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PARAMETER-FREE UNIFORMISATION

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ABSTRACT. This article was motivated by two questions, one in the theory of infinite-time Turing machines and the other a fine-structural question posed by Wayne Richter. The answers to both turn on a question of parameter-free uniformisation, which I prove for all locally countable, limit initial segments of L.

Let α be a limit ordinal. We say that the partial function f on L_α *uniformises* the binary relation $R(x,y)$ on L_α iff for all $p \in L_\alpha$: $p \in \mathrm{Dom}(f)$ iff $\exists y R(p,y)$ iff $R(p, f(p))$. Then L_α satisfies Σ_n *uniformisation* iff every binary relation $R(x,y)$ which is Σ_n definable over L_α with parameters can be uniformised by a function f which is Σ_n definable over L_α with parameters. A stronger statement is *parameter-free* Σ_n *uniformisation*, which asserts the same for relations and functions without parameters. Jensen proved that each L_α satisfies Σ_n uniformisation (see [3]). There are examples of L_α's which do *not* satisfy parameter-free Σ_2 uniformisation (see [1]). Every L_α satisfies parameter-free Σ_1 uniformisation.

We say that β is Σ_1 *stable* in α iff β is less than α and L_β is Σ_1 elementary in L_α. We say that β is *above* x iff x belongs to L_β.

Theorem 1. *Suppose that α is a limit ordinal and L_α is locally countable (i.e., $L_\alpha \models$ every set is countable). Then L_α satisfies parameter-free Σ_n uniformisation.*

Proof. By induction on n. We may assume that n is greater than 1. For the sake of clarity, we assume that n equals 2; the proof below readily generalises to arbitrary n, given the result for $n-1$.

We first discuss Σ_1 Skolem functions. Let U be the set of pairs $(\langle n, x \rangle, y)$ in L_α such that $\varphi_n(x,y)$ holds in L_α, where φ_n is the n-th Σ_1 formula. Then U is parameter-free Σ_1 and can be uniformised by a parameter-free Σ_1 function h_1. Thus the domain of h_1 consists of those pairs $\langle n, x \rangle$ such that $\varphi_n(x,y)$ holds for some y, and for such a pair, $\varphi_n(x, h_1(\langle n, x \rangle))$ holds (h_1 is a Σ_1 *Skolem function* for L_α). It is easy to verify that for any $A \subseteq L_\alpha$, the range of h_1 on $\omega \times [A]^{<\omega}$ is the smallest Σ_1 elementary submodel of L_α containing A as a subset, the Σ_1 *Skolem hull of* A. For any $A \subseteq L_\alpha$ and $x \in L_\alpha$, the Σ_1 Skolem hull of $A \cup \{x\}$ is equal to the set of all $h_1(n, \langle x, a \rangle)$, $n \in \omega$ and $a \in [A]^{<\omega}$.

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As L_α is locally countable, the Σ_1 Skolem hull of A in L_α is transitive for any $A \subseteq L_\alpha$: If this hull contains x, then it also contains an injection of x into ω, all elements of ω and therefore all elements of x. It follows that the Σ_1 Skolem hull of $\{\beta, x\}$ equals the Σ_1 Skolem hull of $\beta + 1 \cup \{x\}$ for any $\beta < \alpha$ and $x \in L_\alpha$.

Lemma 2. *Suppose that y belongs to L_α. Then for some $n \in \omega$, y equals $h_1(n, \beta)$ where β is either 0 or Σ_1 stable in α. More generally, for any $x \in L_\alpha$, there exists $n \in \omega$ such that y equals $h_1(n, \langle \beta, x \rangle)$ where β is either 0 or both Σ_1 stable in α and above x.*

Proof. Given y, let β be least so that y belongs to the Σ_1 Skolem hull of $\beta + 1$ in L_α (= the Σ_1 Skolem hull of $\{\beta\}$ in L_α). Then β is either 0 or Σ_1 stable in α, as otherwise β belongs to the Σ_1 Skolem hull of $\beta' + 1$ for some $\beta' < \beta$ and hence so does y, contradicting the choice of β. Also, y is of the form $h_1(n, \beta)$ for some $n \in \omega$. More generally, given both y and x in L_α, let β be least so that y belongs to the Σ_1 Skolem hull of $\beta + 1 \cup \{x\}$ in L_α (= the Σ_1 Skolem hull of $\{\beta, x\}$ in L_α). As the Σ_1 Skolem hull of $1 \cup \{x\}$ is transitive, it follows that β is either 0 or Σ_1 stable and above x. Also, y is equal to $h_1(n, \langle \beta, x \rangle)$ for some n. □

Now let $\psi_n(x, y)$ be the n-th Π_1 formula with free variables x, y and define $f(n, x) = (m, \beta)$ iff the following hold in $(L_\alpha, \in)$:

(i) Either β is 0 or both Σ_1 stable and above x.
(ii) $h_1(\langle m, \langle \beta, x \rangle \rangle) = y$ is defined.
(iii) $\psi_n(x, h_1(\langle m, \langle \beta, x \rangle \rangle))$ holds.
(iv) $\psi_n(x, y')$ fails for any y' of the form $h_1(\langle m', \langle \beta', x \rangle \rangle)$, $\beta' < \beta$, $m' < \omega$.
(v) $m' < m \rightarrow h_1(\langle m', \langle \beta, x \rangle \rangle)$ is undefined or $\psi_n(h_1(\langle m', \langle \beta, x \rangle \rangle))$ fails.

Clauses (i)-(iii) are easily seen to be equivalent to a Boolean combination of Σ_1 formulas, using the fact that Σ_1 stability is a Π_1 notion. Clause (iv) is vacuous if β is 0 and otherwise, by the fact β is Σ_1 stable and above x, holds in $(L_\alpha, \in)$ iff it holds in $(L_\beta, \in)$; it follows that clause (iv) is Σ_1. Also, clause (v) is a Boolean combination of Σ_1 formulas preceded by a bounded number quantifier; it follows that this clause is Σ_2. So f is a partial function from L_α to L_α which is parameter-free Σ_2.

If $f(n, x)$ is defined, then $\psi_n(x, y)$ holds for some y, as we may take y to be $h_1(\langle m, \langle \beta, x \rangle \rangle)$, where $f(n, x) = (m, \beta)$. Conversely, if $\psi_n(x, y)$ holds for some y, then by the lemma, y is of the form $h_1(\langle m, \langle \beta, x \rangle \rangle)$ for some β which is either 0 or both Σ_1 stable and above x. By minimising the pair (β, m), we see that $f(n, x)$ is defined. Thus $f(n, x)$ is defined iff for some y, $\psi_n(x, y)$ holds, in which case $\psi_n(x, h_1(\langle m, \langle \beta, x \rangle \rangle))$ holds, where $f(n, x) = (m, \beta)$. For each (n, x) define $f'(n, x)$ to be $h_1(\langle m, \langle \beta, x \rangle \rangle)$, where $f(n, x) = (m, \beta)$. Then f' is parameter-free Σ_2 with the same domain as f, and $f'(n, x)$ is defined iff for some y, $\psi_n(x, y)$ holds, in which case $\psi_n(x, f'(n, x))$ holds.

Now suppose that $R(x, y)$ is a binary relation on L_α which is Σ_2 definable over L_α without parameters. Let $R(x, y)$ be defined over L_α by the Σ_2 formula $\exists z \psi(x, y, z)$, where ψ is Π_1. Now let R' consist of all pairs $(x, \langle y, z \rangle)$ such that $\psi(x, y, z)$ holds in L_α; then R' is Π_1 definable over L_α. Choose n so that R' is defined over L_α by the Π_1 formula ψ_n. Then $f'(n, x)$ is defined iff $(x, \langle y, z \rangle)$ belongs to R' for some y, z, in which case $(x, f'(n, x))$ belongs to R'. Let g be the partial function defined by

$$g(x) = y \text{ iff for some } z, f'(n, x) \text{ is the pair } \langle y, z \rangle .$$

Then g is parameter-free Σ_2 and x is in the domain of g iff $f'(n,x)$ is a pair $\langle y,z\rangle$ where $(x,\langle y,z\rangle)$ belongs to R' iff $(x,\langle g(x),z\rangle)$ belongs to R' for some z iff $R(x,g(x))$ holds. So g uniformises R, as desired. $\square$

Remark. The hypothesis of local countability can be weakened slightly. All that was needed for the proof was the transitivity of Σ_1 Skolem hulls, which holds iff L_α is locally countable or is the Σ_1 Skolem hull of $\{\omega_1^{L_\alpha}\}$ in itself.

Applications

Wayne Richter asked the following question: Suppose that α is least so that for some $\kappa<\alpha$, L_κ and L_α have the same Σ_n theory. Then is L_κ a Σ_n elementary submodel of L_α?

The answer is Yes, as was independently verified by Leo Harrington and myself in 2003. Here I give a new proof based on Theorem 1.

Corollary 3. *Suppose that α is least so that for some $\kappa<\alpha$, L_κ and L_α have the same Σ_n theory. Then L_κ is a Σ_n elementary submodel of L_α.*

Proof. Note that α is a limit ordinal; otherwise both κ and α are successor ordinals and $L_{\kappa-1}$, $L_{\alpha-1}$ have the same theory, contradicting the leastness of α. Also, L_α is locally countable, as if $\beta=(\omega_1)^{L_\alpha}$ exists, there are $\kappa<\beta$ such that L_κ is Σ_n is elementary in L_β, again contradicting the leastness of α. So α satisfies the hypotheses of Theorem 1. By that theorem, L_α has a parameter-free Σ_n Skolem function, i.e., there exists a parameter-free Σ_n partial function h_n from $\omega\times L_\alpha$ into L_α such that for any $A\subseteq L_\alpha$, the set of $h_n(m,a)$ for $m\in\omega$ and $a\in[A]^{<\omega}$ is the smallest Σ_n elementary submodel of L_α containing A as a subset, the Σ_n *Skolem hull of A*. Let H be the Σ_n Skolem hull of ω in L_α. Then H is transitive and is Σ_n elementary in L_α. Therefore H is an initial segment of L with the same Σ_n theory as L_α, and it therefore must be either L_κ or L_α. Note that each element of H is a Σ_n singleton in L_α (i.e., the unique solution to a Σ_n formula in L_α) as each element of H is of the form $h_n(m,a)$ where $(m,a)\in\omega\times\omega^{<\omega}$ and h_n is parameter-free Σ_n. If H equals L_α, then κ belongs to H and so we can choose a Σ_n formula $\psi(x)$ whose unique solution in L_α is κ. But then the Π_n theory of L_α is Σ_n definable over L_α: φ belongs to the Π_n theory of L_α iff for some β, φ is true in L_β and $\psi(\beta)$ holds. The latter is a contradiction. So H equals L_κ and therefore L_κ is Σ_n elementary in L_α. $\square$

Our second application is to infinite-time Turing machines. In [2], the *Theory Machine* is defined, which is a Turing machine which runs through the transfinite and at stage $\omega^2\times(\alpha+1)$ writes down the Σ_2 theory of J_α. The machine can continue doing this until it reaches a limit stage Σ such that there is no parameter-free Σ_2 definable partial function from ω onto L_Σ. This ordinal Σ is described as follows.

Corollary 4. *Let Σ be least such that for some $\kappa<\Sigma$, L_κ is Σ_2 elementary in L_Σ (by the previous corollary, this is the least Σ such that for some $\kappa<\Sigma$, L_κ and L_Σ have the same Σ_2 theory). Then Σ is the least limit ordinal so that there is no parameter-free Σ_2 definable partial function from ω onto L_Σ, and therefore the Theory Machine can write down Σ_2 theories for Σ steps.*

Proof. Note that L_α is locally countable for $\alpha\leq\Sigma$, where if $\beta=(\omega_1)^{L_\alpha}$ exists, then there are $\kappa<\beta$ such that L_κ is Σ_2 elementary in L_β, contradicting the definition of Σ. So by Theorem 1, L_α has a parameter-free Σ_2 Skolem function for each limit

$\alpha \leq \Sigma$. For such α let H be the Σ_2 Skolem hull of ω in L_α. Then H is a limit initial segment of L and therefore if α is less than Σ, H equals L_α. It follows that for limit $\alpha < \Sigma$, there is a parameter-free Σ_2 definable partial function from ω onto L_α. There is no such function for Σ, as the range of any such function must be contained in L_κ, where L_κ is Σ_2 elementary in L_Σ and κ is less than Σ. □

Remarks. (i) The previous corollary does not need the full power of Theorem 1. It only needs that for limit $\alpha \leq \Sigma$ there is a parameter-free Σ_2 definable partial function from ω onto the least Σ_2 elementary submodel of L_α. (For any limit α, there is a least Σ_2 elementary submodel of L_α, namely the set of L-least elements of nonempty Σ_2 definable subsets of L_α.) This weaker property holds whenever L_α is the Σ_1 Skolem hull of $\{\beta\}$, where β is the largest Σ_1 stable in L_α (should this exist).

(ii) Regarding the κ and Σ of the previous corollary: κ must be the largest Σ_1 stable in Σ, as any larger Σ_1 stable would yield the same Σ_2 theory as Σ, in contradiction to the leastness of Σ. Also, Σ is not Σ_1 admissible, as otherwise there would be an $\bar{\Sigma} < \Sigma$ such that $L_{\bar{\Sigma}}$ satisfies all Π_2 sentences true in L_Σ (since this set of sentences belongs to L_Σ); but then $L_{\bar{\Sigma}}$, L_Σ have the same Σ_2 theory, again contradicting the leastness of Σ. So in fact there is a bijection between ω and Σ which is Σ_1 definable over L_Σ with parameter κ.

Question. For exactly which limit α does there exist a parameter-free Σ_n Skolem function for L_α?

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