \prec \ldots \prec t$ as in (a) of the condition. As in the usual distributivity proof build $(u, \bar{u}) = (u_0, \bar{u_0}) \geq (u_1, \bar{u_1}) \geq \ldots$ so that $(u_{i+1}, \bar{u_{i+1}}) \in t_{i+1}$ meets D_i and for limit $\lambda \leq \kappa$, $A \cap \alpha, u_\alpha$ decodes the images $\bar{A}, \bar{s}$ of $A \cap \alpha^+, s$ under the transitive collapse of t_λ. Then u_λ obeys the requirement 2. for membership in S_α, with A replaced by $\bar{A}$. But t_λ

preserves A so in fact $\bar{A}$ is an initial segment of A and 2. is satisfied by u_λ. The final condition $(u_\kappa, \bar{u_\kappa})$ extends $(u, \bar{u})$ and meets each D_i, for $i < \kappa$.

Extendibility for the limit coding is as in Beller-Jensen-Welch [BJW] (see David [Da]). In the proof of Distributivity for the limit coding a sequence $\langle p_i : i < \kappa \rangle$ of conditions is constructed and we must verify that p_λ is a condition for limit λ. The only new point is to check that p_{λ_γ} codes the collapse of A relative to an elementary submodel of $t \in \mathcal{A}_s$; again the Condensation Condition guarantees that we can arrange for this elementary submodel to collapse A to an initial segment of itself, as it can be made to equal $t_\lambda[\gamma]$ where $t_0 \prec t_1 \prec \ldots \prec t \in \mathcal{A}_s$ strongly preserve A. Part (b) of the Condensation Condition is used to handle Δ-Distributivity at inaccessibles. $\dashv$

The technique of the proof of Theorem 0.2 of Beller-Jensen-Welch [BJW] also shows that if in addition $I =$ Silver Indiscernibles are indiscernibles for $\langle L, A \rangle$ then A is Δ_1 in a real $R \in L[O^\#]$, R class-generic over L.

Theorem 2 *There is a real R_0, class-generic over L, such that in $L[R_0]$ the Condensation Condition holds for $A = L - cof\, \omega$. Moreover $L[R_0^\#] = L[O^\#]$ and every L-cardinal, with the exception of $\aleph_1^L$, remains a cardinal in $L[R_0]$.*

PROOF: Starting from L first Lévy collapse $\aleph_1^L$ to ω and over the resulting extension M perform a reverse Easton iteration (with Easton support) which at each uncountable regular cardinal κ adds $C(\kappa)$, a closed unbounded subset of κ consisting of ordinals of uncountable L-cofinality (see [Fr]). It is straightforward to show that in $M[\langle C(\bar{\kappa}) : \bar{\kappa} < \kappa \rangle]$ the forcing to add $C(\kappa)$ is $< \kappa$-distributive and, in fact, the same is true for the forcing to simultaneously add $\langle C(\kappa^*) : \kappa^* \geq \kappa \rangle$. So $M[\langle C(\kappa) | \kappa \text{ regular} \rangle]$ has the same cardinals as L, with the sole exception of $\aleph_1^L$. Let R_0 be a real coding this model (obtained from Jensen coding): $M[\langle C(\kappa) | \kappa \text{ regular} \rangle] \subseteq L[R_0]$ and every L-cardinal with the exception of $\aleph_1^L$ remains a cardinal in $L[R_0]$.

Now we verify the Condensation Condition for $A = \{\alpha : L - \text{cof}(\alpha) > \omega\}$ in $L[R_0]$. Let t be transitive, κ regular, $x \in t$. We may assume that card $(t) > \kappa$.

First choose a regular $\alpha > \kappa^+$ such that $t \in L_\alpha[R_0]$ and let $\kappa, R_0, x, t \in u_0 \prec u_1 \prec \ldots \prec L_\alpha[R_0]$ be a continuous tight sequence of length κ^+ where $\text{card}(u_i) = \kappa$ and $\langle u_j : j \leq i \rangle \in u_{i+1}$ for each $i < \kappa^+$. Let $u = \bigcup \{u_i : i < \kappa^+\}$ and note that for regular $\lambda \in u$, $\sup(u \cap \lambda)$ has cofinality κ^+. Since $L[R_0]$

and L have the same cofinalities (above $\aleph_1 = \aleph_2^L$) we have that $\sup(u \cap \lambda)$ has L-cofinality κ^+.

Even more is true: for any pair of cardinals $\delta < \lambda, \lambda$ regular, if $\lambda \in u[\delta]$ then either $\delta = u[\delta] \cap \lambda$ or $\sup(u[\delta] \cap \lambda)$ has cofinality κ^+. The reason is the following. Suppose that $\delta \neq u[\delta] \cap \lambda$ and pick some $i_0 < \kappa^+$ and $\gamma < \delta$ such that $\lambda \in u_{i_0}[\gamma]$ and $u_{i_0}[\gamma] \cap (\lambda \setminus \delta) \neq \emptyset$. Note that then for every $i_0 \leq i < j < \kappa^+$ $\sup(u_i[\delta] \cap \lambda) < \sup(u_j[\gamma] \cap \lambda)$. Thus, $\sup(u[\delta] \cap \lambda)$ is the *strict* supremum of $\sup(u_i[\gamma] \cap \lambda)$, for $i_0 \leq i < \kappa^+$, and therefore has cofinality κ^+. Again we can write L-cofinality instead of cofinality.

Now build $v_0 \prec v_1 \prec \ldots \prec L_\alpha[R_0]$ just like the u_i's but with the additional requirement that u is an element of v_0. We claim that for each $j < \kappa^+$, $v_j \cap u$ preserves $A = \{\beta : L - \mathrm{cof}(\beta) > \omega\}$. Suppose $\beta \in v_j \cap u$ belongs to A and let $\bar{\beta}$ be the image of β under the transitive collapse of $v_j \cap u$. Clearly $\bar{\beta}, \bar{\gamma}$ have the same L-cofinality, where $\gamma = L - \mathrm{cof}(\beta)$ so we can assume that β is L-regular and hence regular (since $\bar{\beta} = \beta$ when $\beta = \aleph_1^L$). Thus $\sup(u \cap \beta)$ has L-cofinality κ^+ and therefore v_j contains a constructible cofinal map from κ^+ into $\sup(u \cap \beta)$; it follows that $\bar{v}_j$ contains a constructible cofinal map from $\kappa^+ \cap v_j$ cofinal into ordertype $(u \cap v_j \cap \beta) = \bar{\beta}$. But $\kappa^+ \cap v_j$ is an element of $C(\kappa^+)$ and therefore has uncountable L-cofinality. So $\bar{\beta} \in A$.

In fact $v_j \cap u$ strongly preserves A : if $\delta < \beta \in A \cap v_j \cap u$ and δ is a cardinal then again we may assume that β is regular. Thus either $\sup(u[\delta] \cap \beta)$ has cofinality κ^+ and belongs to $v_j[\delta]$, hence the old argument applies (with $\kappa^+ \cap v_j[\delta] = \kappa^+$ if $\delta > \kappa$) or $u[\delta] \cap \beta = \delta$ and $v_j \cap u[\delta] \cap \beta = \delta \in C(\beta)$, hence β collapses to δ, which has uncountable L-cofinality.

Finally let $t_i = v_i \cap u \cap t$ for $i < \kappa$. Then of course each t_i strongly preserves A since each $v_i \cap u$ does and t is transitive; we must show that $\langle t_i : i < \kappa \rangle$ is tight. Clearly $\langle \bar{t}_j : j < i \rangle$ belongs to the least ZF^- model containing $\bar{v}_i$ as an element which satisfies $\mathrm{card}(\bar{v}_i) = \kappa$, since $\langle v_i : i < \kappa \rangle$ is tight and $u, t \in v_0$. But $\bar{v}_i$ is of the form $L_{\bar{\alpha}}[R_0]$ and $\bar{v}_i \vDash \bar{t}_i$ has cardinality $> \kappa$. So any ZF^- model containing $\bar{t}_i$ as an element which satisfies $\mathrm{card}(\bar{t}_i) = \kappa$ must also contain $\bar{v}_i$ as an element and satisfy $\mathrm{card}(\bar{v}_i) = \kappa$. So $\langle t_i : i < \kappa \rangle$ is tight. This proves (a) of the Condensation Condition. To obtain (b) perform the same argument, but where $\mathrm{card}(u_i) = \aleph_i$ and u has cardinality κ.

Standard arguments show that R_0 can be found in $L[O^\#]$ (see Friedman [Fr], Beller-Jensen-Welch [BJW]). $\dashv$

Corollary 1 *There is a real $R \in L[O^\#]$ such that R is class-generic over*

L and $\{\alpha : L - cof(\alpha) = \omega\}$ is Δ_1 over $L[R]$ in parameter R. Also every L-cardinal other than $\aleph_1^L$ is a cardinal in $L[R]$.

We now present some other applications of the above techniques. If R is a real and α is an ordinal then we say that α is *quasi R-admissible* if the ordertype of every wellordering that belongs to $L_\alpha[R]$ is less than α. The proof of the main result of David [Da] in fact shows the following.

Theorem 3 *Suppose $\varphi(\alpha)$ is Σ_1 and $L \vDash \varphi(\kappa)$ whenever κ is an L-cardinal. Then there is a real $R \in L[O^\#]$, class-generic over L, such that $L \vDash \varphi(\alpha)$ for every quasi R-admissible ordinal α.*

We can combine Theorem 3 with Theorems 1 and 2 to obtain the following result.

Corollary 2 *There is a real $R \in L[O^\#]$ such that R is class-generic over L and every quasi R-admissible has uncountable L-cofinality.*

PROOF: Let R_0 be as in Corollary 1 and for each $L[R_0]$-indiscernible κ let C_κ be the $L[R_0]$-least CUB subset of κ consisting of ordinals of uncountable L-cofinality. Then $\bigcup\{C_\kappa : \kappa \text{ an } L[R_0]\text{-indiscernible}\} = C$ is a closed unbounded class of ordinals α such that L-cof$(\alpha) > \omega$. Also C may be assumed to contain κ^+ (in $L[R_0]$) whenever κ is a limit point of C. Note that the structure $\langle L[R_0], C\rangle$ is amenable.

Now force over $\langle L[R_0], C\rangle$ using the usual Easton product of Levy collapses, to get a model M such that every uncountable cardinal of M is in C, and then code M by a real $R_1, R_0 = Even(R_1)$. Thus every $L[R_1]$-cardinal $> \omega$ is of uncountable L-cofinality and the latter is a Δ_1 property in parameter R_0 and hence in parameter R_1. By Theorem 3 relativized to R_1 we get the desired real R. Moreover all these steps can be carried out in $L[O^\#]$. $\dashv$

With a little care we can improve Corollary 2 as follows.

Corollary 3 *There is a real $R \in L[O^\#]$ such that R is class-generic over L and the function $f :\ ORD \to L$ defined by $f(\alpha) = [\alpha]^\omega \cap L$ is Δ_1 over $L[R]$ in the parameter R.*

PROOF: Proceed as in Corollary 2, obtaining R_1 such that every uncountable cardinal of $L[R_1]$ has uncountable L-cofinality and $\{\alpha : L-\mathrm{cof}(\alpha) > \omega\}$ is Δ_1 in R_1. Note that we can arrange that if α is quasi R_1-admissible then the L-cardinality of α has uncountable L-cofinality by arranging R_1 to code over $L_{\kappa^+}[R_1]$ a well-ordering of ordertype at least the least element of C greater than κ^+, for $\kappa \in C$.

Now f as defined in the statement of the Corollary is Δ_1 in R_1: it suffices to define $g :$ ORD $\to$ ORD such that g is Σ_1 over $L[R_1]$ with parameter R_1 and $[\alpha]^\omega \cap L \subseteq L_{g(\alpha)}$, for each α. Define $g(\alpha)$ to be the least β such that one of the following holds:

(a) $L-\mathrm{cof}(\alpha) > \omega$ and $\beta \geq \bigcup\{g(\alpha') : \alpha' < \alpha\}$

(b) $\beta \geq g(\alpha')$ for some $\alpha' < \alpha$, $L_\beta \models \mathrm{card}(\alpha) \leq \alpha'$

(c) $\beta > \alpha, \beta$ is quasi R_1-admissible.

Then either (a) or (b) holds for some $\beta \leq (\alpha^+)^L$ or α is an L-cardinal of L-cofinality ω, in which case any β obeying (c) is $\geq (\alpha^+)^L$. So g is as desired. $\dashv$

Our final application concerns immune partitions. A *partition* of the ordinals is a class function F : ORD $\to 2$. We say that F is *immune* if neither $\{\alpha : F(\alpha) = 0\}$ nor $\{\alpha : F(\alpha) = 1\}$ contains an infinite constructible set. The following is proved in Friedman [Fr].

Theorem 4 *There is an immune partition of the ordinals definable in $L[R]$ for some real $R \in L[O^\#]$ which is class-generic over L.*

Corollary 3 allows us to improve Theorem 4 as follows.

Corollary 4 *There is an immune partition of the ordinals which is $\mathbf{\Delta_1}$ over $L[R]$ with parameter R for some real $R \in L[O^\#]$ which is class-generic over L.*

PROOF: First select R_0 as in Corollary 3 and then by Theorem 4 relativized to R_0, choose R_1 generically over $L[R_0]$ coding an immune (indeed $L[R_0]$-immune) partition. This can all be done in $L[O^\#]$. Note that in $L[R_1]$ the immunity of a partition $f : \kappa \to 2$ is a $\mathbf{\Delta_1}$ property. As in Corollary 2 force R_2 so that every quasi R_2-admissible has uncountable L-cofinality. Thus $C=\{\alpha | \alpha \text{ is } quasi R-admissible\}$ is a $\mathbf{\Delta_1}$ closed unbounded class of ordinals of uncountable L-cofinality. For each α in $C \cup \{0\}$ let F_α be the $L[R_2]$-least immune partition of $[\alpha, \alpha')$ where α' is the C-successor to α. Then the union of the F_α's is a $\mathbf{\Delta_1}$ immune partition of ORD. ⊣

Remark. M. Stanley [St] has shown that one can have an immune partition of the ordinals in a cardinal preserving extension of L. Our next result shows that if we want this partition to be Δ_1 in a real parameter then some L cardinals have to be collapsed.

Proposition 3 *Assume that V and L have the same cardinals. Then there is no immune partition of the ordinals which is Δ_1 in a real.*

PROOF: Assume towards contradiction that F is an immune partition of the ordinals which is Δ_1 in a real parameter R. Fix a well-ordering $<$ of $H_{\aleph_\omega+1}$. For $Y \prec H_{\aleph_\omega+1}$ and an ordinal δ let $Y[\delta]$ denote the Skolem hull of $Y \cup \delta$ in $H_{\aleph_\omega+1}$. By the Strong Covering Lemma there is $X \prec H_{\aleph_{\omega+1}}$ of cardinality $\aleph_1$ containing R such that $X \cap \text{ORD} \in L$. We claim that the sequence $\langle X[\aleph_n] \cap \text{ORD} : n < \omega \rangle$ is in L. To see this, note that if Y_n is the Skolem hull of $(X \cap \text{ORD}) \cup \aleph_n$ in $L_{\aleph_{\omega+1}}$ then $Y_n \cap \text{ORD} = X[\aleph_n] \cap \text{ORD}$ and therefore the sequence $\langle X[\aleph_n] \cap \text{ORD} : n < \omega \rangle$ can be computed in L.

Now, let π_n be the transitive collapse of $X[\aleph_n]$ and let $\delta_n = \pi_n(\aleph_\omega)$. Then, by the fact that F is $\Delta_1(R)$ and $R \in X$, it follows that $F(\delta_n) = F(\aleph_\omega)$, for all n. Therefore the sequence $\langle \delta_n : n < \omega \rangle$ contradicts the immunity of F. ⊣

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